Digital Signal Processing
Prof. C.S. Ramalingam
Department Electrical Engineering
Indian Institute of Technology, Madras

Lecture 74: The Discrete Fourier Transform (1) -Review of Fourier methods

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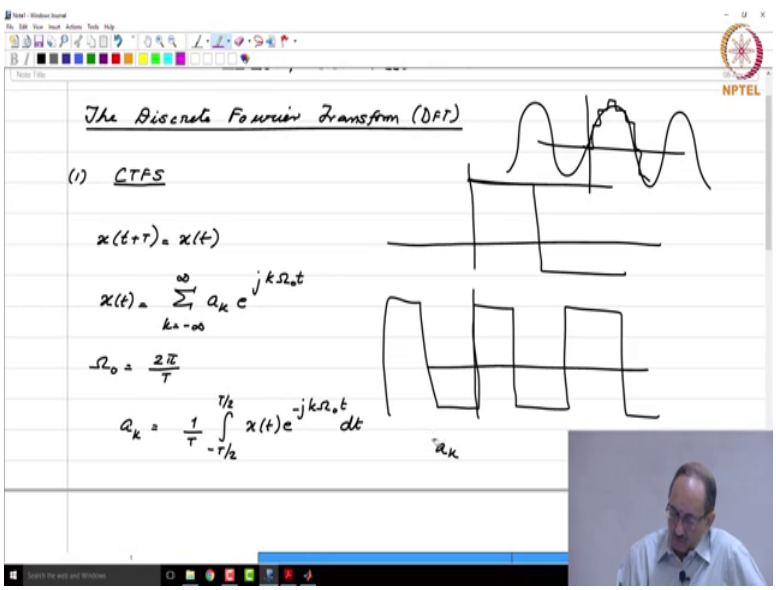

Let us get started here. So, we are now moving on to the last module in the course, namely the Discrete Fourier Transform. And this is commonly abbreviated as the DFT. So, before we begin looking at what the DFT is, it is good to review the various Fourier analysis that we have encountered so far. And, the very first Fourier analysis that you had encountered was the Continuous Time Fourier Series or CTFS and there the assumption was the signal x was periodic, it had a period T and hence it satisfied this relationship.

If this were the case, you were told that it admits a Fourier series expansion and the expansion was given like this, $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}$. Ω_0 was $2\pi/T$, a_k were the Fourier series coefficients. And a_k, the expression was $\frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\Omega_0 t}$. So, this was the assumption about the signal namely periodicity, this was the Fourier series expansion, a_k were the Fourier series coefficients and a_k was given like this; of course, the alternative formula for a_k is 0 to T which is equivalent.

One of the motivations for this is machine representation. That is, you would like to work these things on a computer, $x(t)$ is a continuous time signal and hence that cannot be stored on a machine, because the independent variable takes on a continue of values. The other aspect of course is a_k; a_k is discrete just a sequence which is good, but then you have in general infinite number of them because k still goes from $-\infty$ to $+\infty$ and hence you cannot really store all of the a_k. And, hence this is not amenable to machine representation.

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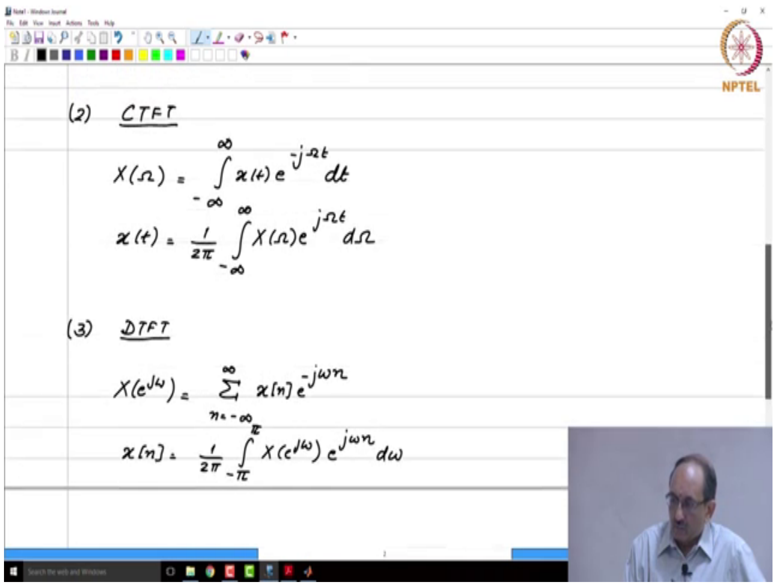

The second Fourier representation that you had seen was the Continuous Time Fourier transform. There the continuous time Fourier transform $X(\omega)$ was given by $\int_{-\infty}^{\infty} x(t)e^{j\Omega t}dt$. And, $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)e^{j\Omega t}d\Omega$.

These are the forward and inverse transform expressions. Again, $x(t)$ is continuous variable signal; t is a continuous variable as before. Only difference now is, $x(t)$ is not periodic. Therefore, you can not represent $x(t)$ on a machine and if you look at the transform, transform is $X(\Omega)$, Independent variable omega is also continuous.

Therefore, once again, neither can you represent the signal on a machine nor it is Continuous Time Fourier Transform, because in both cases, the variable is continuous. The third Fourier series that you had encountered was the Discrete Time Fourier Transform. We are given a sequence $x[n]$ and the Discrete Time Fourier Transform is $X(e^{j\omega})$.

So, this in general, goes from $\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$. So, this was your Discrete Time Fourier Transform and then the IDTFT was $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$.

So, this was the inverse discrete time Fourier transform. We know when we encountered this, we made the remark that the DTFT is nothing new, it is really the continuous time Fourier series in disguise except that this Fourier representation is applied to the 2π periodic function, namely the DTFT.

Therefore, whatever objections we had to the CTFS in terms of machine representation, applies to this. Neither the transform $X(e^{j\omega})$ can be represented on a machine because this is a continuous variable nor the sequence, even though it is discrete, you have infinite number of them. Therefore, this is not amenable for machine representation.

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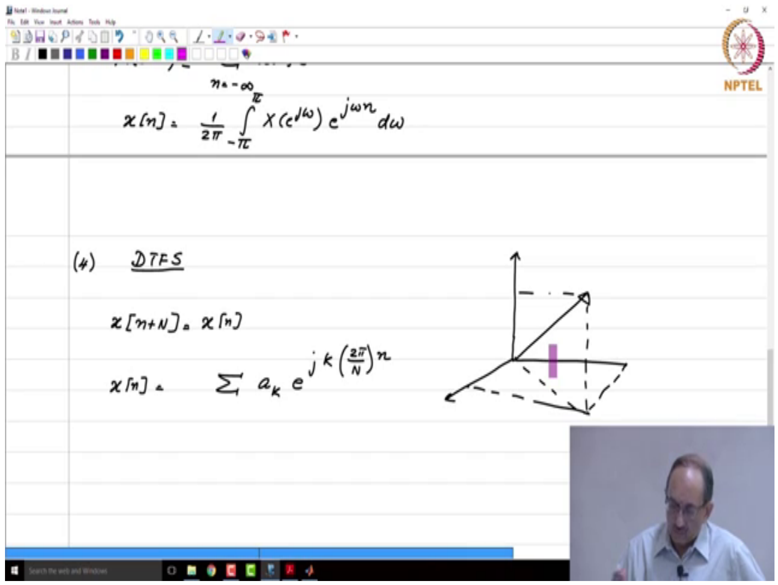

The last Fourier series representation is Discrete Time Fourier Series. I do not know if this was mentioned in signals and systems, was it, yes or no?

Student: No.

No, ok, but it is not too difficult to see what this is. Once you know the principle behind CTFS, DTFS is no different conceptually. So what we have, moment you have Fourier series, you expect some periodicity to be present. In the continuous time case, the signal was periodic with period T.

Now, you are applying the same principle to the discrete time counterpart. You have $x[n+N] = x[n]$. Therefore, this admits a Fourier series expansion and hence $x[n]$ can be represented like this. So, you have $x[n] = \sum a_k e^{jk\omega_0 n}$; ω_0 there was $2\pi/T$. Here the periodicity is N.

Therefore the role of ω_0 that you had seen in the continuous time Fourier series case is now played by the term $2\pi N$; $k(2\pi/N)n$. I do not know how CTFS was introduced to you, was it just given? Ok, this is the Fourier series expansion and this is the formula for the Fourier series coefficients was. That how was it, the way it was introduced or was motivation given from something that you already knew.

Student: Eigen function of LTI system

Ok, so you are told it was eigen functions of LTI systems, all right. So, was that the motivation given.

Student: Like (Refer Time: 09:38) system can we give eigen function as the

Yeah, sure, ok.

Student: So, we wanted to make the any input in terms of CTFS.

Ok. All right, that is one way of approaching CTFS, but really what you have known even before learning about Fourier series is, you have been taught generalized Fourier series even before encountering Fourier series and you have been taught that even before landing up here. Even as part of your school education, you have been taught generalized Fourier series, how about that.

And this, you have been taught about generalized Fourier series even before tenth standard. Sixth, seventh standard I think.

Student: Changes (Refer Time: 11:07) for example.

So, when did you learn about coordinate systems? About that time, correct? All right. And, then later, you were taught about vectors and typically you begin with 3D vectors. So, that it is easy to visualize.

And then, what you are taught is, I have a vector like this and then I have projections like this and then I project it onto the x axis, I project it onto the y axis and I project this vector onto the z axis and I get three coefficients. And then, you can think of this vector as being made up of the projection onto the x axis times the unit vector along that direction.

Similarly, for y and z; therefore, you can think of this 3D vector as being made up of three components whose amplitude in each of the independent vectors is given by the projection of that vector onto that x, y and z axis. And if you had in general a vector in 3D space, you needed three such basis vectors. You needed the i, j and k or a_x, a_y and a_z. Various names given for the three coordinate axis. And, if you only had a 2D vector, you needed only two basis functions. And this can easily be extended to N dimensions, although you cannot visualize beyond three.

If we had four dimensional vector, you need four basis functions and typically these are orthogonal basis functions. And then, to get the coefficient vector for each of the basis functions, all you did was, you projected them onto each of the unit vectors which are orthogonal and then the original vector was now a sum of each of these vectors. Each vector is now multiplied by its coefficient that was obtained by projecting the original vector onto each of these coordinate axes.

So, if you had N dimensions, you needed N basis vectors and you also needed N coefficients. Now, what is happening in CTFS? So, this is periodic with period T and your basis vectors or $e^{jk\Omega_0 t}$ and k itself goes from $-\infty$ to $+\infty$. So, in this case, the number of basis vectors is countably infinite and then you had this a_k. So, each of these basis vectors is now multiplied by these coefficients and then your given signal $x(t)$ is now expressed as coefficient times the corresponding basis vector summed up over all basis vectors.

And to find the coefficient themselves, you need to take or rather you needed to project the vector onto the basis function. And it so happens that you can view projection as what is called as the inner product and inner product for vectors turns out to be multiplying component wise and summing up, but that is not the only definition; that is one of the definition that satisfies the general in the product concept.

If you took up a course on function analysis, you would be talking about vector spaces, norms, inner products and distances and all that is required for a inner product is to satisfy certain axioms. It so happens that, the component wise multiplication of two vectors and then summing up happens to satisfy the definition of inner product for an N dimensional space.

If on the other hand the space were continuous, the inner product is defined something like this. So, what you are doing is, you are taking the signal and then projecting it on to $e^{jk\Omega_0 t}$, which is the k^{th} basis function and the definition of the inner product in this case is x of t times y star of t dt and you integrate over that space.

And it so happens that, the space here is this integral and the limits of the integral is between $-T/2$ to $+T/2$ or between 0 to T. So, $x(t)y^*(t)dt$. So, $x(t)$, remember, $y(t)$ is our basis function which is $e^{jk\Omega_0 t}$, that is why you have $e^{-jk\Omega_0 t}$ here. Because this minus comes because of complex conjugation and the inner product definition in this case happens to be this.

Therefore, all you are doing when you are getting the Fourier series coefficients is that, you are taking the signal and projecting it on to the space of $e^{jk\Omega_0 t}$, as simple as that. And that projection happens to be this integral, that projection which was component wise multiplication for N dimensional case turns out to be this here. Therefore, this representation of $x(t)$ as sum up over all k, $a_k e^{jk\Omega_0 t}$ is nothing but Fourier series expansion, which is another illustration of expressing a signal or a vector in terms of the components that constitute the basis for that space. In this case, the space happens to be infinite dimensional. And notice that, again if you had a three dimensional vector, you could represent it as component along the x, y and z direction and you have your usual Cartesian coordinates.

But, you could also express a vector using, say polar coordinates or spherical coordinates, right. Now the basis vectors are r, θ and ϕ. Or in this spherical coordinate case or ρ, ϕ and z in this cylindrical coordinate case. And now what you would do to express a vector, 3D vector in these coordinates, you would take that vector and project it along the unit vectors in each of these directions.

And now, you have a representation which is different from Cartesian coordinates and yet is a valid representation of that vector in terms of basis functions which are now different. Now, this is not the only representation possible. You can now have a different set of basis functions. For example, you could express it in terms of Walsh basis functions.

Now, Walsh basis function, the one basis function is like this. The next basis function will be like this and then you can have the other basis function like this. I am not trying this exactly, I am just giving you a feel. So, now my basis functions are constant and square pulses. So, this is my basis function. Now I can choose to represent a waveform in terms of these basis functions.

I will do exactly this, now my expansion in this basis function will still be $x(t)a_k$. Instead of $e^{jk\Omega_0 t}$, I will have these basis functions. And, now to find the a_k's, I need to project my signal onto the each of these basis functions which are not different. Therefore, this integral will be different. So, exactly the same idea, but different basis functions. Now, let us see the implication of this. If I have a pure sine wave and the express this in terms of its Fourier series, what will happen is the coefficients that I have, if this were a real valued sinusoid and if I use this, then how many coefficients will you have? You will have two coefficients.

Depending upon the frequency, you will have a_k and $-a_k$ where $k\Omega_0$ is the frequency of this given signal and you would call this as being band limited because there are only two coefficients. The frequency components are frequencies will be $-k\Omega_0$ and $+k\Omega_0$, there are no other coefficients. Therefore, this is what is called as a bandlimited signal.

Now, suppose I have this waveform instead of this, I have this. And now I am expressing this as Fourier series. And now this was one of the first exercises that must have been given to you when you are doing Fourier series and then you had a_k which was a simple formula and one thing that you can surely say about a_k is, it fell off as $1/n$ or $1/k$. But this had, in this will have how many finite set of coefficients or infinite set of coefficients? This will have infinite set of coefficients and this is not?

Student: Band limited.

Band limited. Now let us take this square wave and now do a Walsh expansion,. So, these are the basis functions for Walsh and if I want to express this in terms of Walsh expansion, what do you think will be the number of coefficients that will be required?

Student: 1.

It will be just 1, very good. Now let us take this and now I want to express this in terms of Walsh expansion. If you want to guess how many coefficients you would need, what would be your guess?

Student: Infinite.

Infinite number of coefficients and to get a feel for what is going on, if you truncated this using finite set of quotients, what do you think will be the waveform?

Student: (Refer Time: 23:56).

Very good, it will be like this. So, this will be the approximation that you will get, all right. So, this Fourier series is nothing magical, all it is doing is it is taking one vector and expressing it in terms of the basis functions. So, what basis function you use determines the expansion and Fourier series was the first such thing that happened and people later realized, this is what is going on.

The concept of vector space started to build up in the nineteenth century and then this signal x of t can be thought of as a point in a vector space and hence you can have generalized expansion. People then began to realize that these are nothing but basis functions and then they realize you do not just need to have this kind of basis function, you can have any basis function.

The basis function then from your linear algebra, you would know that, the basis functions need to be?

Student: (Refer Time: 24:56).

No.

Student: Independent.

Independent. If it is orthogonal, then all you need to do is to get the coefficient for that basis function, you need to project it only on that vector. So that, as you add more and more number of basis functions, if you added more basis function, the earlier coefficients remained as they are. And because the expansion is orthogonal, one more basis function if you add, all you need is project the signal onto that newly added basis function, get the coefficient and then keep on increasing the accuracy of the approximation.

Though, therefore, if in general the basis, the space is infinite dimensional. The equivalence is exact, when you have infinite number of coefficients. If you truncate them, you will get the familiar Gibbs phenomenon and that is the error.

So, it is now different from taking a three dimensional vector and then approximating it by two vectors which means, the best approximation would be projecting it onto the xy plane. So, all of this has a very strong intuitive and geometric flavor to this. So, Fourier series is nothing but expansion of a given signal in terms of its basis vectors. Once you realize that, then everything will fall in place, there is no magic.

So, Fourier series will be first and then you had other kinds of things like Walsh and so on. All this background was given to motivate DTFS. Let us look at this. So, this is going from k, going from whatever the number of basis functions are there for this space.

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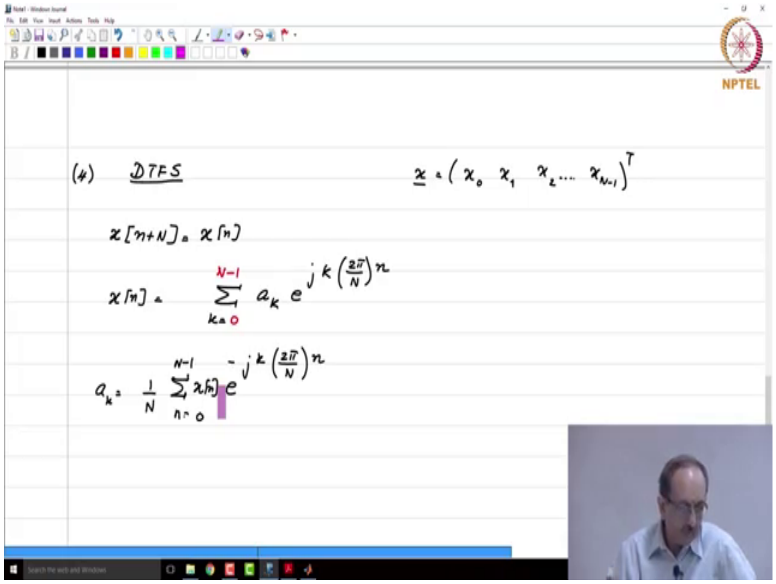

So, the basis functions for this space is consisting of $k(2\pi/N)n$ and a_k is tied to the k^{th} basis function. If you now replace k by $k+N$, what happens to the basis function? It remains the same. And, when you are going to express the signal in terms of its basis functions, you would use as many basis functions as there are in the space.

In terms of continuous time Fourier series, when you took $e^{jk\Omega_0 t}$ and when you let k vary from $-\infty$ to $+\infty$, you generated for each k, a basis function that was independent for every value of k. No value of k that is different would produce something that was what we had seen before. For each and every value of k, the basis function is different and you had to use every single basis function that was there to represent this space completely without any error. Now in this case, the number of basis functions is what.

Student: Finite

Finite and it happens to be N, N and hence you need to go only from 0 to $N-1$ because those are the only independent basis functions that are present. Any value of k outside this interval will give the same basis function. Therefore, as far as DTFS is concerned, this is what the definition is. Now you need to get hold of a_k and you needed to project the given signal onto the basis function. Now what happens here, you can now think of this $x[n]$ as a vector. You can think of this as, I am using the subscript notation here for simplicity; x_0 is really $x[0]$. So, I can think of this as a N component vector.

Student: (Refer Time: 29:34).

Remember, $x[n+N] = x[n]$. So, your only range of values you are interested in this is, from 0 to $N-1$. Anything outside this range is something that already falls in this range. Therefore, to completely specify this sequence, you can think of this sequence from 0 to $N-1$ as a vector with having elements x_0 to x_{N-1}.

Similarly, your $e^{jk\omega_0 n}$ also can be thought of as a vector and to find the component, you need to take the inner product. So, $x[n]$ and when you take the inner product, really it is not $x[n]y^*[n]$ summed up over all n. Really, the inner product in general is, $x[n]y^*[n]$ summed up over all n, that is the general case.

Therefore, I sum this up, n going from 0 to $N-1$ and you can have this factor $1/N$. So, let me now kind of write it more clearly. Therefore, $a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk(2\pi/N)n}$. So, this is really your DTFS and now you can see the connection between CTFS and this.

So, signal is periodic at which the Fourier series expansion and because the basis are only finite, the Fourier series expansion goes only from 0 to $N-1$; a_k are the Fourier series coefficients, a_k are given by this formula. So, this is very similar to the a_k in the continuous time case, where you had an integral, here this is summation. But the underlying conceptual outlook is the same.

All right, so these are the four different Fourier series representation that you have seen so far.