Digital Signal Processing

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Lecture 73:
Sampling (4)
– Reconstruction from samples

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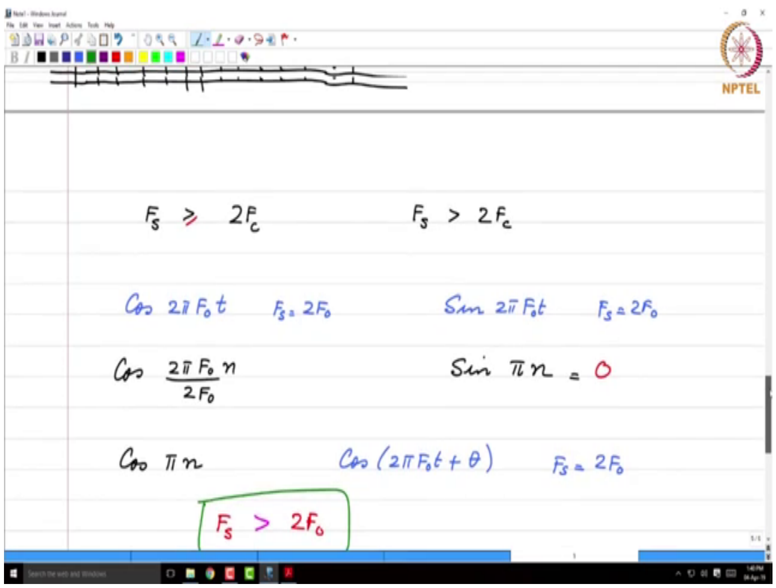

Let us look at another aspect of Sampling. So, we said that $F_s \geq 2F_c$, where F_c is the highest frequency. The question we want to ask is, if it really greater than or equal to or strictly greater. So, let us take this signal, that is, we have $\cos(2\pi F_0 t)$, and then let $F_s = 2F_0$, therefore, this really becomes cos $\cos(2\pi F_0 nt)$, replace t by nT, $T = 1/2F_0$, and we get $\cos(\pi n)$ as the sequence.

Now, suppose you have $\sin(2\pi F_0 nt)$, again remember, we are all making these observations based on impulse train sampling, ideal low pass filters, brick wall filters and so on. And, F_s is again $2F_0$, and clearly in this case, you get $\cos(\pi n)$ which is 0 for all n. So, what is happening here is, if you sample it at the critical frequency, if it is a pure sine, you will be sampling it at its 0 crossings. And, hence, if you had $\cos(2\pi F_0 t + \theta)$, and if you sample it at the critical sampling frequency of $2F_0$, this is nothing but cosine plus sine, $\cos A \cos B - \sin A \sin B$.

So, what will happen is, when you try to recover this signal, you will only be recovering the cosine part. The sine part you will lose. And, hence as far as the theoretical constant goes, you really need F_s to be strictly greater than $2F_0$.

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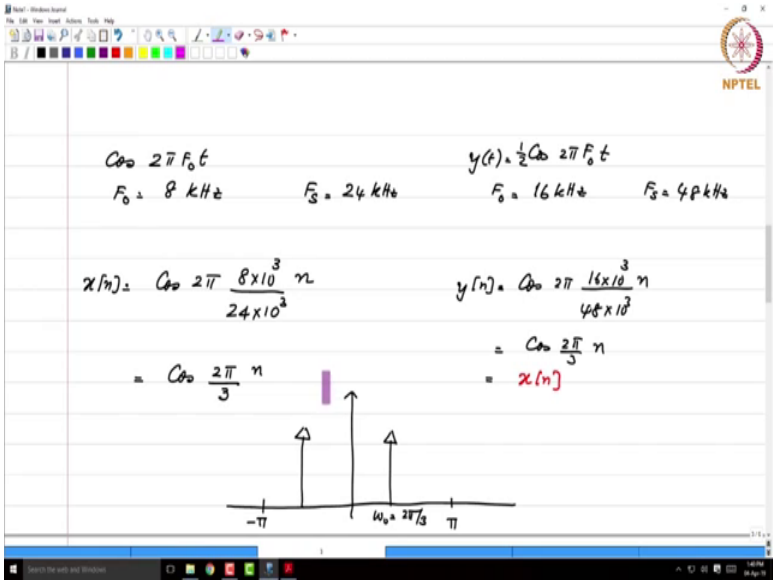

By the way, going back to this just to mention one more point related to sampling it at F_0, rather F_s and $2F_s$.

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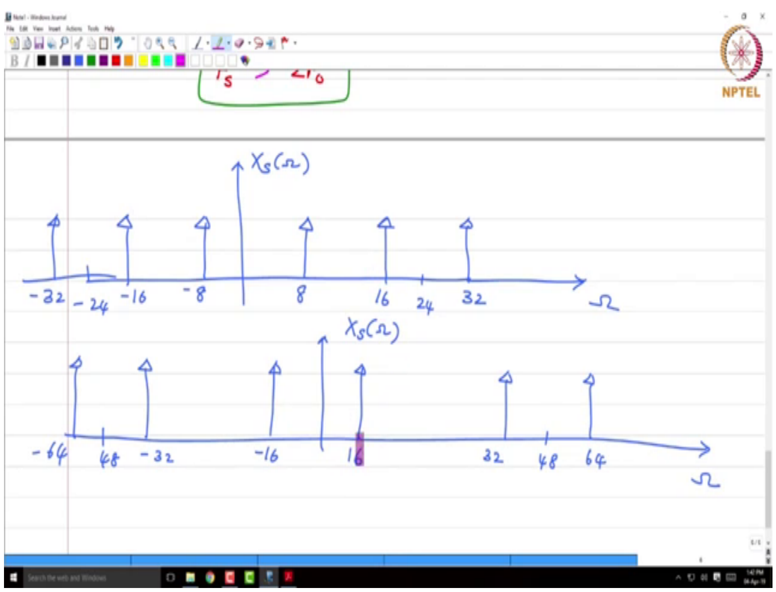

If you look at the corresponding $X_s(\Omega)$, so you have 8 kilo Hertz sampled at 24, that is what we started off with. And let us look at $X_s(\Omega)$. So, this is over sampling, there is not going to be any aliasing. So, $X_s(\Omega)$, so this is 8 kilo Hertz I am plotting this on the frequency scale. So, this is $+8$ and -8, and I am going to sample it at 24 kilo Hertz, therefore this will repeat every 24. So, this repetition will occur at 24. And I will have one impulse at 16, the other impulse at 32 and similarly on this side.So, this will be -24, this will be -16, -32. So, this is $X_s(\Omega)$ for 8 kilo Hertz sampled at 24.

Similarly, the other case was 16 kilo Hertz sampled at 48. So, what will happen is, I have -16 and $+16$, I am sampling it at 48 kilo Hertz, therefore, whatever is sampling at 0 will have to repeat at +48, −48 and so on. And hence, this will be the spectrum of $X_s(\Omega)$ in the second case.

So, this is $X_s(\Omega)$ in the second case. Note that, there is absolutely no ambiguity here. When you are plotting $X_s(\Omega)$, there is absolutely no ambiguity. You will not confuse these two, where confusion happens is, moment you normalized by Ω_s. If you normalize this by 24 and normalize this by 48, both 8 kilo Hertz and 16 kilo Hertz normalized will map to 1/3, times 2π if it is the independent axis, this the Ω axis. So, it is the normalization that causes the problem, whereas $X_s(\Omega)$ by itself there is no confusion here.

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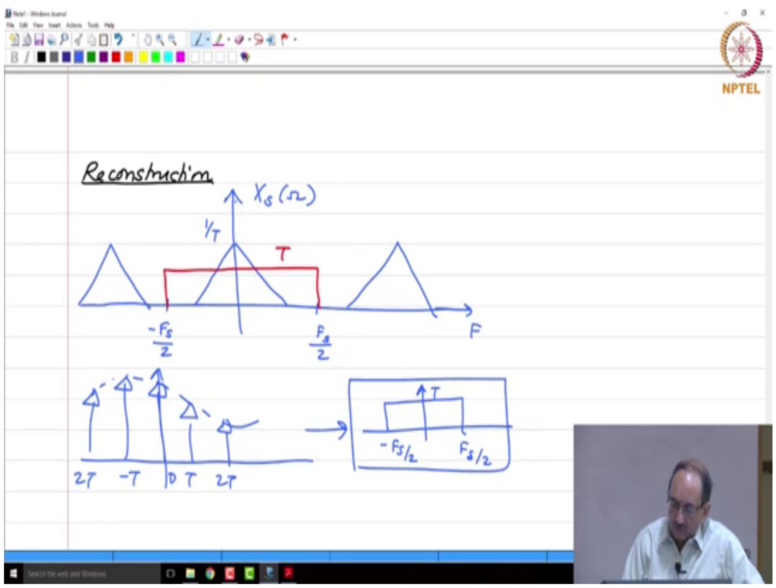

And the last part of sampling is of course reconstruction. So, again let us look at the impulse train sampled continuous time signal. This also must have been part of your earlier course. What we will do now is, we will also relate how reconstruction is done in the discrete time domain. So, what is happening there is, you have this multiple copies of the signal $X_s(\Omega)$ is there, you have scaling by $1/T$, then let me for ease of writing plot this in the in the F domain. So, you need to look at the spectrum only between $-F_s/2$ to $+F_s/2$. If you now put an ideal low pass filter with gain T, it will do two things; one it will counteract the scale factor of $1/T$, then it will completely eliminate all the copies.

So, what you are doing is, you are now applying the impulse train. So, this is the underlying continuous time signal which you have sampled using train of impulses which has spaced kept apart. And to this, you are going to, you are going to feed, the impulse train to this ideal low pass filter with gain of T. And then, you are hoping to recover the underlying continuous time signal exactly.

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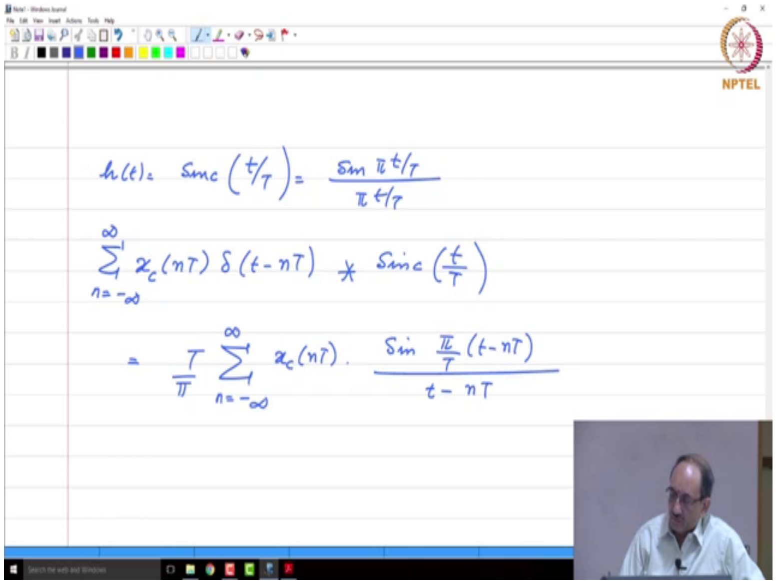

So, this low pass filter has impulse response $sinc(t/T)$, so which is nothing but $\dfrac{\sin(\pi t/T)}{\pi t/T}$. And what is happening is, you have your impulse train coming in, which is $x_c(nT)\delta(t-nT)$. This was your impulse train sampled signal. And then, you are going to feed it to this low pass filter which means in the time domain, you are going to convolve it with this impulse response.

So, you have to convolve it with $sinc(t/T)$. So, this is convolution with impulses is the easiest thing to do. Therefore, this T in the denominator jumps out, you have $\dfrac{T}{\pi}\displaystyle\sum_{n=-\infty}^{\infty} x_c(nT)\dfrac{\sin(\pi(t-nT)/T)}{(t-nT)/T}$, so this is what is happening here. And this is what you must have been told as sinc interpolation.

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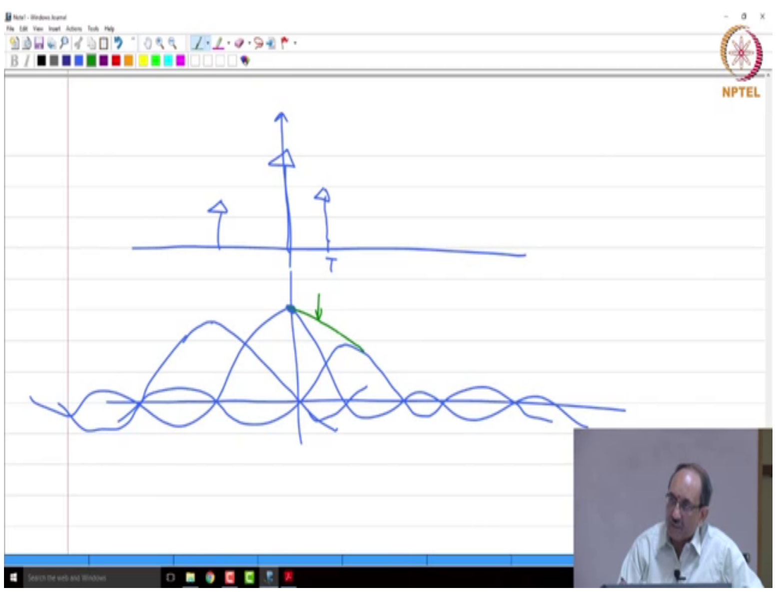

Again, we are looking at theoretical constructs here. Therefore, when the impulse at the origin strikes this reconstruction filter, what will happen is, remember, this the ideal low pass filter which is non castle, therefore, it will respond like this. The next sample comes like this. It will also elicit a exactly the same sinc response, only that it will be delayed by T and it will also be amplitude scaled by whatever this impulse strength is. And hence, if you look at the response due to this, second sample it will be like this.

Similarly, the sample here will trigger a sinc like this. I will show the exact plot later. So, now, what is happening is, note that at the sampling instance, you will find that every other sinc has the zero crossing at these sampling points. Therefore, at the sampling instant, you will have that sample value come out. So, this is true for not only at $T = 0$ 0, but also every multiple of T. In between, you need to add the responses due to all the sinc functions and the underlying continuous curves reconstructed.

So, this point which is in between two samples is the result of adding all the sinc values. So, you get exact reconstruction. At the sample points, all the other sinc functions do not contribute anything, the sample value comes out as it is. In between sampling points, all the sinc functions add up together to give you the exact value, without any loss of information. So, this is the exact reconstruction, and here is the plot from Oppenheim.

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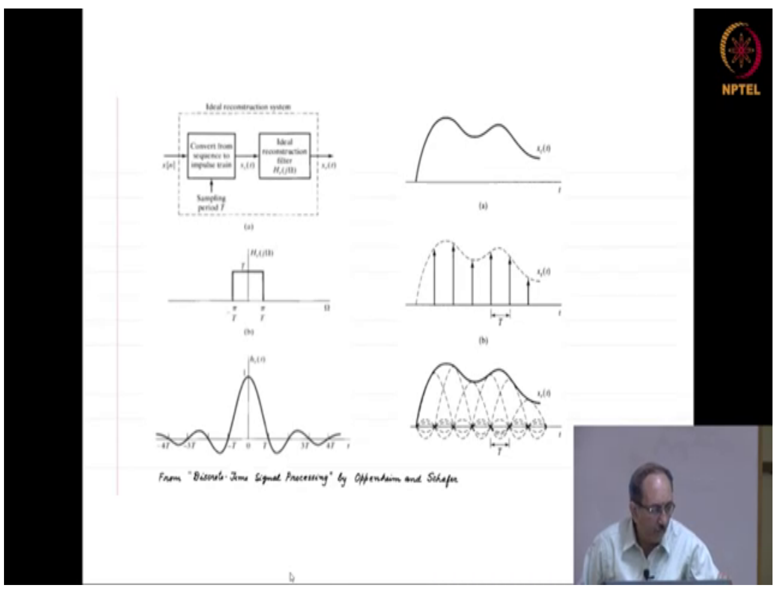

So, we see if I can, if this is. So, this is the underlying continuous time signal. These are the samples. Each sample elicits a sinc response like this. The only difference is the location of the sinc and the amplitude of the sinc. Note that, when you consider this particular sinc, every other sinc goes through zero crossing at that point. In between, they all add up to give you the exact curve without any loss of information. So, this is how the theoretical construct behaves.

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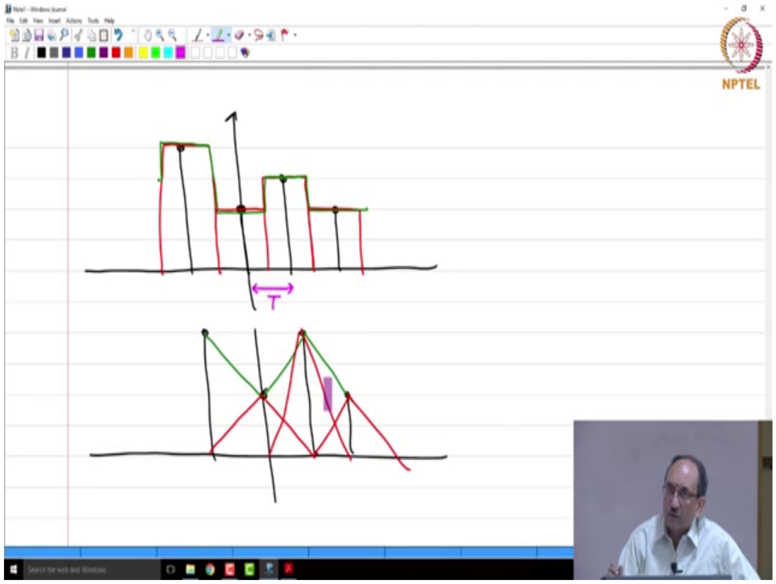

So, this is ideal reconstruction. But what happens in practice is this, I will show the two different ways of reconstructing from samples which is what your actual hardware does. And in this case, I will show the non causal version, you will see what I mean by that. So, if you do sample and hold, so these are the samples that are coming in. And, the sample and hold filter responses like this, this is the non-causal version. What will happen in the causal version is, all you need to do is shift it to the right.

So, this sample comes in, you hold that value. Next sample comes in, you immediately jump to that value. So, the underlying reconstructed curve before further processing is needed is like this. So, this is the waveform that is there, but this waveform has all kinds of discontinuities. So, you need to pass this through a low pass filter, you will get a smooth output and that will be an approximation to the analog signal.

Clearly that works, because all your cell phone is working on samples, telephony samples were transmitted at 64 kilobits per second and yet that the receiving end, you are able to hear continuous time speech, and clearly impulses are not transmitted. Impulse are not transmitted, because first of all impulses are theoretical construct, an impulse has infinite bandwidth. If you transmit impulses train of impulses through a channel, channel is typically a low pass filter, it will band limit the impulse train therefore, all these are only theoretical constrains.

So, what actually happens in practices samples are transmitted and samples are received at the receiver, and they are reconstructed with processes something like this. You do sample and hold, but if you did sample and hold, if you now project this onto the speaker, it will sound buzzy, because it will have all kinds of high frequency components. You have to pass it through a low pass filter and then feed it to the speaker. And note that, samples are actually spaced T apart. So, this is what determines the sampling frequency. So, this is one type of reconstruction.

The other type of reconstruction is, if you had something like this, it is natural to first join these samples by straight line and then do low pass filtering. This is better, because it is only corners but not discontinuities. And now if you pass this through a low pass filter, again you will get a smoothed out version. This clearly has less high frequency component than in the previous case, because this has first derivative discontinuous, therefore, this spectrum falls off as $1/|\Omega|$. Whereas, this second derivative is discontinuous and hence the spectrum falls off as $1/|\Omega|^2$.

So, therefore, this false off much more rapidly and hence it requires less stringent low pass filter to eliminate the high frequency components. And, it is easy to see that, the impulse response of this reconstruction filter is a triangle like this, because if this sample has this impulse response and if this sample has this triangular impulse response, if you add these two edges, you will get this vertical line. So, this is linear interpolation.

So, linear interpolation can be thought of as the underlying filter having a triangular impulse response. Again, I am showing only the non-causal version. In practice, the causal version of this will be implemented. So, this is better than the previous case, because you require a less stringent low pass filter, but again there is no free launch to have come up with a filter that is like this requires more processing. But, if this is all done in the discrete time domain, it is simple, but we are we have to do this in the continuous time domain.

So, you need to come up with a filter that has this kind of impulse response. And then, you will reconstruct. Rather than sinc interpolation, you are doing either sample and hold which is interpolation using this kind of filter versus this kind, and this is better. And, this is what is fed, after you take these processes through a low pass filter and then feed it to your speaker. So, this is how reconstruction is done.

So, with this, we complete sampling, and then we can move to the last part of the course, namely discrete Fourier transform.