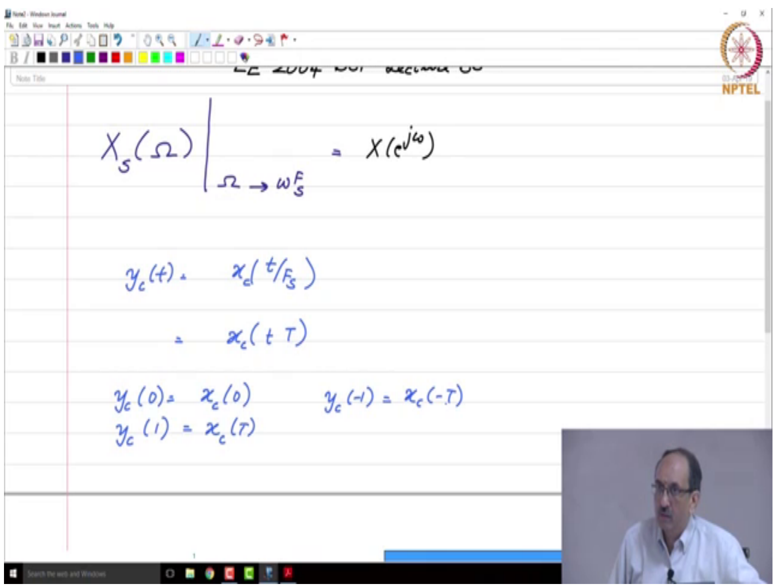
Digital Signal Processing
Prof. C.S. Ramalingam
Department Electrical Engineering
Indian Institute of Technology, Madras

Lecture 70:
Sampling (3)
-Examples of going from $X_s(\Omega)$ to $X(e^{j\omega})$

(Refer Slide Time: 00:23)



Let us continue with sampling. So, $X_s(\Omega)$ is the impulse train sampled signal's spectrum. Two things happen, scaling by $1/T$ and periodic repetition with period Ω_s. And this, if you replace Ω by ωF_s, then you get the DTFT. So, what this step does is, it takes the Ω_s periodic spectrum and you scale it by F_s.

And, scaling by F_s converts the Ω_s periodic spectrum to a 2π periodic spectrum which is what the DTFT is. And, that is the connection between the sequence whose values are samples of the underlying continuous time function. The samples are taken T apart. But, when you plot it as a sequence, you plot them as a sequence of numbers with the independent axis being $n = \{0, \pm 1, \pm 2, \ldots\}$.

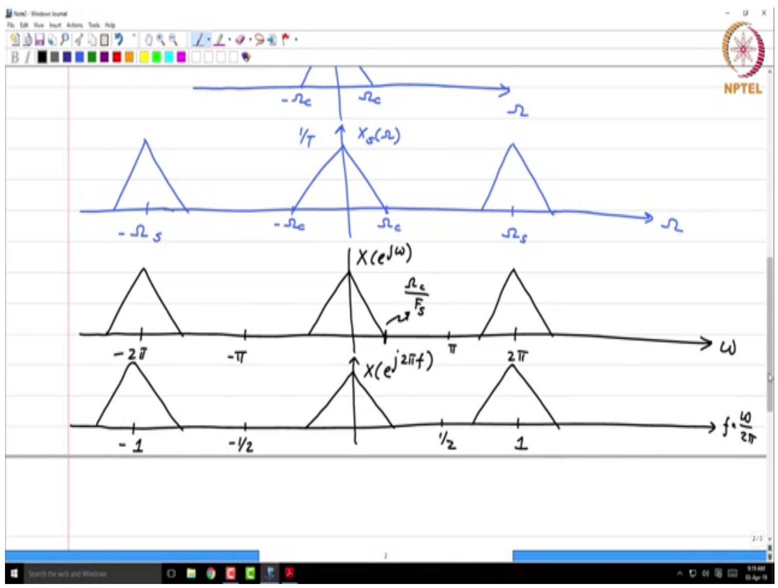
And, that sequence of numbers has DTFT $X(e^{j\omega})$ which is 2π periodic. And, that spectrum is intimately associated with the impulse train sampled signal's CTFT, which is Ω_s periodic by a scale factor of F_s which is the compression of the frequency axis. Compression if F_s is greater than 1 and expansion if F_s is less than 1. And this is all consistent. Remember, if you are going to scale the frequency axis by F_s, in the time domain, you will inversely scale.

If F_s is the scale factor in the frequency domain, $1/F_s$ will be the scale factor in the time domain. Therefore, this is really $x_c(tT)$ and notice that $y_c(0) = x_c(0)$. And $y_c(1) = x_c(T)$, $y_c(-1) = x_c(-T)$.

So, this compression by a scale factor of F_s in the frequency domain, when I say compression, I am assuming it is the case where F_s is greater than 1 which is the typical case that appears in practice. Therefore, this compression by a factor of F_s in the time domain leads to an expansion in the time domain. And, the expansion in the time domain is such that the expansion factor causes the samples that were originally T apart to now be 1 second apart.

So, everything falls in place and this is consistent to the fact that the sequence are placed one apart. Because, the independent axis is n and the samples are one apart, $\{0, \pm1, \pm2, \ldots\}$. Hence, you can think of $y_c(t)$ as doing an expansion in the time domain and then taking samples one second apart. If you did that, then your spectrum will be 2π periodic. So, all of this is consistent with what we have seen before.

(Refer Slide Time: 04:29)

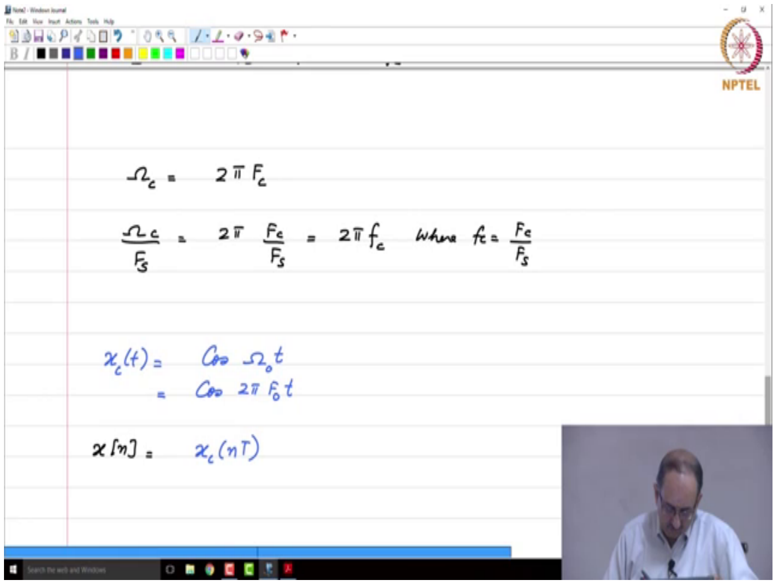


And, just to complete the picture, so this is already in my notes and recapping this point. So, you have a bandlimited signal $x_c(t)$ whose spectrum is this. And then, when you sample this, you will get $X_s(\Omega)$. So, you have a scale factor of $1/T$ and then this periodically repeats, the periodicity is Ω_s.

And, now we want the spectrum of the underlying DTFT. So, this now is $X(e^{j\omega})$, this is also periodic only that its periodicity is 2π. Of course, the way I have drawn this, for ease of comparison, I have made this 2π line up with Ω_s. So, clearly the second of the third plots are not to the same scale.

And the other way of looking at this rather than looking at the Ω scale, you can also look at the F scale which is nothing but $\Omega/2\pi$. And this point continues to be Ω_c. So, we are compressing by a factor of F_s. Therefore, this frequency which was Ω_c, now becomes Ω_c/F_s.

(Refer Slide Time: 07:08)



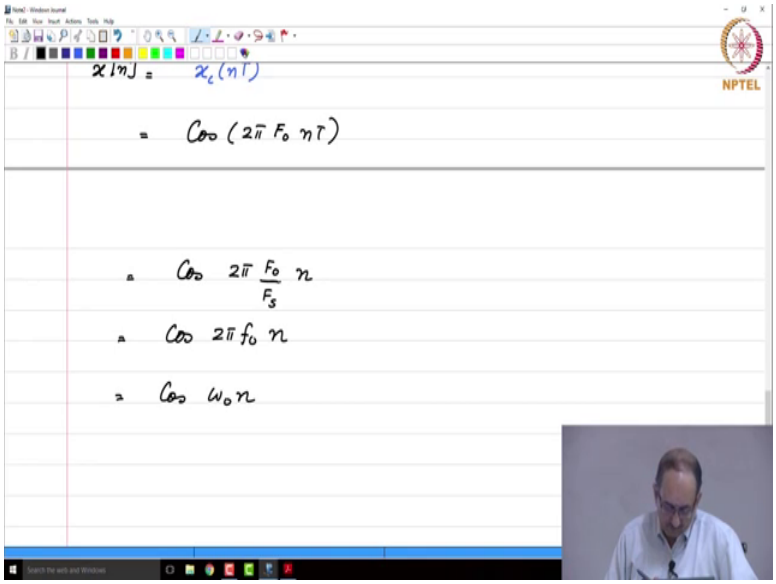
Note that, $\Omega_c = 2\pi F_c$. And hence, $\Omega_c/F_s = 2\pi F_c/F_s$. And this can be written as $2\pi f_c$, where $f_c = F_c/F_s$. And hence, if your independent axis is f rather than ω and ω and f are related by a scale factor of 2π therefore, this becomes $+f_c$.

So, this is how the analog frequency in Hertz is mapped to the digital frequency f by this normalization. And, f sometimes is also called as the normalized Hertz. Therefore, whatever your sampling frequency is and if you know the true frequency in Hertz, if you divide that by the sampling frequency, you will get the normalized frequency.

Again everything is consistent, ω is dimensionless number because it is the argument to the exponential. Therefore, f also is dimensionless because it is ratio of F_c to F_s; F_c is in units of Hertz, F_s is in units of Hertz. Therefore, the f scale is again a dimensionless quantity. And just to illustrate this with a sinusoid, if we had $x_c(t) = \cos(\Omega_o t)$.

This in turn is $\cos(2\pi F_o t)$, because $\Omega_0 = 2\pi F_o$. And then, if you think of the sequence $x[n]$ which is obtained by taking uniform samples of the underlying continuous time signal. Remember, this $x[n]$ is not the impulse train sampling, it is just the sequence of numbers taken T apart therefore, this is $x_c(nT)$.

(Refer Slide Time: 10:07)

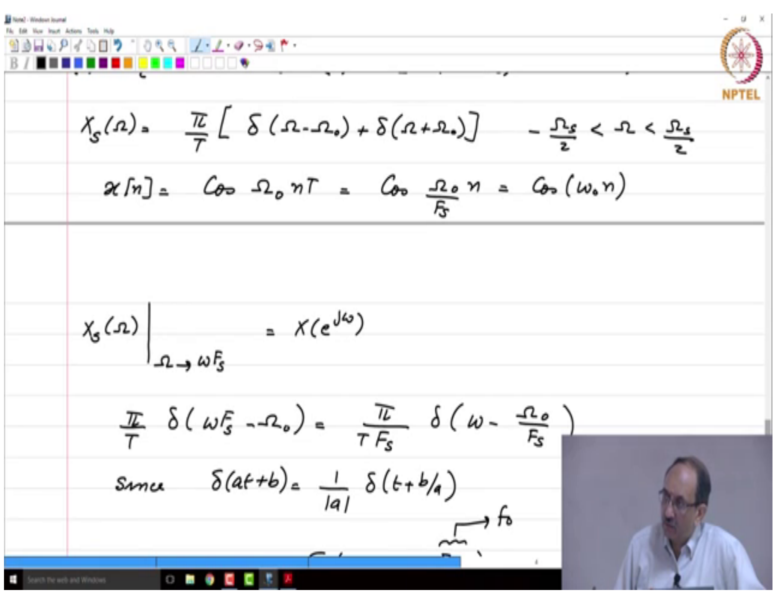


And, this is nothing but $\cos(2\pi F_o nT)$, replace t by nT. So, this is this which in turn is $\cos(2\pi F_0 n/F_s)$, T after all is $1/F_s$. And, we know that $F_0/F_s = f_0$ and $2\pi f_0 = \omega_0$. And hence, if you have a sinusoid with frequency Ω_0 and if you sample it at T apart, you get a sequence which is $\cos(\omega_0 n)$.

And $\omega_0 = 2\pi f_0$ and $f_0 = F_0/F_s$. So, you take the true frequency, normalize it by the sampling frequency to get the normalized frequency. And, then 2π times the normalized frequency gives you the radian frequency.

Now, we will do few examples in which, we are going to take the underlying continuous time signal and then get its continuous time Fourier transform. Then, we will periodically repeat it and scale it by $1/T$. And, then we will replace Ω by ωF_s. This should give us $X(e^{j\omega})$ based on whatever we have developed so far. And we will apply this to actual examples and then verify we get what was known to us before.

(Refer Slide Time: 11:59)



So, that is a first example, we will take $x_c(t) = \cos(\Omega_0 t)$. This has continuous time Fourier transform $\pi\Big[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)\Big]$. This is the continuous time Fourier transform $X_c(\Omega)$.

Now we look at $X_s(\Omega)$, you need to do two things; scale it by $1/T$ and then periodically repeat it. Therefore, this becomes $\frac{\pi}{T}\Big[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)\Big]$. So, this seems identical to the previous step except for the amplitude scale factor, what else needs to be put in here? T.

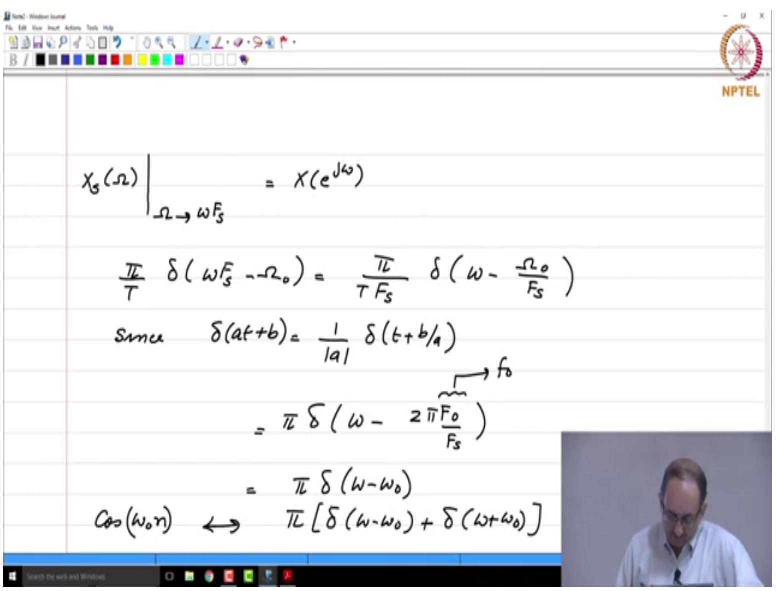
Student: minus (Refer Time: 13:11).

So, this is periodic repetition. So, you have to either periodically repeat the impulse or restrict the range. So, this is valid from $-\Omega_s/2 < \Omega < \Omega_s/2$. So, this is the fundamental copy of the periodic repetition of the spectrum. Now, we are going to take $X_s(\Omega)$ and then we will replace Ω by ωF_s.

And, if everything were done right, we should get $X(e^{j\omega})$. Let us look at this particular factor $\frac{\pi}{T}\delta(\Omega - \Omega_0)$, that is, $\Omega - \Omega_0$. We are going to take Ω and replace Ω by ωF_s. We will just concentrated on the first factor and do the simplification.

We also know that $\delta(at + b)$ is $\frac{1}{|a|}\delta\left(t + \frac{b}{a}\right)$. So, now, let us apply this formula to this previous term. So, now, you have $\frac{\pi}{TF_s}\delta\left(\omega + \frac{\Omega_0}{F_s}\right)$.

(Refer Slide Time: 15:25)



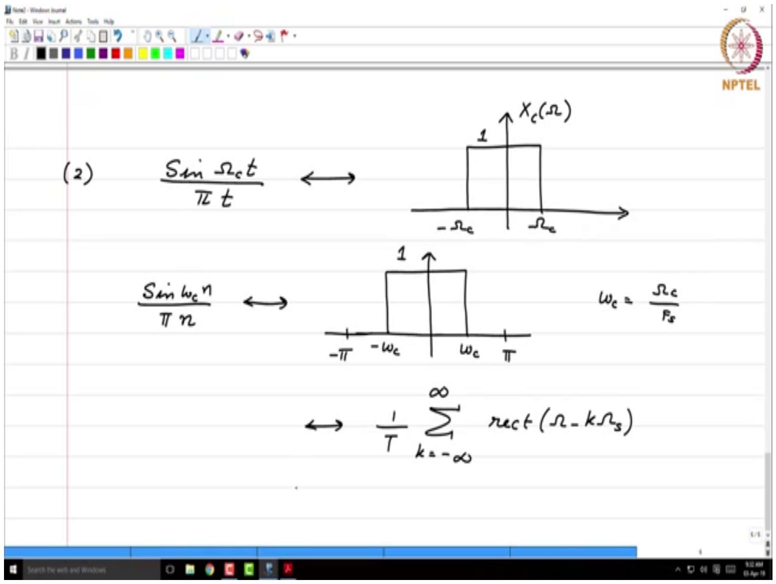
This in turn becomes π, denominator is $TF_s = 1$. And now you have $\Omega - \Omega_0$ after all is $2\pi F_0/F_s$. $F_0/F_s = f_0$. And $2\pi f_0 = \omega_0$; therefore, this simplifies to $\pi\delta(\omega - \omega_0)$. And this should correspond to the samples taken T apart.

Therefore, this is $\cos(\Omega_0 nT)$, which in turn is $\cos(\Omega_0 n/F_s)$, which we just earlier saw this was nothing but $\cos(\omega_0 n)$. Therefore, the samples of the signal taken T apart gives us the discrete time sequence $\cos(\omega_0 n)$. And, the first term turns out to be $\pi\delta(\omega - \omega_0)$.

Similarly, the second term will turn out to be $\pi\delta(\omega + \omega_0)$. And, hence we get the familiar transform pair

DTFT pair, $\cos(\omega_0 n)$ is nothing but $\pi\Big[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\Big]$. So, what the theory that was worked out for the general case has been illustrated for this specific example.

(Refer Slide Time: 17:22)

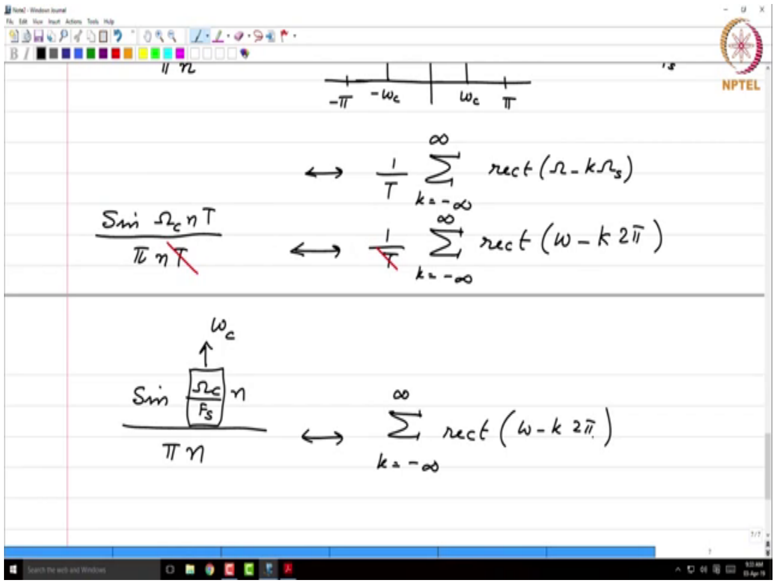


Let us look at the second one. $\dfrac{\sin(\Omega_c t)}{\pi t}$ is your underlined continuous time signal. This continuous time Fourier transform is the ideal low pass filter with cutoff Ω_c. So, this is really $X_c(\Omega)$. Now what we are going to do is we are going to sample this. And, when we are done with this, we are trying to relate this to this pair, $\dfrac{\sin(\omega_c n)}{\pi n}$.

This is the samples of the continuous time ideal low pass filters impulse response. And, starting from the earlier figure, if everything went right, we should derive this transform pair. And this is between $-\omega_c$ to $+\omega_c$ and ω_c is nothing but Ω_c/F_s. And if you replace t by nT, this becomes $\dfrac{\sin(\Omega_c nT)}{\pi nT}$.

And now, so this is the sequence. The corresponding impulse train sampled signal will be $\sum_{k=-\infty}^{\infty} rect(\Omega - k\Omega_s)$, where $rect$ stands for this rectangular function between $-\Omega_c$ to $+\Omega_c$. So, this is the transform of the impulse train sampled sequence in continuous time. And now, when you, all you need to do is you need to scale this by 2π to get the 2π periodic spectrum.

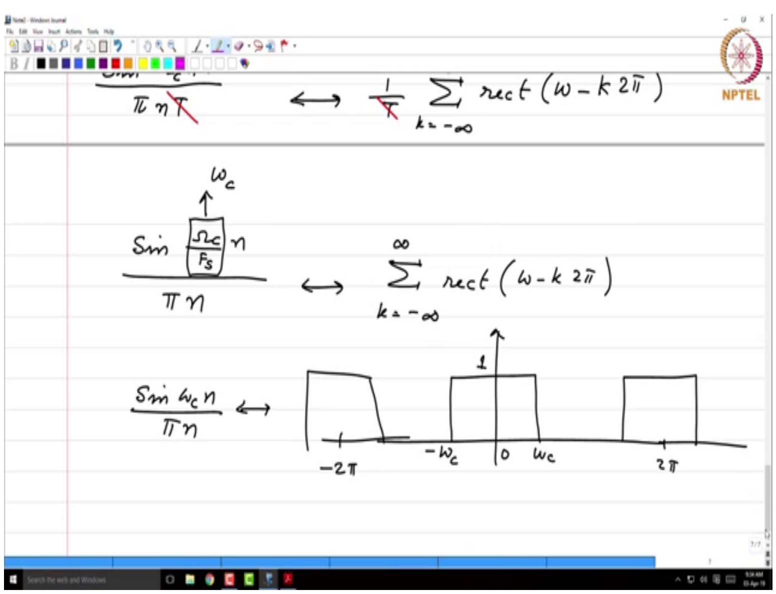
(Refer Slide Time: 20:25)



And the samples are $\frac{\sin(\Omega_c nT)}{\pi nT}$. So, now, this becomes $\frac{1}{T}\sum_{k=-\infty}^{\infty} rect(\omega - k2\pi)$. So, what I have done in this step is, going from here to here I have a scaled by 2π. And this should correspond to the discrete time Fourier transform of this sequence.

And this sequence is obtained by taking samples of the underlying continuous time function T apart. And notice that, these two terms get cancelled. And hence, you have sine, T after all is $1/F_s$, $\frac{\sin(\Omega_c n/F_s)}{\pi n}$. This Ω_c/F_s by definition is now ω_c and hence this has $\frac{\sin(\omega_c n)}{\pi n} \longleftrightarrow \sum_{k=-\infty}^{\infty} rect(\omega - k2\pi)$.

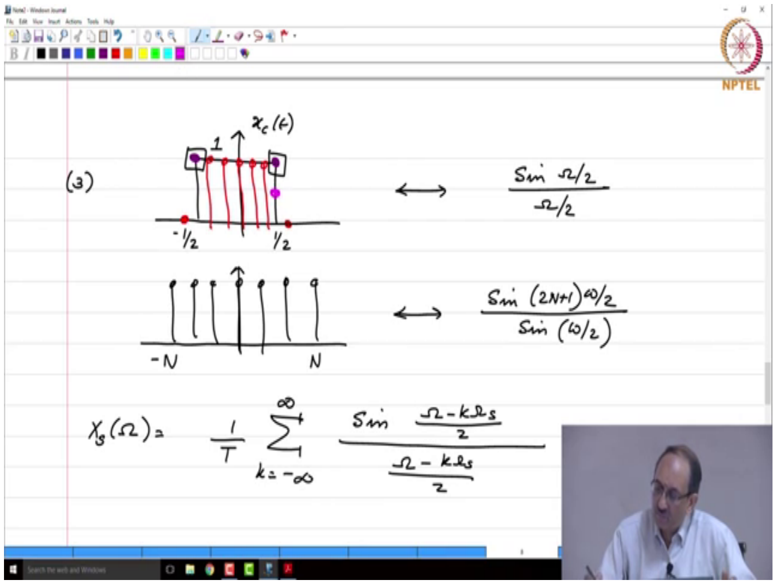
(Refer Slide Time: 22:01)



Therefore, $\frac{\sin(\omega_c n)}{\pi n}$ has DTFT that is periodically repeating with cut off ω_c, ω_c is nothing but Ω_c/F_s.

So, again this gives us back our familiar transform pair. We know that $\dfrac{\sin(\omega_c n)}{\pi n}$ is the ideal low pass filter in the discrete time domain.

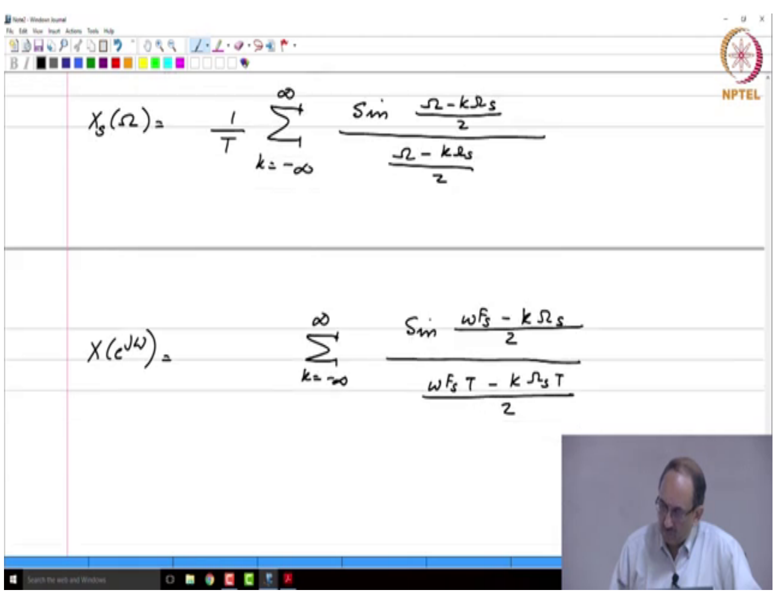
(Refer Slide Time: 22:48)



And the final example we will look at is this. So, this is $x_c(t)$ and its transform of course is *sinc*. So, this is nothing but $\dfrac{\sin(\Omega/2)}{\Omega/2}$. This is not band limited. And hence, if you sample this, you will get aliasing. And what we are trying to do is, we are trying to relate this discrete time sequence between $-N$ to $+N$.

So, this is our discrete time sequence. And we know that this transform is $\dfrac{\sin(2N+1)\omega/2}{sin(\omega/2)}$. Therefore, if you take samples of this and then form this sequence, you should get this as the spectrum starting from this.

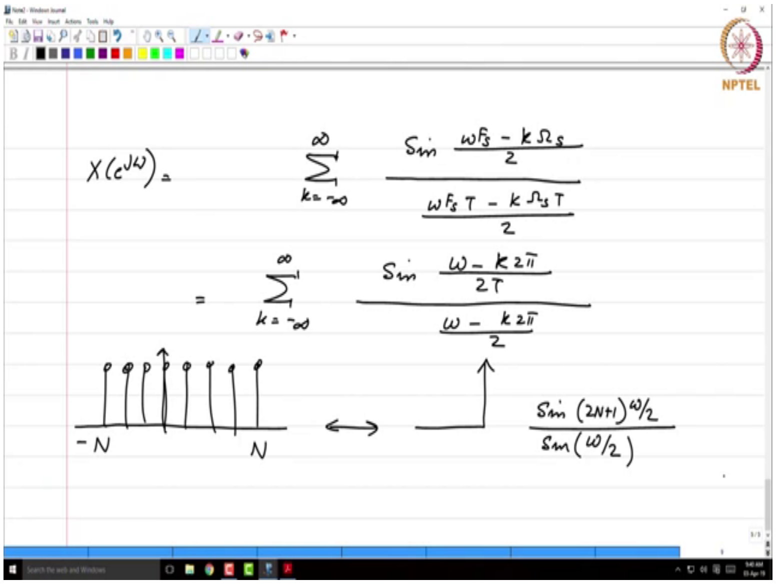
(Refer Slide Time: 24:32)



So, what we need to do is, the impulse train sampled signal spectrum is $X_s(\Omega)$ and that is nothing but $\frac{1}{T}\sum_{k=-\infty}^{\infty}\left(\ \right)$. And then you need to repeat this $\frac{\sin(\Omega - k\Omega_s)/2}{(\Omega - k\Omega_s)/2}$. So, this is what the sampled signal spectrum is. And $X(e^{j\omega})$ is scaled, replacing Ω by ωF_s.

So, this is now $1/T$. So, let me take care of the $1/T$ factor slightly differently. So, now, I have sine, wherever Ω is there, I am going to replace it by ωF_s, $\sin(\omega F_s - k\Omega_s)/2$ and then I have let me multiply this T here. So, this becomes ωF_s, I will multiply by T, $(\omega F_s T - k\Omega_s T)/2$.

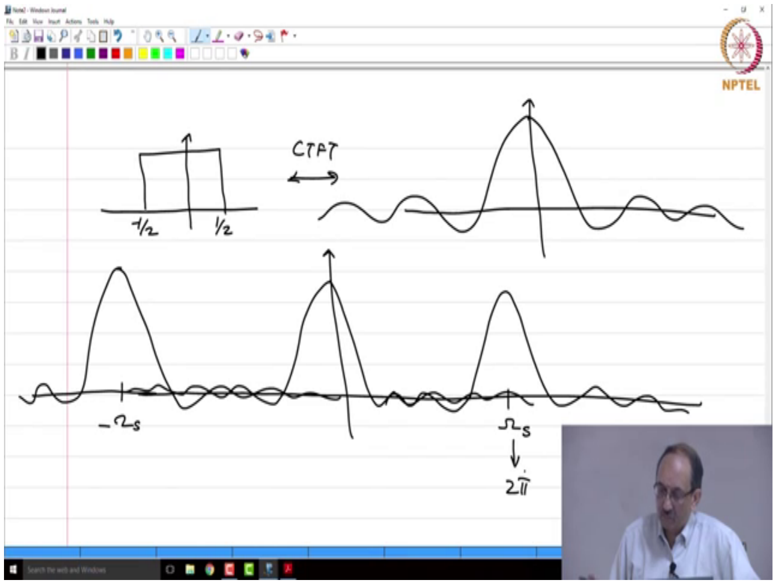
(Refer Slide Time: 26:18)



So, now this becomes k going from $-\infty$ to $+\infty$ sine. So, F_s after all is $1/T$. Therefore, this becomes ω times, let me multiply numerator and denominator by T. So, this becomes ω times, if you multiply $\Omega_s \times T$, you will get $\Omega_s \times T = 2\pi$. And $F_s \times T = 1$. So, this becomes $(\omega - k2\pi)/2T$. And $\sum_{k=-\infty}^{\infty} \frac{\sin(\omega - k2\pi)/2T}{(\omega - k2\pi)/2T}$ is the expression for this so, where this is between $-N$ to N.

So, this is exactly this expression. And this expression has been obtained by taking the aperiodic sinc repeating it periodically with period Ω_s, scaling it by $1/T$ and then scaling the frequency axis by F_s. So, these are the steps that were done. And they should really be equal to $\frac{\sin(2N+1)\omega/2}{sin(\omega/2)}$. So, it does not seem to be like that at first glance, but let us get a feel for what is going on here.

(Refer Slide Time: 28:27)



The underlying continuous time signal is this, between $-1/2$ to $1/2$. And this is $\dfrac{\sin(\Omega/2)}{\Omega/2}$; the continuous time Fourier transform which is aperiodic. Now what is happening is, when you sample, you will cause periodic repetition of this. And therefore, what is happening here is, amplitude scaling by $1/T$ is always there, the other thing is periodic repetition.

So, this is the fundamental copy and then you have the repetition on the left hand side. Similarly you have repetition like this, remember each of this is the aperiodic sinc. And this repetition is $\Omega_1 s$. Later, this will get map to 2π, when you scale by F_s. Therefore, this and this are equivalent.

So, this is the dirichlet kernel, which is 2π periodic and you can think of that as taking the underlying analog sinc, namely $\dfrac{\sin(\Omega/2)}{\Omega/2}$, periodically repeating it and then adding it up. If you add all these copies, you will get the 2π periodic dirichlet kernel. So, that is the connection between the aperiodic sinc and the dirichlet kernel. This is nothing but the aliased version of the analog sinc.

So, to really show how this is exactly this, mathematically starting from this to this, probably some more some other technique is needed. But, we know that these two must be the same based on the sampling theorem development. Other thing to note is that, in this form, you do not see N. So, where is N being, I mean implicitly playing its role here?

Student: (Refer Time: 31:27) sample faster then we have to go (Refer Time: 31:33).

Yes. So, that is implicitly captured in the parameter T, all right. So, N is there implicitly by in the parameter T. The other important assumption that is made here going from this to this is that, when you take samples, you are taking samples like this. The next sample is here and the next sample is here, that is, after this sample, the next sample falls on this point and similarly on this side it falls on this point.

So, clearly, the question might arise, why not take a point here. And taking a point at the very two ends means that you are sampling at the point of discontinuity. And, this development assumes that you are not sampling there. Again, if you had, if you remember a statement I had made some time back, you realize why this is the assumption that is being made. This assumption is needed, that is, you do not sample at the points of discontinuity is needed because?

Student: (Refer Time: 33:41) greater than (Refer Time: 33:43).

Right, you are saying it will be greater than 1, why is that?

Student: (Refer Time: 33:47).

Student: (Refer Time: 33:49).

So, you are now, you are thinking of Gibbs phenomenon.

Student: Yes sir.

So, what is the answer?

Student: (Refer Time: 34:01).Student: (Refer Time: 31:27) sample faster then we have to go (Refer Time: 31:33).

Yes. So, that is implicitly captured in the parameter T. So, N is there implicitly by in the parameter T. The other important assumption that is made here going from this to this is that, when you take samples, you are taking samples like this. The next sample is here and the next sample is here, that is, after this sample, the next sample falls on this point and similarly on this side it falls on this point.

So, clearly, the question might arise, why not take a point here. And taking a point at the very two ends means that you are sampling at the point of discontinuity. And, this development assumes that you are not sampling there. Again, if you had, if you remember a statement I had made some time back, you realize why this is the assumption that is being made. This assumption is needed, that is, you do not sample at the points of discontinuity is needed because?

Student: (Refer Time: 33:41) greater T

Ok.

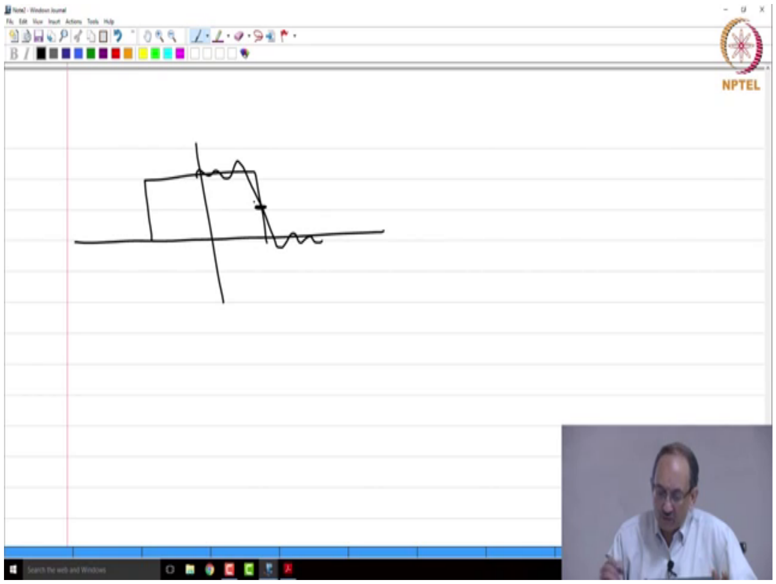
Student: (Refer Time: 34:03).

All right. So, you think if I had sample there, the sample value would be larger.

Student: Quite.

Your thinking of Gibbs phenomenon is correct, but the fact that the amplitude has to be larger is wrong. At the discontinuity, to what value will the spectrum converge to average value? Therefore, if you recall, you are not recalling the Gibbs phenomenon exactly.

(Refer Slide Time: 34:41)



So, this overshoot will be like this. Overshoot will be to the left of the discontinuity and undershoot will be to the right of the discontinuity. So, this 9 percent overshoot and 9 percent undershoot will happen to the left and to the right. At the value of the discontinuity, it will exactly go through the average value. And hence, if you take samples, at the point of discontinuity, to be consistent with this development, this sample value has to be, instead of this, and this the sample value to that is consistent with this expression on the other side will have to be half.

Therefore, in this theory, we assume you do not sample at the point of discontinuity. If you want the expressions to match, you have to make sure the sample at the value of discontinuity is really half the value. Or rather the average value to the left and to the right of the discontinuity.