Digital Signal Processing
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Lecture 07:
Elementary Signals (2)
The Unit Impulse (contd.)

Keywords: elementary signals, unit impulse, dirac delta function, unit doublet

Let us get started for the day; we were looking at the delta function continuous-time dirac delta function. And, then we were looking at its definition from the sequence of functions point of view and that point of view gave rise to some paradoxes. So, let us continue along the same lines and then see one more point that might seem surprising if you have not seen this before.

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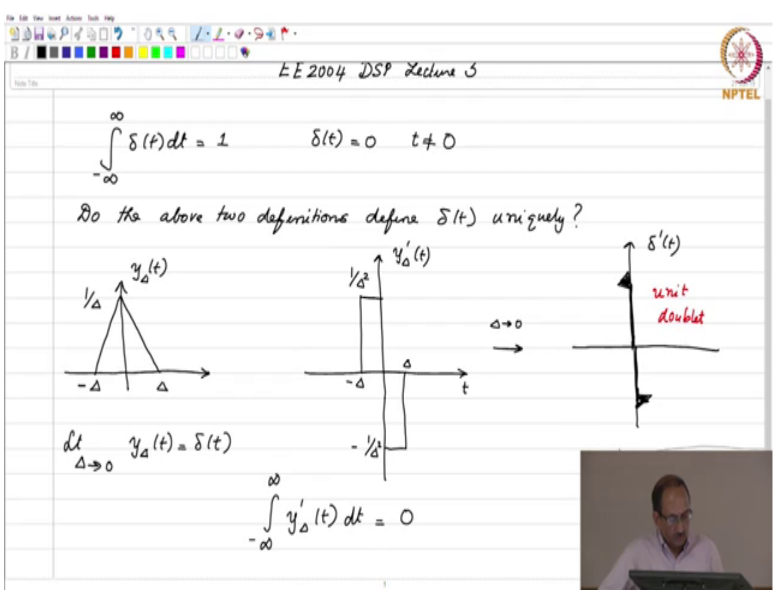

So, you are usually told that the delta function definition is $\int_{-\infty}^{\infty} \delta(t)dt = 1$. So, the area under the function is 1, that is what we are used to and then $\delta(t) = 0,\ t \neq 0$. So, the question is do the above two definitions define $\delta(t)$ uniquely? So, this is something worth asking and for this, let us again go to the sequence of functions approach. And, then see whether we can infer something more; actually we have been using the sequence of functions approached to show that what problems can exist. In this case, we will use the sequence of functions approach for a plausibility argument.

So, for this let us look at this. So, this was one of the functions which in the limit appeared to converge to the delta function. So, I think we call this as $y_\triangle(t)$. And, then somehow we got ourselves convinced that this looks like the delta function in the limit, it becomes skinnier and skinnier, taller and taller and area is always 1. And in the limit, this seems to satisfy these two properties. Now let us look at the derivative of this. So, let us look at $y'_\triangle(t)$. So, this has this form which is easy to see and the height of course, is the slope and the slope is $\dfrac{1}{\triangle^2}$; difference in y coordinates by difference in x coordinates.

And, in the limit as $\triangle \to 0$, this seems to approach this function. So, for each of these rectangular pulses, each part becomes skinnier and skinnier, taller and taller. And, in the limit this is how this is depicted. This the function, $y_\triangle{}'(t)$ in the limit and this is what is called as the unit doublet, $\delta'(t)$. And this is the derivative of the delta function, again we are using sequence of functions to suggest a plausibility argument. Since, we are comfortable with sequence of functions converging to delta, we are using the same approach here. Now, clearly what is the area of $y_\triangle{}'(t)$? It is 0.

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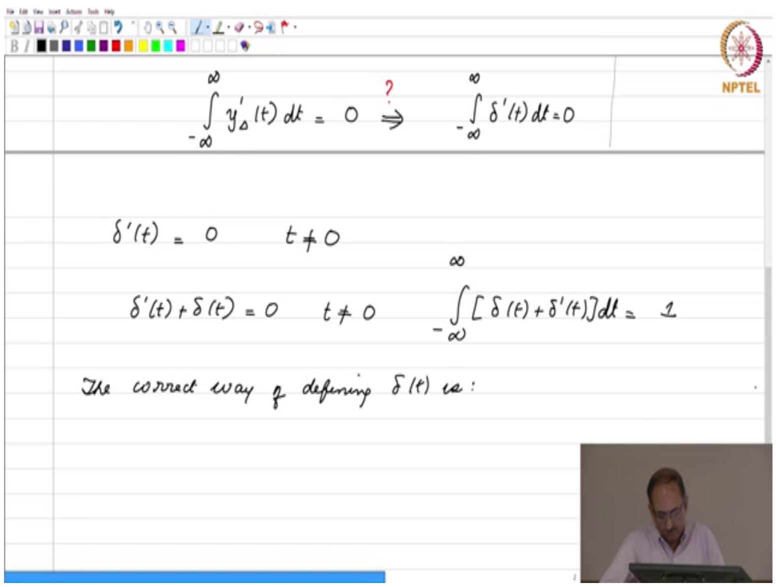

And, so this seems to imply that $\int_{-\infty}^{\infty} \delta'(t)dt = 0$, based on the limiting ideas we have been used to in this context before. So, at least this seems to suggest this and what can you say about $\delta'(t)$ for $t \neq 0$? Remember, if $\delta'(t)$ is this, in the limit as $\triangle \to 0$, this becomes skinnier and skinnier taller and taller. So, what can you say about $\delta'(t)$, whenever $t \neq 0$? It is 0.

So, looks like the area of the $\delta'(t)$ is 0 and $\delta'(t) = 0$ for $t \neq 0$, seems reasonable based on what we looked at so far. Now what about this, $\delta'(t) + \delta(t)$? 0, very good.

$$\delta'(t) + \delta(t) = 0 \text{ for } t \neq 0, \quad \int_{-\infty}^{\infty} [\delta(t) + \delta'(t)]dt = 1$$

So, clearly $\delta'(t) + \delta(t)$ also satisfies these two equations. So now, you see the problem with this. So, this begs the question what is the correct definition of $\delta(t)$.

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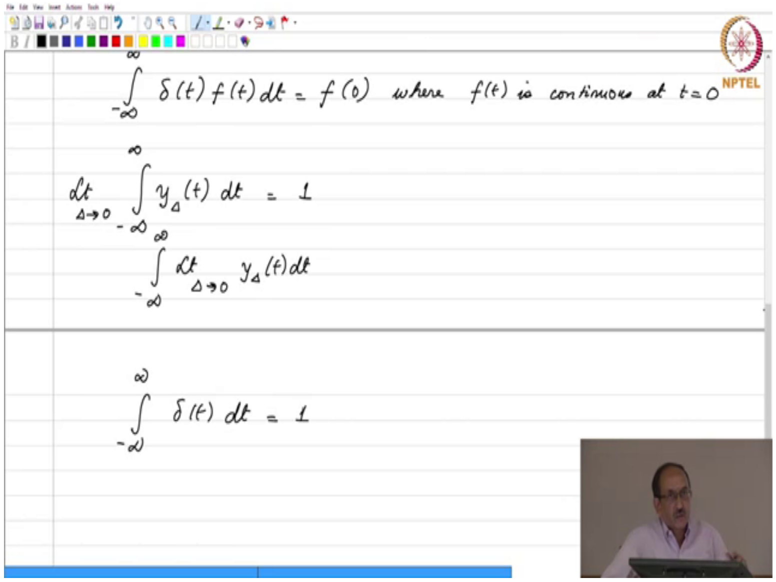

So, the correct way of defining the delta function is as follows

$$\int_{-\infty}^{\infty} \delta(t)f(t)dt = f(0)$$

where, $f(t)$ is continuous at $t = 0$. So, this is how $\delta(t)$ should be defined, any function that satisfies this is the delta function. Now, just to give you the larger picture so, mathematician John Kaczynski, he developed the theory of delta function based on sequence of functions in a very rigorous and careful manner. And, Laurel Schwartz, French mathematician developed a theory of $\delta(t)$ based on what he called as the theory of distributions. And, his approach is what is widely used today and to understand that requires a lot more mathematical machinery.

As far as we are concerned, we are happy with the sequence of functions approach to get some feel. But, then all that I have done so far is to make you aware that, that is much more to $\delta(t)$ that goes on then what we are used to and, we do not have the mathematical training to appreciate it fully. We will use the general properties as listed here and as you might have already encountered some of it before. Inside of integrals, delta function is fine provided it is in product with a function that is continuous at the location of the impulse. And, $\delta(0)$ is really not defined, if you look at the theory developed, $\delta(0)$ is not defined whereas, in engineering textbooks you will find the $\delta(0)$ is ∞ and so on.

So, all these things are not careful development of the concept. The correct way of calling delta function is in some books it is called as a functional; in some books it is called as a generalized function or a distribution. So, these are terms that should remind us, this is not an ordinary function. And $\delta(t)$ is not the only generalized function or a distribution; there are other members of this set. But, delta function is the one that is most famous, most well known and that is what we will be using for our purposes in this course.

Inside of integrals, it is fine when it is in product with the function that is continuous at the location of the impulse. So, typically we will do symbolic manipulations with delta and if you are not sure as to what the underlying theory is we might make mistakes in the symbolic manipulation. So, you have to be very aware of the limitations in our understanding of $\delta(t)$. Reason why, you are running into difficulty is, for example, suppose you consider function like $y_{\triangle}(t)$. And, $\int_{-\infty}^{\infty} y_{\triangle}(t)dt = 1$ for all $\triangle$ for the $y_{\triangle}(t)$ that I have shown. And what we are doing is we are applying the limit as $\triangle \to 0$.

And, then the flaw is in this step integral; obviously, is a limiting process, here you have $\lim_{\triangle\to 0}$. What we are doing is we are interchanging two limiting processes and then we are taking the limit inside the integral sign. And, then we think that $\lim_{\triangle\to 0} y_{\triangle}(t) = \delta(t)$ and then we come up with $\int_{-\infty}^{\infty} \delta(t)dt = 1$. So, the problem arises because we are interchanging two limiting processes and without being sure whether we are allowed to do that or not. So, under ordinary rules of calculus, there is no known function that satisfies the definition of the delta.

Even this definition that is; even this definition of delta no known ordinary function can satisfy this. So, that is why you required people like Laurel Schwartz to come up and develop a whole new theory. And, the theory of distributions has developed by him is considered to be one of the landmark achievements of the last century. So, there is lot more going on here, if you want to learn more about the delta function at a more understandable level you can look up the book, First Course in Fourier Analysis by Kammler.

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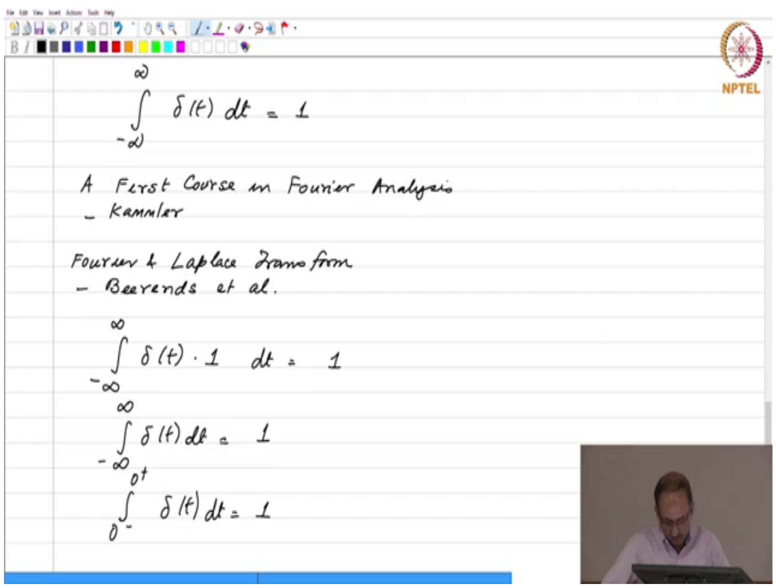

So, this is one book you can look up and then another book is Fourier and Laplace Transform, by four Dutch authors; now both these books are by Cambridge University Press. So, you can get some feel for how mathematicians handle delta function, unlike the loose and non-rigorous way that we handle delta in engineering textbooks. Even though we call it as a function, it is not a function. As long as you are aware of that and things that are happening with delta that is more than meets the eye, then you are fine.

$\int_{-\infty}^{\infty} \delta(t)f(t)dt = f(0)$ so, that immediately gives us the fact that the area under the delta function is 1. So, replace $f(t)$ by 1. And since this function is evaluated at the origin, the right hand side is 1 and sure enough $f(t) = 1$ for all t is indeed continuous at $t = 0$. Therefore,

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

So, the so called area under $\delta(t)$ is 1 actually follows from this. And, what we are normally used to is that we are given these two definitions and then $\int_{-\infty}^{\infty} \delta(t)f(t)dt = f(0)$ is taken as a consequence of the earlier definitions that we are used to and we call this as the sifting property. Really, this is the fundamental definition and the others are consequences.

The other thing that we are used to when we deal with delta functions is that until you have delta functions in your armory, the derivative of a function at a point of discontinuity does not exist. But, then we see that $\int_{-\infty}^{\infty} \delta(t)dt = 1$ and since this exists only in the infinitesimal interval between 0^- to 0^+, $\int_{0^-}^{0^+} \delta(t)dt = 1$.

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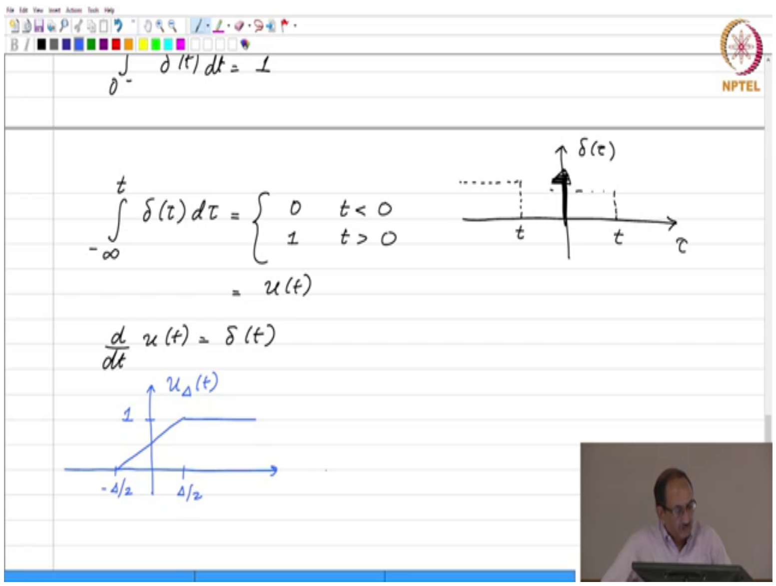

And, then what we do is we start to look at things like this, $\int_{-\infty}^{t} \delta(\tau)d\tau$. And the picture that we have in mind when we are doing this is we have this impulse sitting at the origin. So, this is $\delta(\tau)$ and $-\infty$ to t is nothing, but the running integral of this. So, we have two possibilities here; if your $t < 0$, then the running integral value is?

Student: 0.

0 therefore, we say that this is 0 for $t < 0$. The other possibility is of course, is the value of $t > 0$ in which case they running integral value is?

Student: 1.

1. So, $\int_{-\infty}^{t} \delta(\tau)d\tau = \begin{cases} 0 & t < 0, \\ 1 & t > 0 \end{cases}$ is nothing, but our usual?

Student: unit step function.

Unit step function, right. Therefore, we have this relationship, $\int_{-\infty}^{t} \delta(\tau)d\tau = \begin{cases} 0 & t < 0, \\ 1 & t > 0 \end{cases} = u(t)$. And, then we bring in what is well known to us then we say $\frac{d}{dt}u(t) = \delta(t)$, correct?

Student: (Refer Time: 19:31) defined what $u(t)$ (Refer Time: 19:33).

Yeah. So, this does not define what $u(t)$ is at $t = 0$. But so, why is that a problem?

Student: (Refer Time: 19:41).

You mean, you are now asking whether this follows. This not being defined at t equal 0 is not an issue really. And, to go from here to here, what are you invoking?

Student: (Refer Time: 20:04).

Ok, I mean, what first principles? What is the name?

Student: (Refer Time: 20:19).

So, this is fundamental theorem of calculus, correct ok. So, we are doing symbolic manipulations here. We are treating this as an ordinary function, fundamental theorem of calculus applies to ordinary functions, $\delta(t)$ is anything, but we are happily using this, right? Anyway, we leave it to the mathematicians to establish these things rigorously, and the picture that the engineer has of this is this. So, this is $u_\triangle(t)$.

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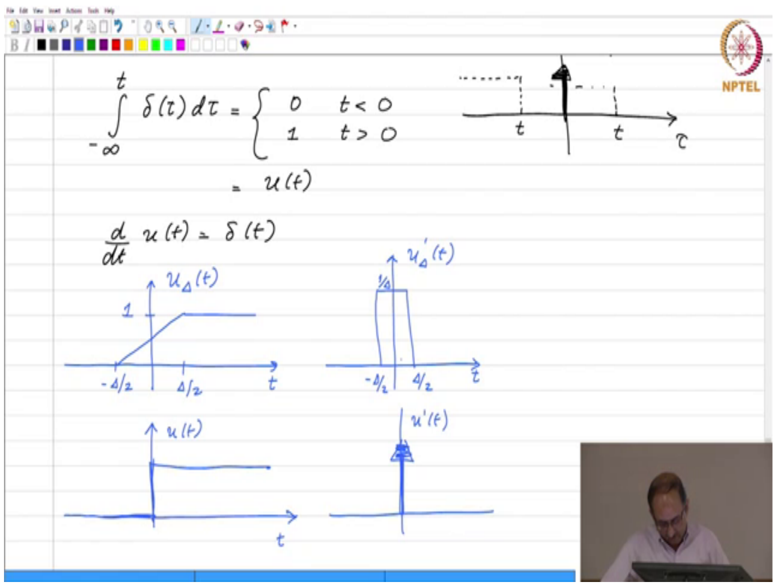

And $u'_\triangle(t)$ is between $-\dfrac{\triangle}{2}$ to $\dfrac{\triangle}{2}$ and height is $\dfrac{1}{\triangle}$. And, then we take limit as $\triangle \to 0$. In the limit as $\triangle \to 0$, this sloping part becomes more and more vertical. So, in the limit, this becomes like this. So, this indeed becomes $u(t)$ and we are quite convinced that $u'(t)$ is indeed the delta function.

These are the kinds of pictures that we have as far as these operations are concerned at least in engineering textbooks. All I want you to be aware of is you need a lot more rigor than establishing these things by these simple minded pictures. And, just to make sure you have understood all these things properly, if you know the answer to this question very good, if you do not know it is worth thinking about till the answer to this becomes clear and there are no confusions about this.

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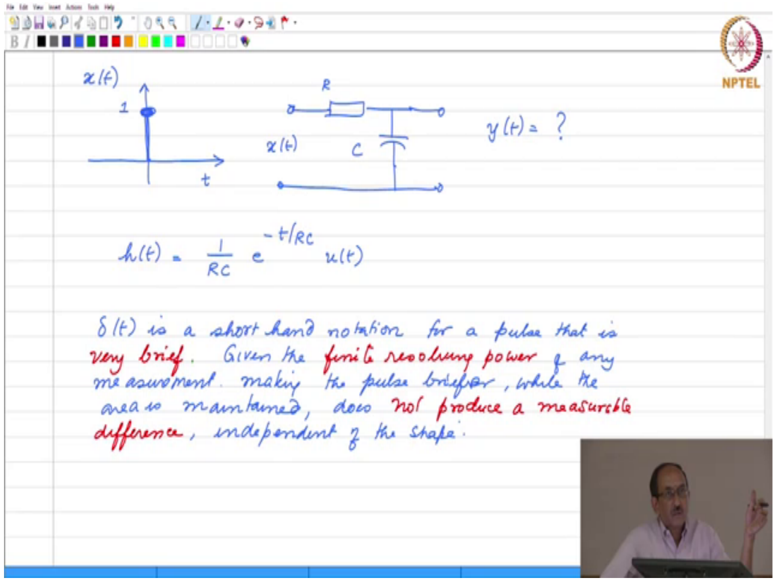

So, this is $x(t)$ and then this signal is applied to this circuit. So, R can be anything, it can be 1Ω, C can be 1F. So, this is my $x(t)$ and then the question is what is $y(t)$? The impulse response is

$$h(t) = \frac{1}{RC}\, e^{-t/RC} u(t).$$

So, if the answer has to what $y(t)$ is very clear to you, then you have understood all these concepts very well. If not, based on whatever we have seen so far, go back and try to find what $y(t)$ is for this particular input and this particular circuit.

Just to wrap up $\delta(t)$, the way that we would like to understand $\delta(t)$ is, $\delta(t)$ is a short hand notation, because we are engineers we deal with practical systems and signals. $\delta(t)$ is a short hand notation for a pulse that is very brief. In practice, all measuring equipments have finite resolving power. You cannot measure anything in practice with infinite precision. So, all practical equipment have finite resolving power. To within the limits of this resolving power, making this pulse any briefer, independent of its shape as long as the area is maintained, you will not be able to observe any difference in the measured output. That is all.

So, the shorthand notation for delta is for a pulse that is so brief, making it briefer will not make any difference in practice because, all practical measurements have finite resolving power. So, it is in that context, we are using $\delta(t)$. Deriving or developing an underlying theory that is sound is completely different and that is what the mathematicians have done. So, given the finite resolving power of any measurement making the pulse briefer while, maintaining the same area, does not produce a measurable difference, independent of the shape.

So, just to put this in context, Professor Sarathi has this high voltage lab which is in the same corridor as ESB 127 and it is a huge very high ceiling lab, where we get transformers from all over the country. And, there he does testing and one of the things that he does is impulse response. So, what is being done is an extremely high voltage discharge that is really brief is applied to the system and then measurements are taken. And, that extremely high voltage discharge that is so brief is the equivalent for an impulse in practice.

So, this is really used in practice for measurements such as the one that I just now mentioned and theoretically what you do is analyze such signals as impulses. So, that is where the use is.