Digital Signal Processing
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Lecture 67:
Linear Phase (4), Sampling (1)
-Linear phase IIR filtering and their realizability

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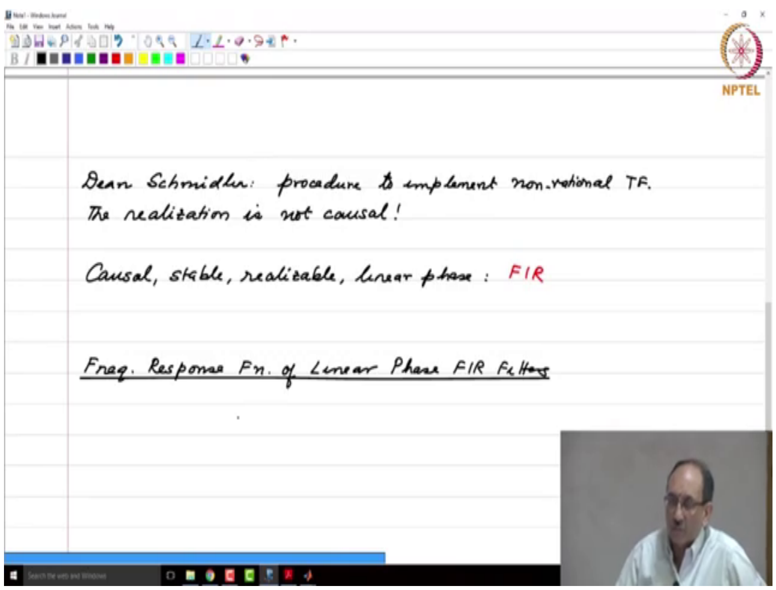

Now let us look at the last segment of linear phase FIR filters. So, we look at the frequency response function of linear phase FIR filters. So, all we are going to do now is, we are going to take advantage of the symmetry properties that are present and then, simplify the frequency response expression, that is all. And then put it in a form that will recall an earlier expression for the frequency representation that we have seen. And, one of the representations for the frequency response that we have seen is amplitude times $e^{j(\text{continuous phase})}$.

So, we will put the frequency response expression for Types I, II, III and IV in that form. We will derive Type I and then Types II, III, and IV. You can immediately guess the expression from the form of the answer for Type I.

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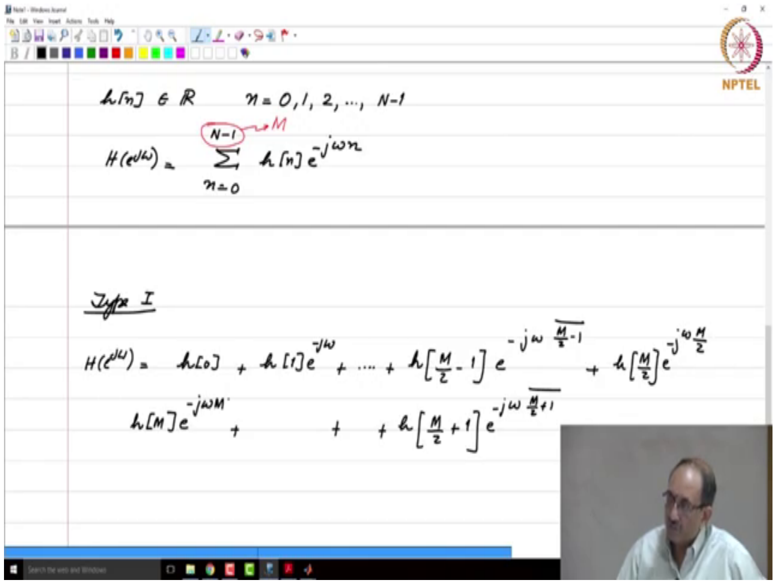

Therefore, we have $h[n]$. So, this is real valued and this is for $0, 1 \ldots, N-1$. And $H(e^{j\omega}) = \sum_{n=0}^{N-1} h[n]e^{-j\omega n}$. And, in this particular section related to these expressions, it is customary to denote $N - 1 = M$. So, whenever you see M, it really stands for $N - 1$.

So, let us look at Type I. Therefore, $H(e^{j\omega})$ is $0, \ldots, N - 1$, so that is $0, \ldots, M$. So, this is $h[0] + h[1]e^{-j\omega} + \ldots$ and so on. And remember, this is Type I, which means the length N is odd and the symmetry is even. Therefore, if N is odd, then M which is $N - 1$ is even. Therefore, symmetry sample is $h[(N - 1)/2]$ which falls on a sample and $(N - 1)/2$ is the same as $M/2$.

So, we look at the penultimate term to the center sample. The center sample is $h[M/2]$. The previous sample therefore, is $h\Big[\dfrac{M}{2} - 1\Big]e^{-j\omega(\ )}$. So, this is nothing, but $\Big(\dfrac{M}{2} - 1\Big)$. And then you have $h\Big[\dfrac{M}{2}\Big]$. The next index of course is, $h\Big[\dfrac{M}{2} + 1\Big]e^{-j\omega(M/2+1)}$ and so on. And finally, you are left with $h[N - 1]$, $h[N - 1]$ of course is $h[M]$, $h[M]e^{-j\omega M}$. So, this is the expression written out.

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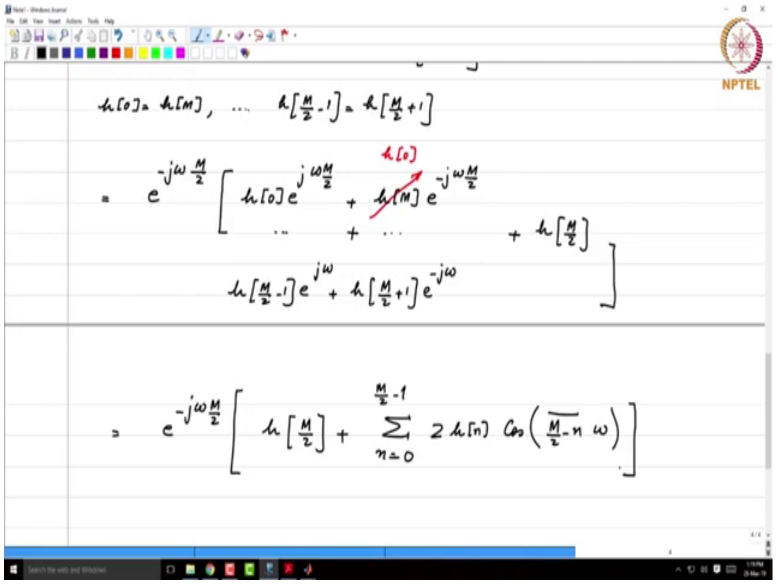

And now, we make use of the fact that, $h[0]$ is the same as $h[M]$ and so on. And $h\left[\frac{M}{2}-1\right]$ is the same as $h\left[\frac{M}{2}+1\right]$. And the center sample stays as it is without any counterpart. Now, let us take $e^{-j\omega M/2}$ outside. So, now this becomes $h[0]e^{j\omega M/2}$. If you focus on the last term, this is $h[M]e^{-j\omega M/2}$. But this is the same as $h[0]$. So, this is the kind of pattern that is going to emerge.

Similarly, you will have $h\left[\frac{M}{2}-1\right]$. And you have e to the, you have taken $e^{-j\omega M/2}$ outside here. So, what will be left when you take $e^{-j\omega M/2}$ is, $h\left[\frac{M}{2}-1\right]e^{j\omega}+h\left[\frac{M}{2}+1\right]e^{-j\omega}$. And, you have a whole bunch of terms that pair up like this. And, one term that is without a pair is this.

And now you see the pattern that has emerged. So, you have $e^{-j\omega M/2}$, n going from 0 to, we will see what the upper index is. The upper index clearly, we have to account for this center term $h\left[\frac{M}{2}\right]+\sum_{n=0}^{(\ )}$ to this term is accounted for therefore, the highest index has to go to $(M/2)-1$.

Therefore, the upper limit is $(M/2)-1$. This if you simplify, it becomes $2h[0]\cos\left(\overline{\frac{M}{2}-n}\ \omega\right)$. And let me write down the final expression and you will be able to verify each of these terms follows this pattern. So, this is $e^{-j\omega M/2}\left[h\left[\frac{M}{2}\right]+\sum_{n=0}^{\frac{M}{2}-1}2h[0]\cos\left(\overline{\frac{M}{2}-n}\ \omega\right)\right]$.

You can quickly verify that this is true. For example, if you put $n=0$ here, right, you will get $2\cos\left(\frac{M}{2}\ \omega\right)h[0]$, which is exactly this. If you put $n=0$, you will get the first pair, right. Put the upper limit $(M/2)-1$. If you put $n=(M/2)-1$, this will become $\cos(\omega)$. This n will become $(M/2)-1$. So, that is exactly this term. $(M/2)-1$; this plus this will give you 2 times this $\cos\omega$. So, this is the general expression for all the terms.

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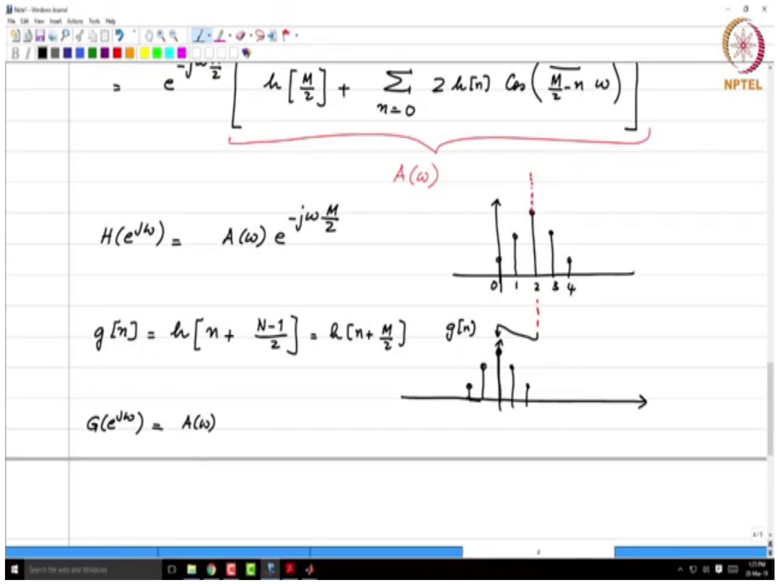

And if you see this, this can be thought of as $A(\omega)$. Because this after all is a real valued quantity. And hence, you have $H(e^{j\omega})$ for Type I, given by $A(\omega)e^{-j\omega M/2}$. So, this is the form for the frequency response for Type I. And this is amplitude times $e^{j(\text{continuous phase})}$.

And in this case, the continuous phase happens to be precisely linear because, that is what the filter satisfies because of the symmetry conditions. Remember, this is Type I, which means even symmetry and odd lengths. As a simple example, if this were an example of a Type I filter of length 5. This is the center sample which is the point of symmetry.

Suppose you form another filter $g[n]$, you take the original filter $h[n]$ which is Type I and advance it by $(N-1)/2$ samples. Basically what we are doing is? This center sample which was occurring at $(N-1)/2$, is now shifted to the left so that this center sample is now located at $n = 0$.

Therefore, you are looking at $g[n]$ which is $h\Big[n + \dfrac{N-1}{2}\Big]$, which is the same as $h\Big[n + \dfrac{M}{2}\Big]$ because, $N-1$ after all is M. And hence, $g[n]$ is this sequence, where this sample about which there is even symmetry which was occurring at $(N-1)/2$ has now been moved to align with $n = 0$. What can you say about $G(e^{j\omega})$? It will be $A(\omega)$, because g and h are related by a shift of $M/2$ samples.

If you shift $g[n]$ now to the right by $M/2$, it will acquire a linear phase term $e^{-j\omega M/2}$. Therefore, you can think of $A(\omega)$ as being the frequency response of the filter $g[n]$.

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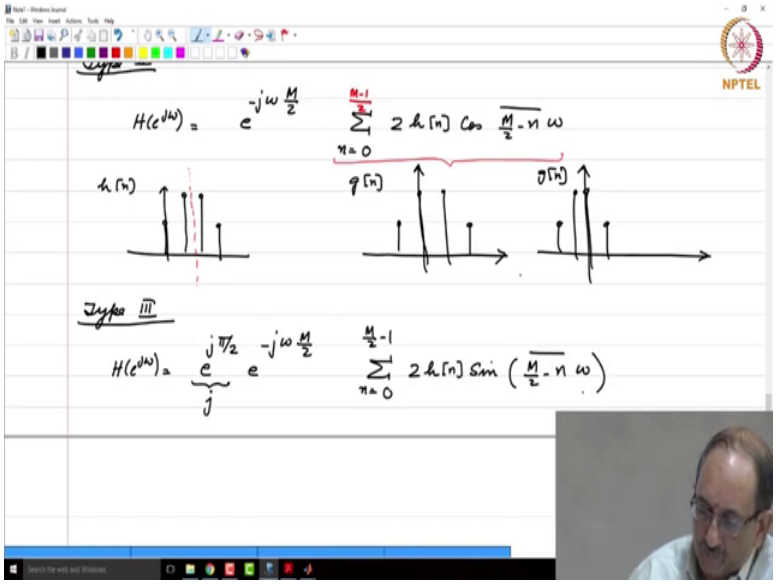

And once you see the form for Type I, Type II is almost can be guessed. It can almost guess the response for Type II. Remember, Type II is even symmetry, but even length which means there is no center sample. Therefore, $H(e^{j\omega})$ in this case turns out to be, $e^{j\omega M/2}(\ )$, before you had $h\left[\frac{M}{2}\right]$ sample in the expression here.

There is no sample for Type II, that corresponds to this. Therefore, that term will be missing. You will find that the expression is again $2h[n]\cos\left(\overline{\frac{M}{2} - n\ \omega}\right)$, the summation goes from 0; before it went to $(M/2) - 1$. Remember, M was even in that previous case. Now, M is odd therefore, the upper limit goes to $(M/2) - 1$.

So, this the only change. You can think of this as being $A(\omega)$, the amplitude response. Is there a filter or a sequence $g[n]$ having frequency response $A(\omega)$? The answer is NO, because remember, we are now looking at Type II. Type II is even symmetry and even length. Therefore, here is an example Type II filter and the point of symmetry is this which is midway between samples. And, if you call this as $h[n]$, no way can you shift it and have a sequence whose Fourier transform is real value.

Because, if you shift it to the left, either it will have the case like this. One second. It will either be like this or like this. Question?

Student: (Refer Time: 16:30).

Now this is $(M-1)/2$. Why don not you check it? I think this is correct. So, the definition of M is still the same, this need not be an integer. Is that the reason why you are thinking there should be a $(M-1)/2$? This is just the argument of the question. So, that can take on any value. It is not an index of a sample that necessarily has to be an integer. Therefore, I am glad you asked that question. So, this is not constrained to be an integer and this is indeed correct, why do not you verify?

I am almost sure, this is right. So, coming back to this, you can have $g[n]$ like this or $g[n]$ like this and neither of these sequences has $A(\omega)$ as its fourier transform. The point is, there does not exist a sequence that has $A(\omega)$ as the DTFT. Unlike in the Type I case, where you had $g[n]$ when it was the original sequence of shifted by $(N-1)/2$ samples to the left, it gave rise to an example sequence like this whose Fourier transform was $A(\omega)$.

And moment you have seen Type II, Types III and IV are similar. So, you have $H(e^{j\omega})$. Remember, Type III is generalized linear phase, β corresponds to $\pi/2$, therefore, you have $e^{-j\omega/2}$. You also have $e^{j\pi/2}$. This is type III.

So, the symmetry is odd and the length also is odd and the center sample $h\left[\dfrac{M}{2}\right]$ is always 0, because of odd symmetry. Therefore, that term doesnt come into the equation. So, you are left with n going from 0 to again $(M/2)-1$. $2h[n](\ )$ and now because of odd symmetry, rather than cosine, you can expect sine. So, this is $2h[n]\sin\left(\overline{\dfrac{M}{2}-n\ \omega}\right)$.

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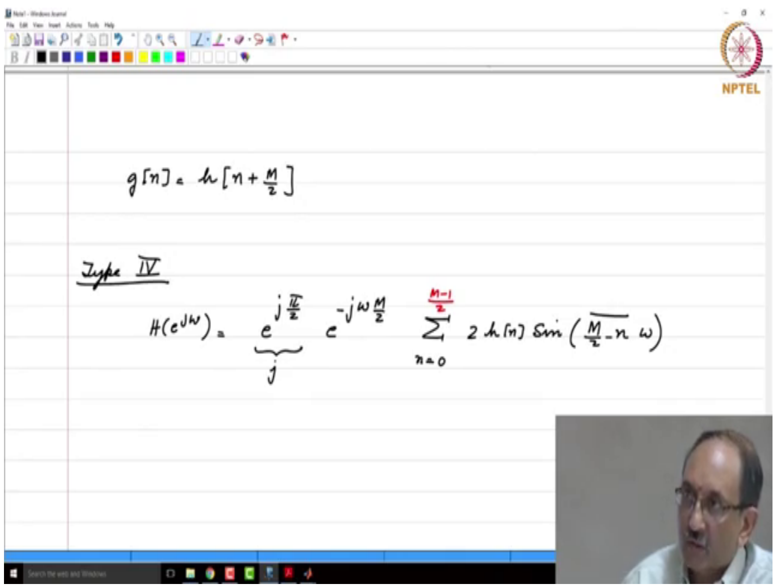

And similar to the Type I case, if you define $g[n]$ to be $h\left[n+\dfrac{M}{2}\right]$. Then, you do have a filter $g[n]$ that has frequency response given by $A(\omega)$. Really, it is $jA(\omega)$, because $g[n]$ will be real and odd. Therefore, the Fourier transform has to be imaginary and odd therefore, the four Fourier transform will be $jA(\omega)$. And, once you have seen the pattern going from Type I to Type II, it is easy to see what is going to happen when you go from Type III to Type IV.

So, that is $H(e^{j\omega})$, this thing remains the same $e^{j\pi/2}$ giving rise to the factor of $je^{-j\omega M/2}$. There is no center sample. Even if center sample were there, in the Type III case, it turned out to be 0. So, this is once again $(M-1)/2$. Again it has to be sine, $2h[n]\sin\left(\overline{\dfrac{M}{2}-n\ \omega}\right)$.

So, these are the frequency response expression for the four types of linear phase filters. Later, when you take a course, if you at all take a course on digital filter design, when we come to a FIR filter design using the Chebyshev approximation, these four expressions will be written in a slightly different form and these two expressions can be shown to be equivalent. And the reason for writing this in another form is that, there is an associated theorem that requires the expression to be in that form.

It so happens at these four types of linear phase filters do satisfy that requirement. But before to, you can be sure that they do satisfy the requirement, you have to write this in the different form that is required for that theorem. So, this is the easier form, the other form is slightly more involved. Once you see that, it can be written in that form, all it requires is getting the algebra right. So, with this, we come to the end of our discussions on linear phase filters.