Digital Signal Processing
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Lecture 63:
Linear Phase (3)
-Symmetry is sufficient but not necessary for IIR filters to have linear phase

Let us continue our discussion on the phase response, we were looking at implications of group delay and we saw that the output is a delayed version of the input and the group delay being $(N-1)/2$. Therefore, if you want to align the filtered output to the input, you have to advance the output by $(N-1)/2$ samples so that you can align that with the input or equivalently delay the input by $(N-1)/2$ sample so that you can compare the unfiltered input with the filtered counterpart.

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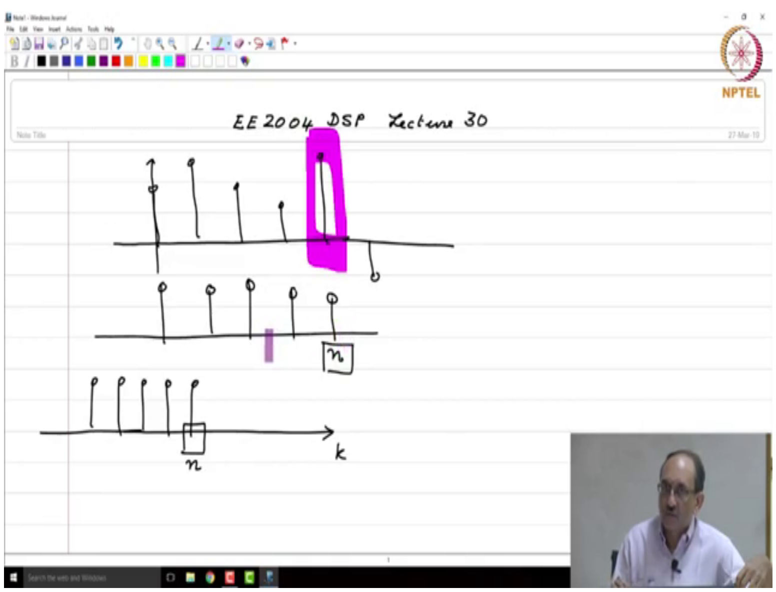

We were looking at this averaging example, suppose you had an input sequence like this and if you want to filter it, if you want to average it by say five samples. Remember, when you are going to convolve the input sequence with a filter, if this were the averaging filter, and if this is say 1/5. Remember, when we convolve, we first make this as k, the dummy index and then we reflect this therefore after reflection, this becomes like this. And then we shift it, when we shift this, the original origin sequence point becomes n.

So, this was what we had mentioned when we are doing convolution. Therefore, when you convolve a given sequence with this kind of filter where we are in this particular case, averaging 5 samples. If this is the averaging filter, this is the point at which the output is taken. When you do regular convolution, when you convolve this sequence with this 5 point averaging filter. For this position of n, you will get one number and this number that you get by averaging five contiguous samples is assigned here.

However, intuitively, you feel that you should assign it to the center sample. Therefore, you take the output and then advance it by $(N-1)/2$ samples, if you want to get rid of the group delay introduced by the filter, that is all. When you do the actual filtering, the output is assigned to this particular sample. And if you want to align that filtered and the unfiltered sequences, you have to shift by $(N-1)/2$ samples and then compare and then you can see the difference between the two.

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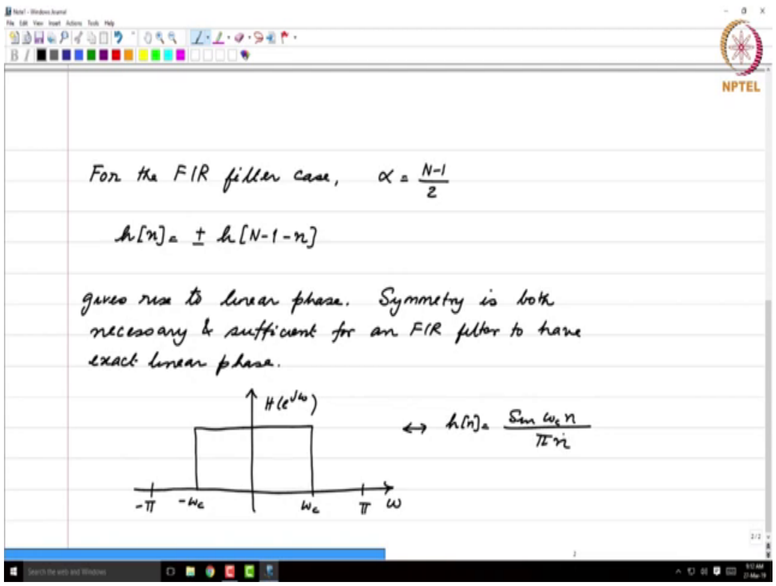

For the FIR filter case, which is what we have been considering. We started off with imposing linear phase and then we have to solve that equation and get non-trivial solutions. and the solutions that we got were $\alpha = (N-1)/2$ and $h[n] = h[N-1-n]$. So, we had $\alpha = (N-1)/2$, and then $h[n] = h[N-1-n]$ and this was plus or minus. For types I and II, it was plus even symmetry. For types III and IV, it was minus which was odd symmetry.

So, this gives rise to linear phase. So, this is either symmetry or anti symmetry and we will in general label these as symmetry for this particular context. Again, we will make this statement; that symmetry is both necessary and sufficient for an FIR filter to have exact linear phase. Symmetry is both necessary and sufficient for an FIR filter to have exact linear phase. What about the IIR case? Let us get a feel for that.

Let us consider the ideal low pass filter, $-\pi$ to π and then you have cut off $-\omega_c$ and $+\omega_c$. So, this is $H(e^{j\omega})$ which is the frequency response of an ideal low pass filter, and then we know that this impulse response is $\dfrac{\sin(\omega_c n)}{\pi n}$. So, this as given here is zero phase.

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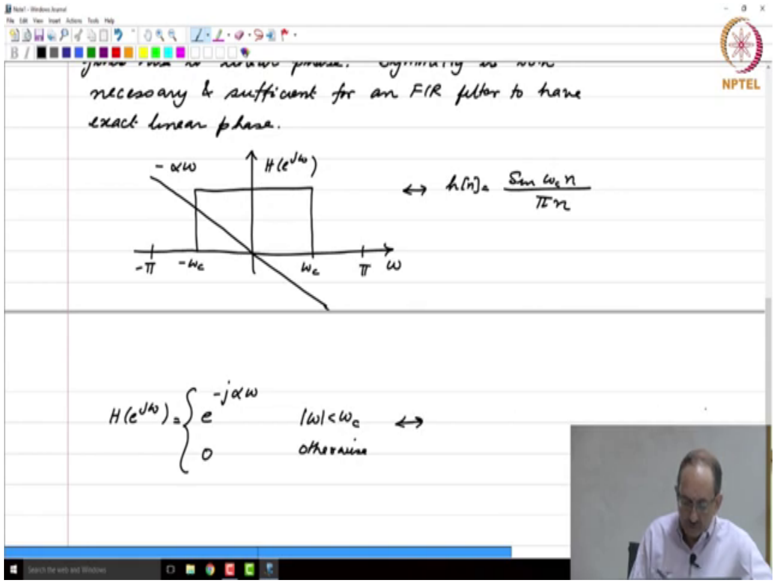

Now, let us introduce a linear phase term. So, let us introduce a linear phase term like this and this slope is $-\alpha\omega$. Therefore, we now have $H(e^{j\omega})$ as $e^{-j\alpha\omega}$ times 1 between $-\omega_c$ and $+\omega_c$. So, this is $e^{-j\alpha\omega}$ for $|\omega| < \omega_c$ and 0 otherwise.

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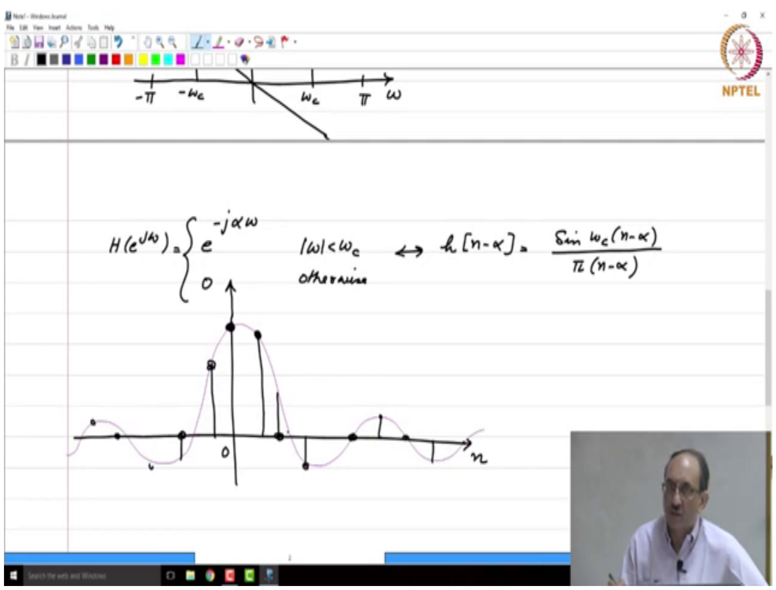

So, this in turn means that the corresponding impulse response is $h[n-\alpha]$ which turns out to be $\sin(\omega_c(\ ))$. Wherever n is there, we are going to replace n by $n-\alpha$, πn gets replaced by $\pi(n-\alpha)$. What we are about to see is that for the IIR filter case, this clearly is IIR filter, because the impulse response is $\dfrac{\sin(\omega_c n)}{\pi n}$ which exists for all n. We will show that for the IIR filter case, you can have linear phase exactly linear phase and yet not have symmetry.

So, in this particular case, the phase is clearly linear because that is the phase term that we have introduced. Therefore, there is no question that about the linearity of this phase term. Now, let us look at the corresponding time domain response. Let me first draw the underlying continuous time envelope of the sinc and then mark out samples; this is for the zero phase case.

So, this is how these samples are, the envelope is. And now what I am going to do is, for the zero phase case, I am going to plot the samples like this. Clearly, in this case, there is symmetry and there is symmetry about $n = 0$. And, now if you recall shifting by a fraction, let us now consider a case where alpha is some integer plus fraction and the interpretation is, you shift the underlying continuous time envelope and then resample at the original points. That is the interpretation that we have seen so far.

So, now let me take this and then shift it like this. So, I need to shift this also so, now this is the shifted envelope. And we are going to sample at the original sampling instance. And now if I do that, remember, I have retained the original sampling points. Therefore, this sample value is this, this sample value is really this and this sample value is this.

So, this rough sketch. In my notes, I will put the exact MATLAB plot. And now if you look at these samples, clearly they are not symmetric. They are not symmetric at all and yet this filter has exactly linear phase because this is what we started off with. We imposed $e^{-j\alpha\omega}$. And this filter is linear phase IIR and yet does not possess symmetry. Therefore, this is an example of the case where you can have an IIR filter with linear phase, but without symmetry. And hence, symmetry is sufficient, but not necessary for having linear phase in IIR case.

For certain values of α, clearly this will be symmetric. For example, if α were an integer, then all you are doing is you are taking the original sinc and then shifting it by integer samples and hence it will be symmetric about that shift. So, if α were an integer, this will still possess symmetry, because all you are doing is taking the original sequence that was centered in the origin and shifting it by some other integer samples. And the new shift point will be the point of symmetry. Is there any other case for which you can still have symmetry?

If you, very good. If you shift it by integer plus half sample, then the sequence will still be symmetric. And to illustrate that if I shift it by half a sample, here, I have shifted it by half sample. And now if I take the samples, I will get samples like this and this will still be symmetric, except that the point of symmetry will fall exactly in between samples.

So, this is similar to what was happening in types II and IV where the point of symmetry was midway between samples. So, for integer or integer plus half, this continues to be symmetric. For all other values of α, symmetry is not there, but no matter what α is, this is linear phase and this is IIR. So, this is illustration of the fact that symmetry is sufficient, but not necessary for linear phase for the IIR case.