Digital Signal Processing

Prof. C.S. Ramalingam
Department Electrical Engineering
Indian Institute of Technology, Madras

Lecture 61:
Zero - Phase Filtering, Linear Phrase (1)
-Linear Phase
-FIR filters with symmetry

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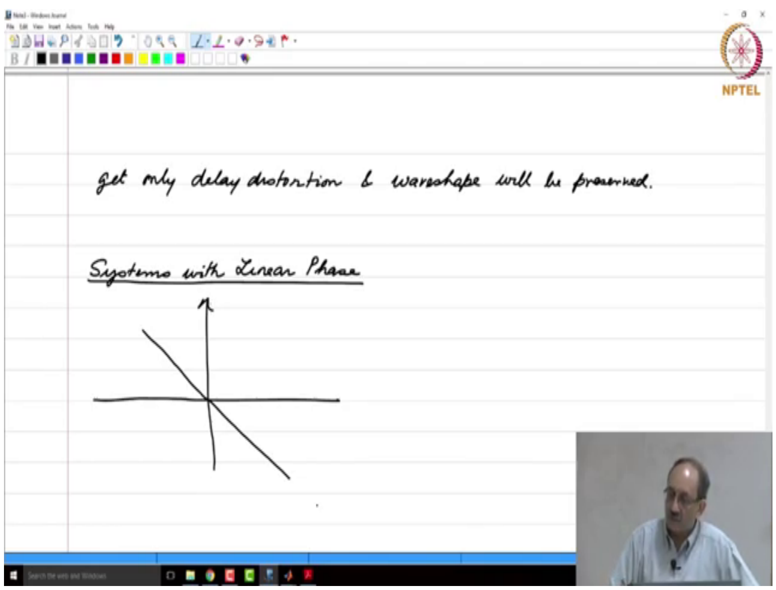

Therefore, let us look at systems with linear phase. So, this is one example where the system has precisely linear phase. Note that the slope has to be negative because, we want the group delay it be positive. Group delay has the physical implication of the delay that is important to the input therefore, group delay has to be positive therefore, the slope is negative.

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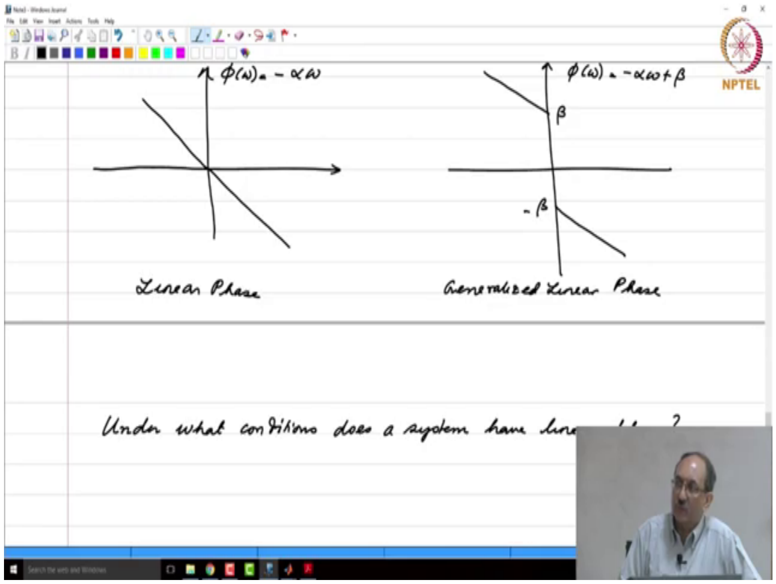

So, this is what is called linear phase and the phase response $\phi(\omega)$ has the form $-\alpha\omega$. This is another kind of system with linear phase, but this is now called as generalized linear phase. And this is denoted by this, and this $\phi(\omega)$ is given by $-\alpha\omega + \beta$.

And if beta were 0, generalized linear phase becomes linear phase. Therefore, what we are going to do is. we are going to impose linear phase on the system and then we will ask the question what condition is satisfied when the system has generalized linear phase.

Student: (Refer Time: 02:13).

Remember, we are talking about systems with real valued impulse response. Therefore, the phase has to be odd symmetric and hence we are talking about such systems for one half of the frequency response and we are extending it from $-\pi$ to π by odd symmetry. So, by this, we are assuming real valued impulse response.

So, we ask the question, under what conditions does a system have linear phase? So, this is the question that we asked.

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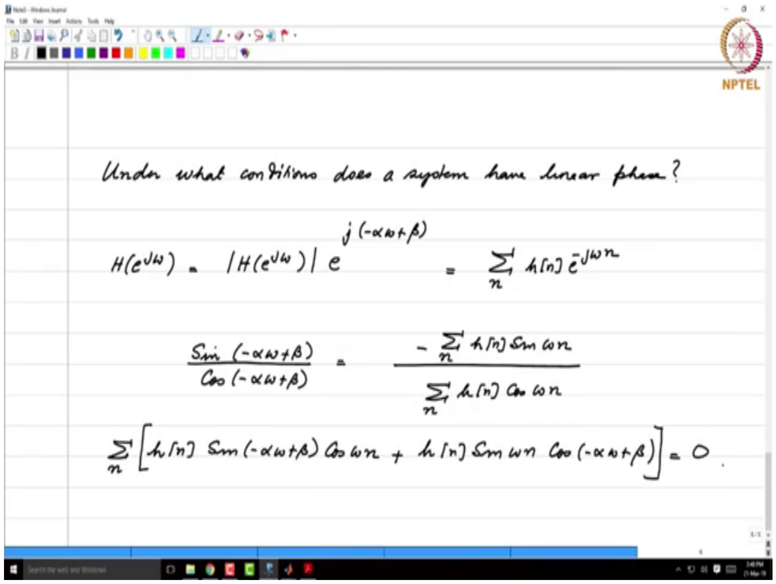

So, what we do is, we write the frequency response as magnitude times $e^{j(\text{angle})}$, but now the phase angle is not something that is arbitrary, but it is precisely linear phase. So, we are imposing this phase on the frequency response and we will see what condition has to be satisfied if this is imposed. And, we also know that the frequency response is given by $\sum_n h[n]e^{-j\omega n}$.

For now, I have written the summation as sum over all n, rather than writing it in the usual fashion n going from $-\infty$ to $+\infty$. The implication of this we will see as we go along. Now, let us equate the tangent of the angle on both sides, right. So, this will be $\frac{\sin(-\alpha\omega+\beta)}{\cos(-\alpha\omega+\beta)}$. So, this is the tangent of the angle of on the left hand side. This of course, is $\frac{-\sum_n h[n]\sin(\omega n)}{-\sum_n h[n]\cos(\omega n)}$. So, this is the tangent of the angle of the term on the right.

Now, we can cross multiply. So, this is $\sum_n \left[h[n]\sin(-\alpha\omega+\beta)\cos(\omega n) + (\ )\right]$, then I can cross multiply this with this and then take this to the other side, right. If I cross multiplied this and this I got this, if I cross multiply this and this I will get the term on the right hand side, I will take the right hand side to the left hand side. And because of the minus sign here, I will get a plus sign here and this will be nothing but $\sum_n \left[h[n]\sin(-\alpha\omega+\beta)\cos(\omega n) + \cos(-\alpha\omega+\beta)\sin(\omega n)\right]$. This must be 0, cross multiply and take collect all terms to one side.

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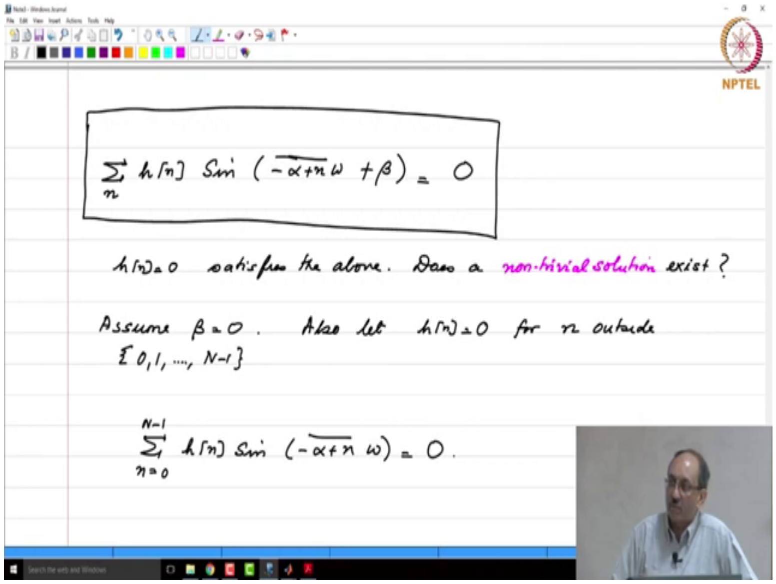

So, this is $\sum_n \Big[h[n](\ )\Big]$. So, this of the form $\sin A \cos B + \cos A \sin B$. So, this is $\sin(A+B)$ therefore, this is $\sum_n h[n]\sin(\overline{-\alpha+\omega}\ n+\beta)$, it must be 0. Oh sorry, this is $\sin(\overline{-\alpha+n}\ \omega+\beta)$, now this is ok. So, if the system is to have linear phase, then this equation must be satisfied because we start off with linear phase and had arrived at this condition. Now we need a solution to this equation and one solution that is immediate, that is true which the mathematician would state, would be, what is one immediate solution to this? $h[n] = 0$.

So, that is the trivial solution. It is mathematically valid, but practically useless. So, now, we are looking at non-trivial solution, right. So, $h[n] = 0$ satisfies the above. The question is, does a non trivial solution exist? So, now, if you are given this problem and say you have to solve it; rather than somebody else giving this problem to you, this is the problem that you are grappling with, so, you are so passionate about this day in and day out. You think about this, you want to solve this. If you keep thinking about some problem long and hard sooner or later, solution will emerge or you will go to sleep, all right.

But if you are passionate, you will be thinking about this day and night and suddenly some sparkle come and hit you and you will be able to make some headway. So, if you are grappling with this problem, you will try to simplify things, so, whether it to get a feel for what the approach might be. So, the first thing that you are likely to do is, let us assume $\beta = 0$.

So, which means you solve this problem for the linear phase case. Then, in general, n from the basic definition of the DTFT should go from $-\infty$ to ∞. Now, you are going to restrict yourself to the case n going from 0 to $N-1$. So, we are now looking at the FIR filter case. So, also let $h[n] = 0$ for n outside this set and hence you are finally left with this problem that you are going to solve, $\sum_{n=0}^{N-1} h[n]\sin(\overline{-\alpha+\omega}\ n+\beta) = 0$. So, this is a simplified version of the problem. Now we want to see whether some head we can be made.

So, a bunch of terms are adding up to 0, right. Bunch of terms are adding up to 0 which means some cancellations are going to happen. So, the brain wave that hits the person who desperately wants to solve this because he or she is so passionate about finding a solution, what occurred to that person was something along these lines.

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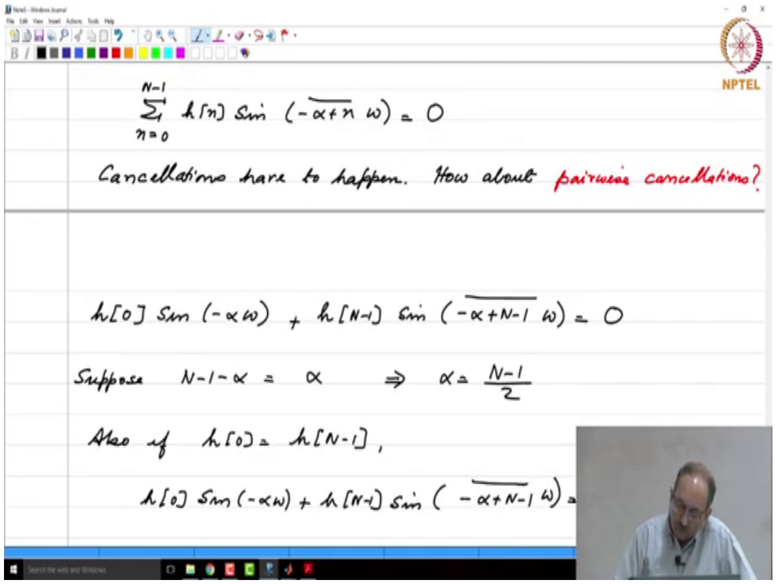

The person realized that, cancellations have to happen. How about pair wise cancellations? So, this is the first kind of foothold into solving this problem. And the person then thought along these lines, maybe pair wise cancellations are possible and the person decided to choose the first and the last term. So, you put $n = 0$, then $h[0]\sin(-\alpha\omega) + h[N-1]\sin(\overline{-\alpha + N - 1}\ \omega) = 0$.

So, can these two terms cancel is the question. You know that $\sin(-\theta) = -\sin(\theta)$. Therefore, suppose, $N - 1 - \alpha$ which is this term, if this were equal to α. Suppose if this were the case, immediately this would imply that $\alpha = (N-1)/2$. Therefore, for this particular value of α, the argument of the sign, they have opposite signs. So, this is $\sin(\theta)$ this is $\sin(-\theta)$.

So, if they have to cancel, you also need, if $h[0] = h[N-1]$, then these two terms will cancel. Because the co-efficient are the same, the argument of the sin have equal and opposite angles. Therefore, if this is true, then $h[0]\sin(-\alpha\omega) + h[N-1]\sin(\overline{-\alpha + N - 1}\ \omega)$ under these two conditions; $\alpha = (N-1)/2$ and $h[0] = h[N-1]$.

So, this is the first real progress you have made in trying to solve this. Now the next biggest step comes. If this were true, then following the same pattern, the second term and the penalty made term have to cancel.

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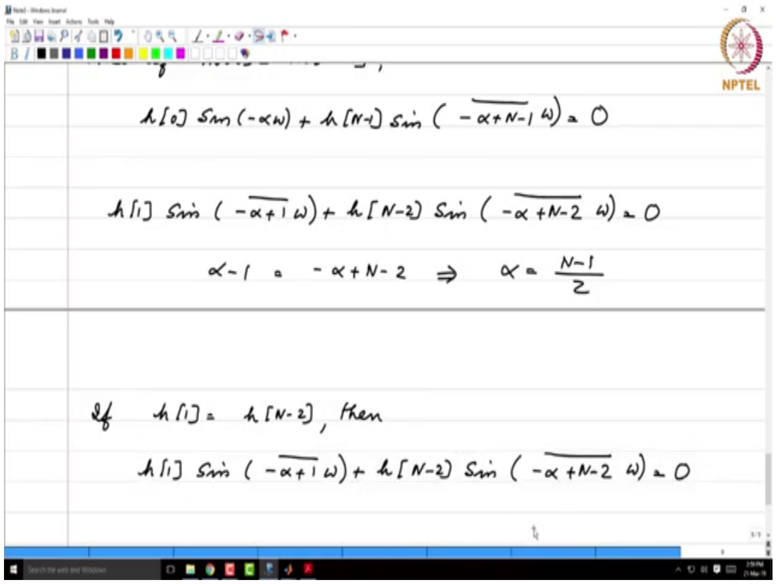

Therefore, you need $h[1]\sin\left(\overline{-\alpha+1}\ \omega\right) + h[N-2]\sin\left(\overline{-\alpha+N-2}\ \omega\right) = 0$. Again, you try to follow the same logic therefore, you want $1-\alpha$ to be the same as $-\alpha+N-2$; am I getting this right?.

Student: (Refer Time: 15:23).

So, $-\alpha+N-2$, I should put $n=1$ and then $N-2$, right. So, here I put $N-2$ and here I have put n to be 1, I want them to have.

Student: $\alpha-1$ (Refer Time: 15:42)

Oh yeah, so this is $\alpha-1 = -\alpha+N-2$, sure enough. Now this is the most crucial step. So, what does this imply, this implies $\alpha = (N-1)/2$, so, this is, the person must be see giving a huge size of relief. Why is that? Because if now α where something else, then all this carefully laid plans of mice then would have gone astray. So, now, α mercifully turns out to be $(N-1)/2$ and again you need $h[1]$ to be the same as $h[N-2]$. Therefore, if then this again cancels, right? This is satisfied. Now, we have really made significant progress.

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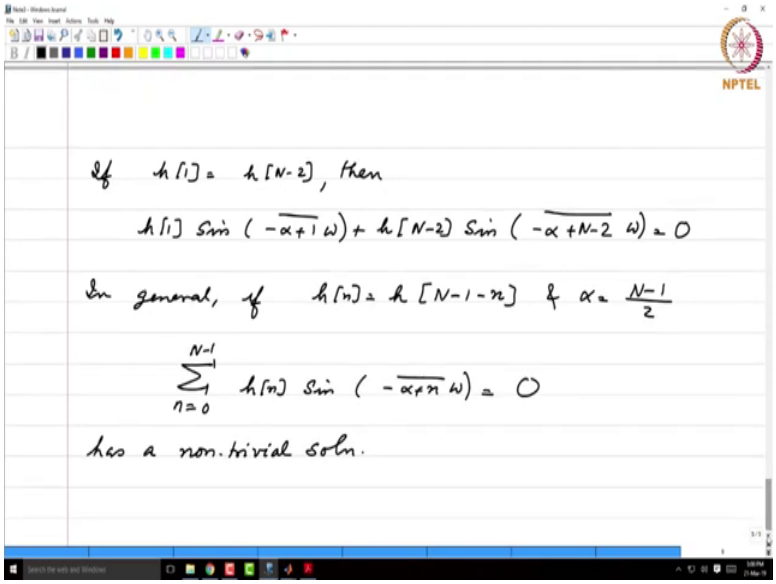

Therefore, in general, if $h[n] = h[N-1-n]$ and $\alpha = (N-1)/2$, then $\sum_{n=0}^{N-1} h[n] \sin\left(\overline{-\alpha+n}\ \omega\right) = 0$ has a non-trivial solution. We will now just illustrate this with a simple example and then conclude this session.

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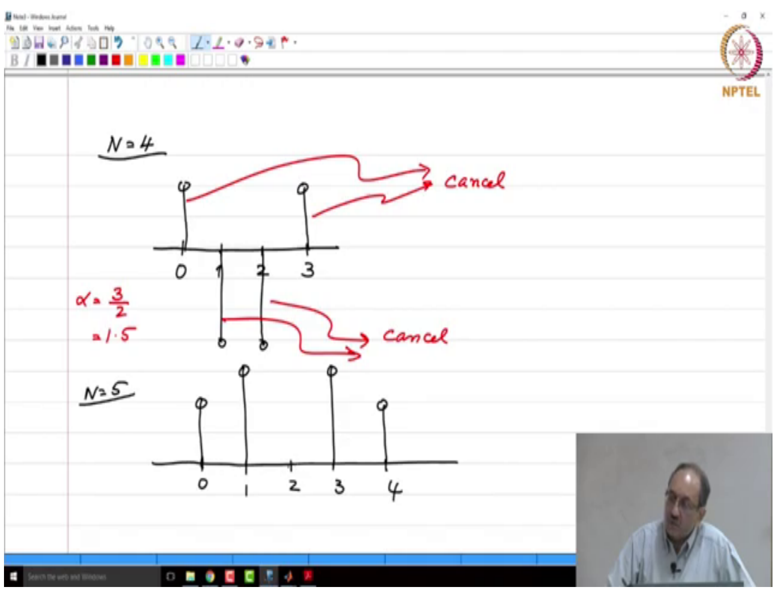

Suppose, $N = 4$, so the quotient go from $0, 1, 2, 3$, because you need to go from 0 to $N-1$. So, $h[0]$ must be the same as $h[3]$, $h[1]$ must be the same as $h[2]$.

So, this and this cancel, this and this cancel. So, now, these will get canceled for $\alpha = (N-1)/2$. So, this is $3/2$ or 1.5. Now let us look at $N = 5$. If we have $N = 5$, this goes from $0, 1, 2, 3, 4$. So, $h[0]$ must be the same as $h[4]$, $h[1]$ must be the same as $h[3]$, $h[2]$ does not have any pair, so, it can be some arbitrary number.

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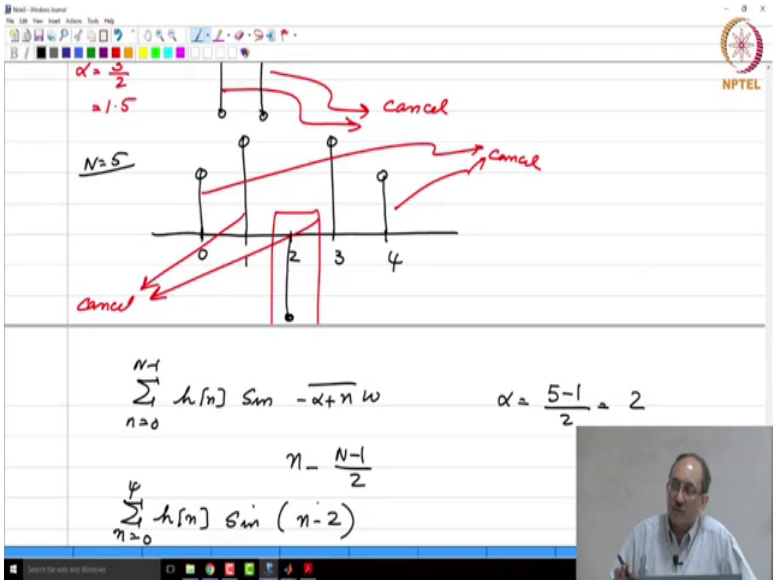

We will see that, we will see. So, this and this cancel, this and this cancel. Now, this is left without a pair, but remember, we are looking at $\sum_{n=0}^{N-1} \sin\left(\overline{-\alpha+n}\ \omega\right)$.

So, this is really $n-\alpha$,$\alpha=(N-1)/2$. In this particular case, α is $(5-1)/2$ which is 2. Notice that, this sample corresponds to $n=2$. Therefore, you have the case, you have $\sin(n-2)$ for this particular example. Therefore, the center sample which corresponds to $n=2$ independent of the value of the second sample rather the $n=2$ sample, this does not need a pair to cancel. Because, whatever $h[2]$, is it is going to get multiplied by $\sin(0)$. Therefore, this does not need a pair to cancel.

Therefore, this works for both n equal to odd as well as n equals even. And just one more comment before we wrap up, we have shown that this is true for $\beta=0$. Does this immediately provide a solution for some other value of β as well?

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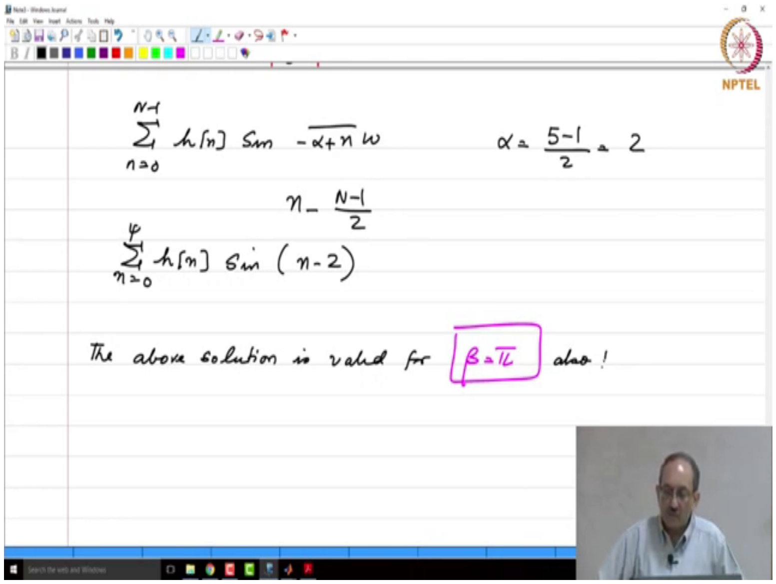

Student: $+2\pi$.

$+2\pi$ is a trivial modification.

Student: π.

$+\pi$, right. So, the above solution is valid for $\beta = \pi$ also. So anything mod 2π, we are only looking at values of β that is mod 2π. It is true for $k\pi$, k being both odd and even, but they are all mapped to $\beta = \pi$ case.

So, now, we will see whether there are other possibilities. If you put a condition on ω, then you are restrict in the frequency response. You dont want to put a restriction on the frequency response. That is why you are restricting these values, α and β, the parameters that you have to play with. Now let us see in the next class, whether there are other values of β that yield solutions.