Digital Signal Processing
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Lecture 59:
Allpass Filter, Group Delay
- Group Delay
- Physical interpretation of group delay

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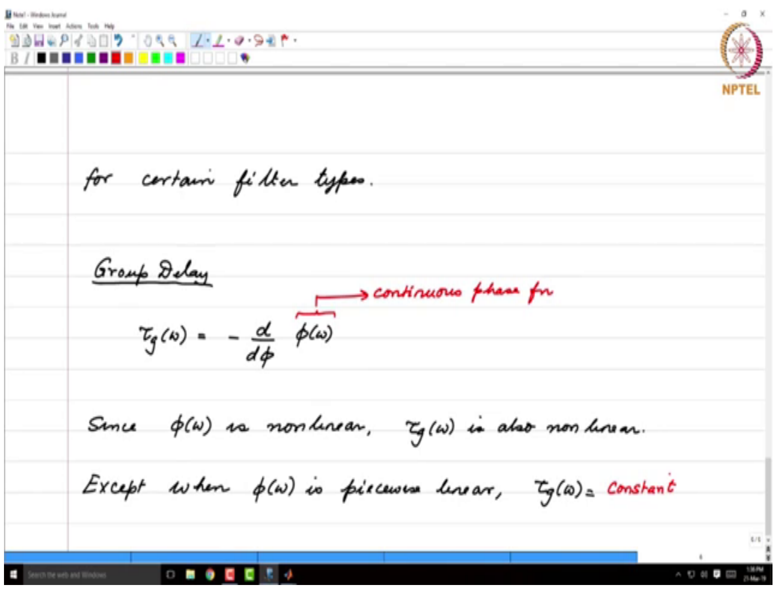

Now, let us move on to the next important topic of Group Delay, which is very closely tied to the phase response. So, we will define what group delay is. So, this is the notation that is used for group delay and this is nothing but the negative derivative of $\phi(\omega)$, where $\phi(\omega)$ is the continuous phase function. So, this is the unwrapped phase and for systems with rational transfer function, you can always write the frequency response as amplitude response times e to the j continuous phase response.

So, that is always possible and hence for the class of systems that we are interested in, $\phi(\omega)$ being a continuous phase function is the actual behavior and the group delay is the negative derivative of $\phi(\omega)$. And, since in general, $\phi(\omega)$ is non-linear, $\tau_g(\omega)$ is also non-linear.

Except when $\phi(\omega)$ is piecewise linear, and we have seen that $\phi(\omega)$ can be piecewise linear when you have zeros on the unit circle, right. We have seen specific examples where you can have linear phase and the examples that we have seen so far is zeros on the unit circle. So, when $\phi(\omega)$ is piecewise linear, then $\tau_g(\omega)$ is constant. The only question that might come up in your mind is for example, for the cases that we have seen earlier, the phase response is something like this.

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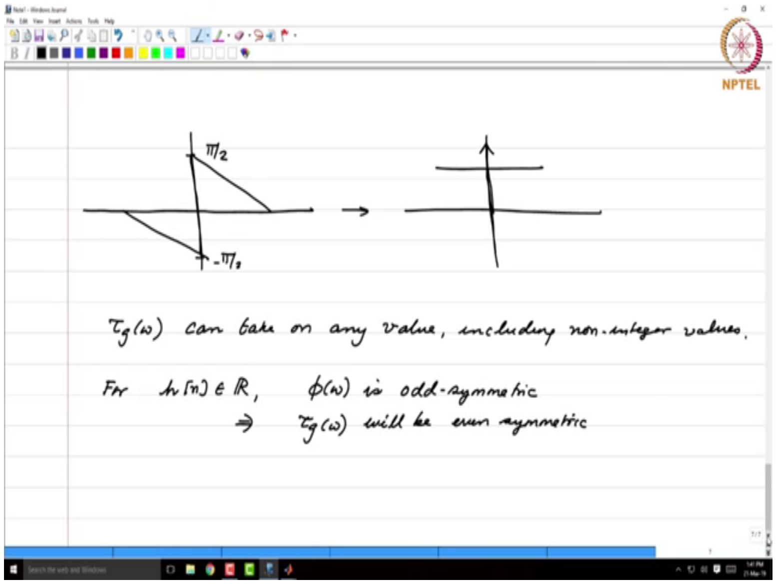

And we have this jump and this jump is of value π and note that phase unwrapping cannot get rid of jumps of π in the magnitude times $e^{j(\text{angle})}$ response. However, if you write this as amplitude times $e^{j(\text{angle})}$, you will get rid of these jumps of π because this jump of π will get absorbed in the amplitude therefore, there are no issues.

But even in this representation, if you differentiate this, this will give you a constant group delay. But at this point, at $\omega = 0$, you will have an impulse because of this discontinuity and this might cause some worry saying well there is an impulsive component to the derivative. But note that, at this frequency value at $\omega = 0$, because the zero is on the unit circle, the magnitude response goes to 0. Therefore, there is no component in the output at this frequency and hence this impulse appearing is not an issue.

And if you look at this from the other representations point of view, namely amplitude times $e^{j(\text{angle})}$, that phase angle does not contain any jumps. Therefore, whichever representation you look at, there are no issues in terms of the group delay. And, in general, $\tau_g(\omega)$ can take on any value including non integer values and the reason why I am mentioning this is we are going to give a physical meaning to group delay.

And, this delay need not necessarily be an integer. Therefore, group delay can take on fractional values. But this should not causes any problems because we know what it means to impart fractional delay to sequences. And, for systems with real valued impulse response, for $h[n]$ that is real valued, $\phi(\omega)$ is odd symmetric. This implies that $\tau_g(\omega)$ which is its derivative will be. What will be the symmetry of $\tau_g(\omega)$?

Student: Even symmetric.

Will be even symmetric. Again this is satisfying because you expect the delay to be the same for $+\omega_0$ and $-\omega_0$. For $+\omega_0$ and $-\omega_0$, which corresponds to a component at ω_0, you want the delay to be the same for real valued signal, so everything is falling in place.

So, this is satisfying which is what you would expect for systems with real valued impulse response. You do not want $+\omega_0$ and $-\omega_0$ suffering different delays for a real valued system. And that falls out from the symmetry property of phase angle and it is corresponding derivative.

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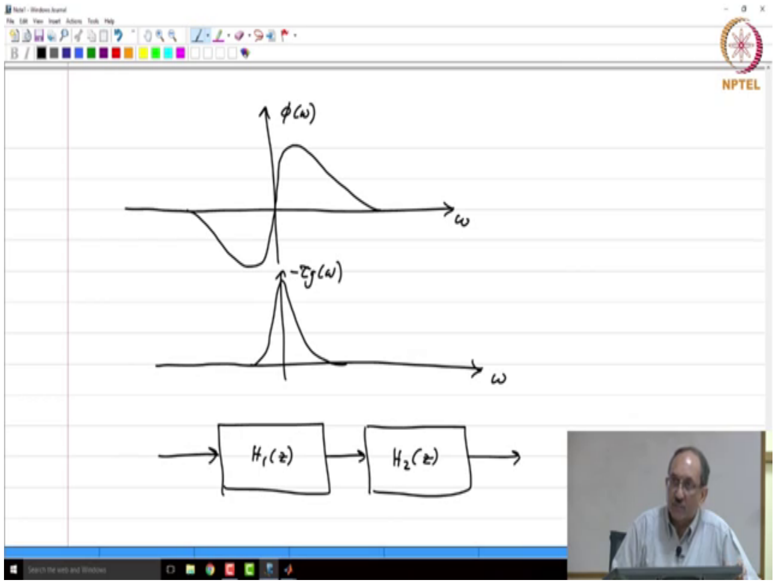

The other thing that we have seen is, if there is a zero or pole close to the unit circle, then the phase response when you go around the unit circle when you cross a pole or zero that is near, the phase response will change very rapidly. Therefore, we have seen this kind of phase response, right. The zero or pole being very close will cause a very rapid change and the corresponding group delay will have a spike, because that after all is the derivative of the group delay, all right.

Actually, this is minus tau, the way I have drawn, this is $-\tau_g(\omega)$. Anyway, it will have a spike near or a pole or a zero because the phase change is rapid, you will have a very rapid increase in the group delay. Now, let us try to get a feel for what this group delay does in practice.

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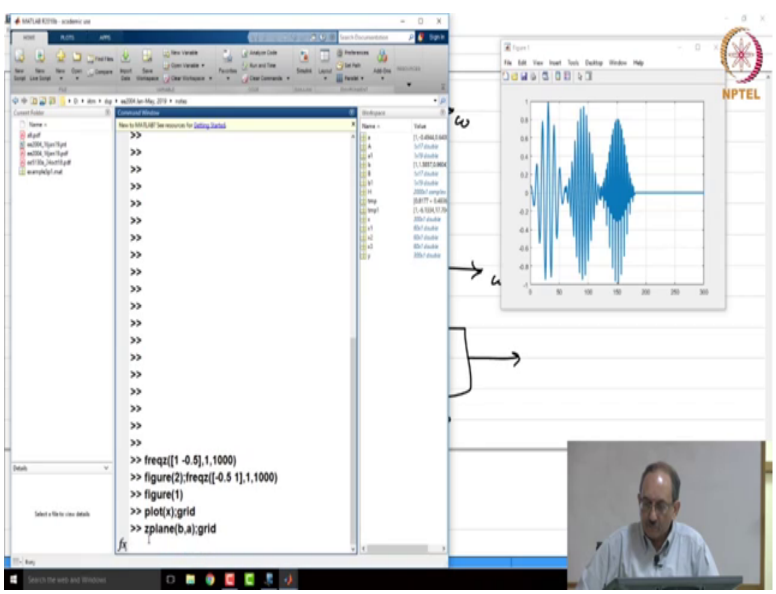

So, let us look at a practical example, I have an input which is this and this is 300 samples long. But beyond this point, it is all zeros. In the beginning, it has three pulses, the envelope of these three pulses is the same and inside it has oscillations. So, the first pulse has the lowest oscillation, the second pulse has an intermediate frequency of oscillation, and the third pulse has the highest frequency of oscillation, so this the input.

Now, what I am going to do is I am going to pass this through a system and I want to observe the output behavior. This system I am going to pass through has actually two parts, so this is $H_1(z)$, the second system is $H_2(z)$. $H_2(z)$ happens to be allpass, so it cannot change the overall magnitude response.

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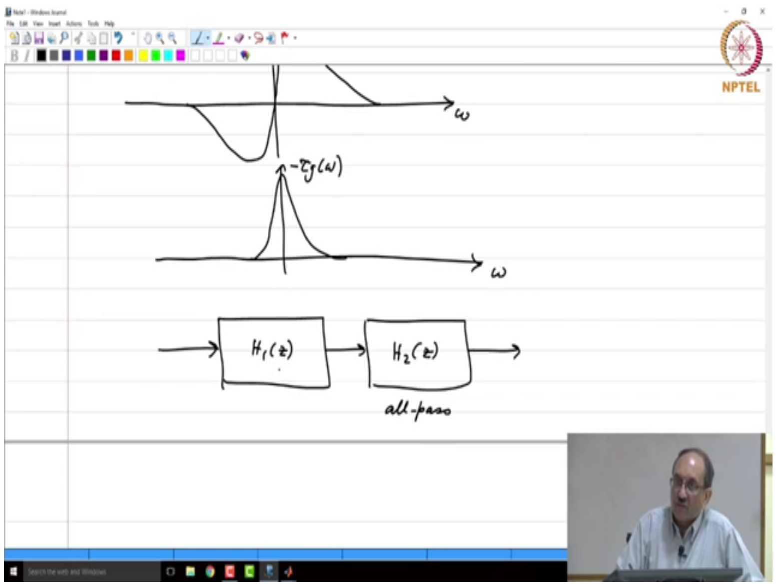

And, this has a certain frequency response based on it is poles and zeros. Now, let us look at what the first system is, let us look at what $H_1(z)$ is.

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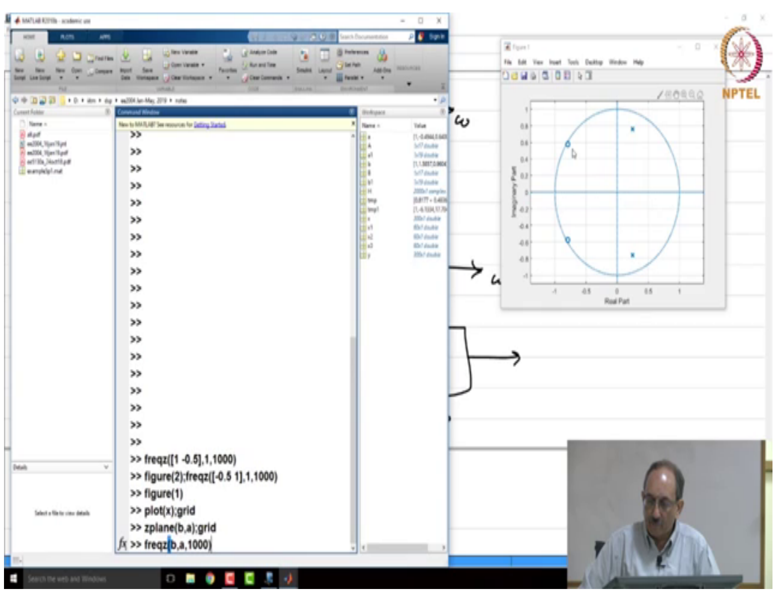

So, this is what the first system is, and you can expect gain around this frequency region to be high, because you are getting close to this pole. This zero is almost on the unit circle therefore, you can expect a frequency response dip here and sure enough if you plot the frequency response of this system.

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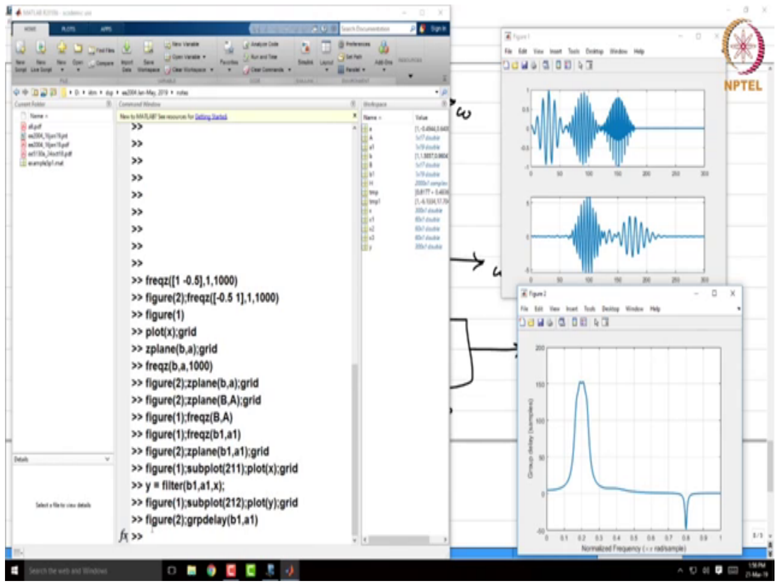

So, this is what the frequency response is. Pole-zero plot and the corresponding frequency response, so, around this region, there is a peak and around this region, there is a dip. So, this the overall frequency response corresponding to this pole-zero plot and this the phase response. And notice that in this region, the phase response changes more rapidly compared to elsewhere because of this pole.

And, similarly in this region, the phase response changes even more rapidly because this is very very close to the unit circle. So, closer to the root is to the unit circle, more rapid will be the phase response change. So, this change in phase response is due to this pole, this change in phase response is due to this zero. So, this is the system and it is frequency response; pole-zero plot and it is frequency response. Now the second system is allpass.

So, now let me plot the pole-zero plot of the allpass system. So, so this is the pole-zero plot and this is indeed allpass. So, I have a set of 4 poles that are very very close to the unit circle equally spaced and you see the number 2 here, it means it is a second order zero. Actually all of them are double poles, all of them are double zeros. Because of numerical precision in MATLAB, not all of them are being marked as second order zeroes and second order poles.

Ideally they are second order; all of them are second order poles and second order zero therefore, you should have the label 2 for each and every root and this is allpass. Now, you can look at the frequency response. So frequency response, there should be no surprises here. So, this is $freqz(B, A)$, so this is the corresponding frequency response.

This magnitude if you notice, this is 10^{-3}, so that is why you see something here and really if you zoom out, this fluctuation will not be seen. You are seeing this fluctuation as if it is something significant, because the scale is very very narrow; it is zoomed in. If I zoom out, this will be a constant and this change of 10^{-3} will hardly be visible. And again this is happening because the poles are not exactly at the location there they have to be, because of numerical precision.

So, this is the allpass section; its magnitude is all one and this is the phase response. And, the phase response is changing very very rapidly. This is where the 4 poles, and 4 zeros are; all the 4 poles are very close to the unit circle, the zeros are at the corresponding reflected position. Because the pole is close to the unit circle, they reflected zero also is at the very close location to the unit circle. And because of that, this phase response is changing extremely and rapidly, and this is the constant magnitude response.

Ideally this should be 1, you see this because of some numerical errors. If you zoom out, this will be 1 and this is a phase response. Now, if we cascade both these systems together, suppose I cascade them together, I have to cascade the earlier system shown with the allpass. The vectors $b1$ and $a1$ have that. And, if you recall, this was the frequency response that was shown earlier. This remains unchanged, because it is in cascade to the allpass system.

So, this is exactly what it was earlier, but what is significantly different is this. This is the phase response of the overall system which is the sum of the phase response of the simple system that was shown earlier with two poles and two zeros cascaded with the allpass system that we just saw. And it is dominated by this rapid change in phase over a small change in frequency value. And if you look at the corresponding pole-zero plot.

So, this is what the corresponding pole zero plot is. So this is exactly the allpass section that was there earlier and this is the $H_1(z)$ frequency response. And now what we are going to do is, we are going to take the input that I had shown earlier with three pulses, with three different carrier frequencies. And then we are going to pass it through this system and then we are going to see what the output is going to be.

Let me, so this is what I had shown earlier, the input. Now, I am going to form the output y, I am going to filter it. The system is $b1, a1$ which is the overall system which is the cascade of H_1 and H_2 and the input is x. So, this is what the output y is going to be after filtering it through this system whose pole zero plot is shown here and the input x is what is shown in the first plot.

Now, let me plot y and I am going to plot it below the first, all right. So, this is what the output is. Now, let us get a feel for what is going on. Remember, there was a dip in the frequency response due to the zero, all right. If you recall, there was a zero very close to the unit circle and the frequency response had a valley. And this pulse which had the highest frequency of oscillation, it so happens that the frequency response as a sharp null at that frequency and hence this gets killed. No issues, all right.

So, this gets killed which is why you do not see that happening here. What is the other striking thing that you see when you look at. So, this is gone, but among the two pulses that are remaining, what is it that you see?

Student: (Refer Time: 20:44).

The first pulse and the second pulse, the order has been reversed right. The first pulse and the second pulse the order has been reversed. First pulse occurred earlier in time compared to the second pulse, at the output, it is occurring later and this is exactly because of group delay. Now, if you look at this, this starts at time index of 0. Now if you look at this, roughly this starts at time index of 150.

So, this has suffered the delay of 150 samples. Now let us again use MATLAB's help and then plot group delay. As I said MATLAB can do anything you want short of those two items, eating your lunch and writing your quizzes. So, now, let us plot group delay.

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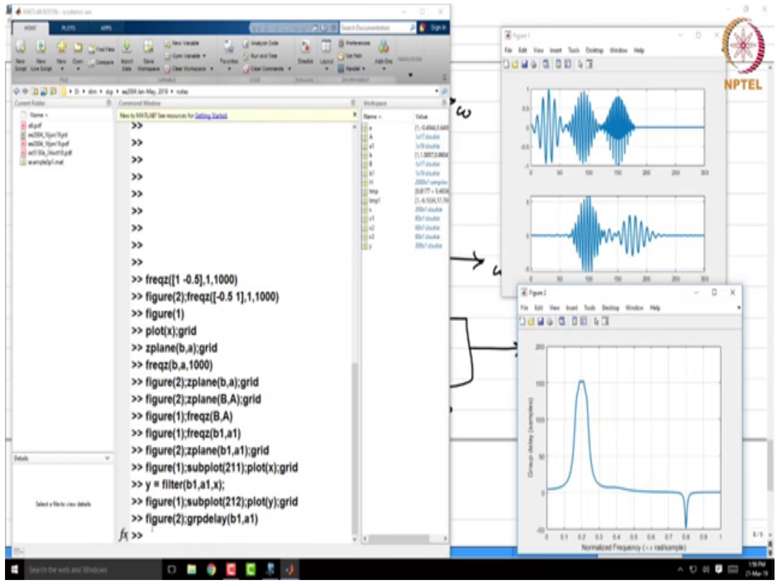

So, this is the group delay here, and remember the allpass section had a set of 4 poles and zeros at a certain location which corresponds to a certain frequency region. And at that frequency region, you had the greatest change in phase response, you can expect the group delay to have a spike around that region. And it so happens that it is exactly at 0.2 in MATLAB's notation or it is actually 0.1 in the conventional notation and the delay that you see here is 150 samples.

So, this 150 is no accident, this 150 samples is exactly the delay that the first pulse underwent and occurred 150 samples later. It so, happens in the first pulse frequency location is such that it is around that area frequency, its frequency content is exactly where you have the sharpest change in phase and therefore the largest group delay. And this group delay is manifesting itself as that component appearing 150 samples later at the output, so this is why it is called group delay.

And hence you see in this example dramatically the order of the two pulses gets reversed; first pulse and second pulse switch order. And if you look at this, the second pulse is practically unchanged, there is a very slight delay and the reason is at this particular frequency, the group delay is not much at all. Yeah, question.

Student: (Refer Time: 23:57).

Yes, so if you, that is a very good observation you have made. This is exactly where the zero is going to be therefore, there is not going to be any frequency component here. So, the point related to that question is very important what he raised. If group delay physically means delaying of the signal by so many samples, then does it mean if the group delay is negative, the system can advance your input which means this is non-causal which is not possible.

But, the important point to note is for all practical filters in the pass band, the group delay will always be positive. And if they go negative, it will be in the stop band which really does not matter, there is no physical interpretation.