Digital Signal Processing

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Lecture 57:
Phase Response (2)
-Phase response (cont'd)
-Alternative representation of frequency response: continuous-phase representation
-Phase response of a single complex zero/pole

Let us continue to look at some more aspects of Phase Response. What we had seen earlier was if you cross a zero on the unit circle, the phase response will change by π.

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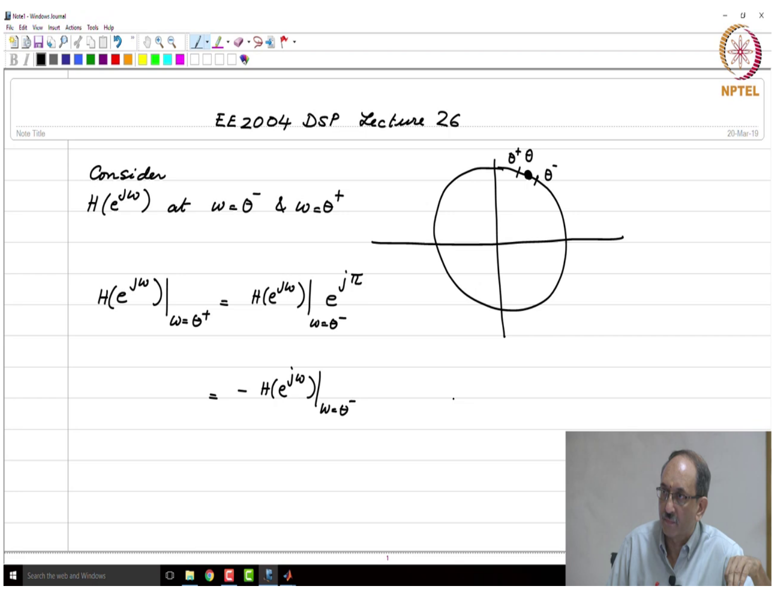

Let us get some more feel for this. So, we have a zero on the unit circle and this is at an angle of θ. Let us consider $H(e^{j\omega})$ at $\omega = \theta^-$ and at $\omega = \theta^+$. So, we are looking at a point here and at a point here. So, this is θ^-, this is θ^+, ϵ close to θ. When you compare the distances to all other zeros and poles, then when you go from θ^- to θ^+, will the distances to all the other poles and zeros, will those change much or will there not be any change?

There will not be any change because you are moving from θ^- to θ^+. So, as far as every other zero and pole is concerned, the change will not be much; it will be a very negligible change. The change would be negligible for both the products of the distances as well as to the change in the angle, as far as all the other poles and zeros are concerned.

Therefore, we will now try to relate the frequency response at θ^- and at θ^+. So, we want to look at of $H(e^{j\omega})$ at $\theta = \theta^+$ and we want to relate this to the frequency response at θ^-. As far as all the other poles and other zeros are concerned, nothing much is going to change. However, when you consider the zero at θ and since the zero is on the unit circle, when you cross the zero, you are going to acquire a phase change of π.

Therefore, the frequency response at θ^+ is related to the frequency response at θ^-. The only difference is it will acquire a phase change of π, otherwise everything else remains the same. And $e^{j\pi}$ of course, is -1 and hence this can be written as the frequency response at $\omega = \theta^-$ but with a minus sign. Therefore, what you can infer from this is that, when you cross a zero on the unit circle, you acquire a sign change.

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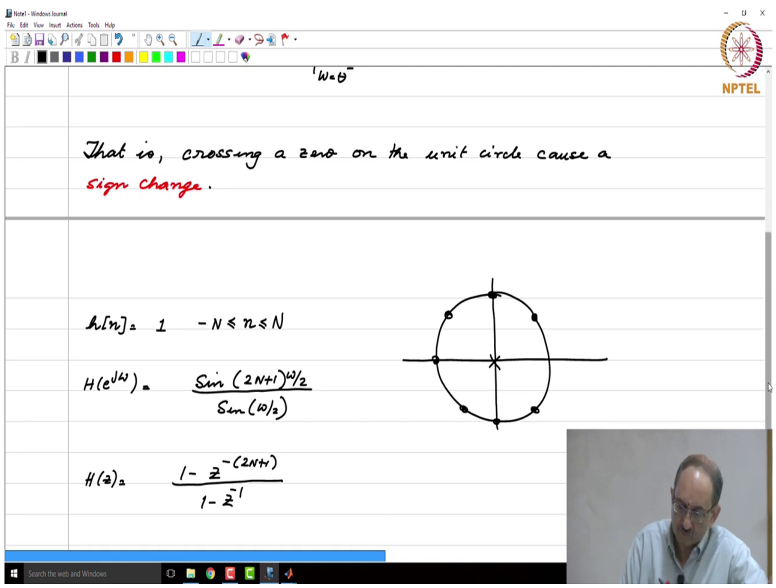

So, this is an important point. So, crossing zero on the unit circle causes a sign change. Let us now get a feel for this with a specific example. Suppose you have $h[n] = 1 \; -N \leq n \leq N$, then we know that $H(e^{j\omega}) = \dfrac{\sin(2N+1)\omega/2}{\sin \omega/2}$. We have seen this quite a few times before. We have also seen the Z-transform of this.

So, this is $H(z) = \dfrac{1 - z^{-(2N+1)}}{1 - z^{-1}}$. Therefore, this has roots equally spaced on the unit circle for the numerator; the zeros. And it has one non trivial pole at $z = 1$, which we have seen before gets canceled. Therefore, as a simple example this could be the pole-zero plot for a certain value of N.

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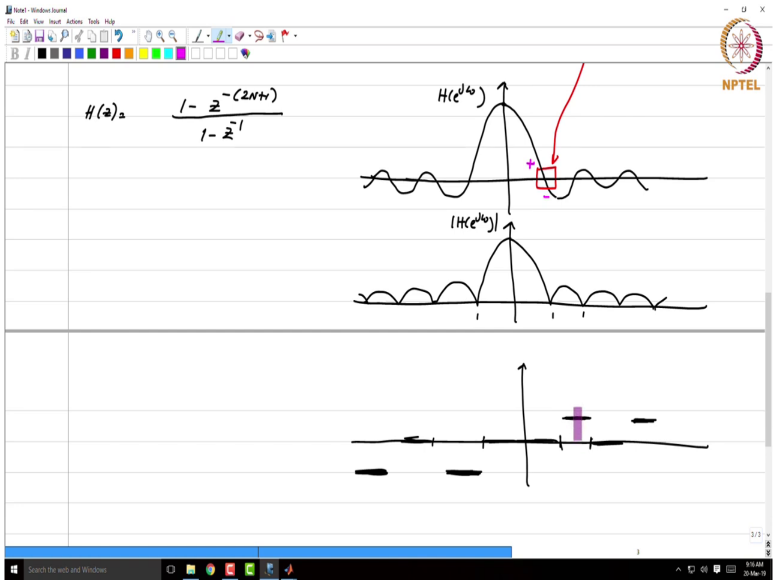

Now, again we have seen this geometric interpretation before we are revisiting this. But now, we are focusing on a point that is related to the idea of acquiring a sign change when you cross a zero. So, when you plot $H(e^{j\omega})$ which is this function and we know what the shape is from the geometric interpretation. We are here on the unit circle, you will have a certain frequency response.

And then when you go towards the first zero, the frequency response will drop and when you are on the zero, this frequency response at this particular point will be 0. And now you are crossing this. Remember we are actually plotting $H(e^{j\omega})$ which we can plot, there are no issues because in this particular case, $H(e^{j\omega})$ is this real valued function. When you cross this zero, the frequency response becomes like this. Let me just complete the picture.

And now you see crossing this zero has now caused a sign change. The frequency response was positive here and once you have cross the zero, the frequency response has now become negative. Therefore, this is the intuition behind crossing a zero acquiring a phase change of π really means a sign change. And of course, this is true for every zero that you cross.

The same frequency response can also plotted in a slightly different manner. In this case, because this was real valued, you can plot it like this. But, you can also plot it in the usual sense. And what we mean used to is, we been used to plotting magnitude response and phase response. Therefore, if you know plot this particular frequency affirm that is magnitude for this particular case then, the response will look like this. So, this is magnitude, what about phase?

So, between these two points, $H(e^{j\omega})$ is the positive quantity, non negative quantity. And the phase of a real number that is non negative, the phase is 0. Therefore, the phase response it is like this; zero. And now when you consider this interval, the amplitude is negative and the phase corresponding to the negative number is π.

And hence in this interval, the phase response is π. And by convention, we plot the phase for a negative number, when ω is greater than 0 to be $+\pi$ because equally well you could have marked this as $-\pi$. And similarly when you look at the next lobe which is positive, again the phase response is 0 and for the next negative going lobe, the phase response is again π.

And because this is a real valued sequence, magnitude response is even and phase response is odd. Therefore, once you have plotted the phase response for ω greater than 0 to be like this, you have to complete the picture from 0 to $-\pi$ as the one that is consistent with an odd symmetric plot. Therefore, in this region, you will have $-\pi$, again 0 and again $-\pi$ and so on.

So, this is the same frequency response that is plotted as magnitude versus frequency and phase versus frequency. In this particular example because, it was real valued, you can also plot $H(e^{j\omega})$ just by itself. On the other hand, if it were complex, you need to plot the real part versus frequency and imaginary part versus frequency or you can plot magnitude versus frequency and faces versus frequency. Yeah, question.

Student: (Refer Time: 12:51) to the other (Refer Time: 12:53).

Yes. So, you mean to say so, this being $-\pi$ and this being $+\pi$ is, yeah. So, that is why I had mentioned that by convention for ω greater than 0, you show this as $+\pi$. It is just a convention, yes, what you are saying is correct. This could as well be flipped and still will be consistent.

Student: (Refer Time: 13:21).

Yes, you can in this particular case, you can make it even symmetric. But because, the sequence is real valued, in general, the phase response is odd, we make it as an odd symmetric sequence. So, what you saying works only for this particular case, not in general all right. So, what we have seen in the discussion of phase response is, when you cross a zero, the phase change is π.

Therefore, when you plot magnitude versus frequency and phase versus frequency, the phase will jump by π where ever you cross the zero. Now, let us look at another representation of frequency response, where there are no jumps of π.

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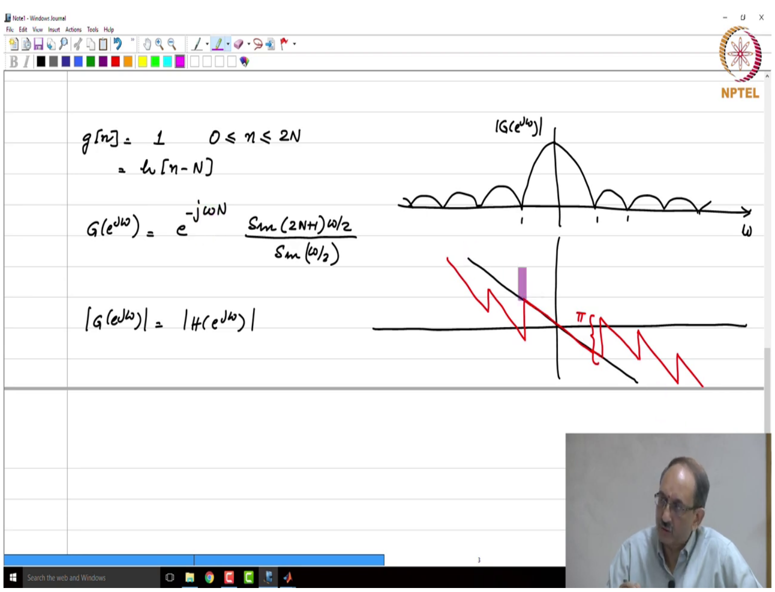

Suppose you had this particular sequence, suppose you had $g[n] = 1,\ 0 \leq n \leq 2N$ and you can easily see this is nothing but $h[n-N]$ where $h[n]$ is the $h[n]$ that was shown in the earlier example. And now you can easily see that the frequency response $G(e^{j\omega})$ is using the delay property, $e^{-j\omega N}\dfrac{\sin(2N+1)\omega/2}{\sin(\omega/2)}$.

As far as magnitude response goes, nothing changes. Because, the delay has introduced just a phase factor; magnitude response remains the same. And hence this is as before. So, this is magnitude of the new sequence identical to the earlier case. Now, as far as phase goes, this particular term $e^{-j\omega N}$.

If you look at the phase of this term alone, this is a linear phase term with slope ω. The value is $-\omega N$; so, this is a linear phase term. So, if you just focus on this alone, then the phase response will be like this. But the overall phase response given this magnitude response is the phase response due to this term and the phase response due to this term.

Therefore, the overall phase response is the addition of this phase response to this linear phase term. And if you look at this, so between the case where you are in this region, nothing changes because you are going to add 0. However, when you come to this part, then you acquire a jump of π and then the linear phase term continues. So, this how the phase response looks.

So, this is the odd symmetric completion of the figure and this jump is a jump of π, all right. Yeah, is there a question?

Student: (Refer Time: 18:13).

Yeah. So, the phase response is just this; $-\omega N$ and so, this is a linear function so, that is why I have drawn this black curve which is linear with that particular slope with slope of $-N$. So, phase response is not limited to be between $-\pi$ and π, we are going to actually come to that concept in a minute. The frequency can be anything, but as far as the signal is concerned, you replace frequency by frequency mod π because the resulting signal is now different.

But when you are looking at the phase response, it is precisely given by this term which is $-\omega N$ and $-\omega N$ is exactly this; there are no issues here. But we are going to the point that you have in mind in a minute. So, this the overall phase response for this given magnitude response and this shows jumps of π. However, there is another representation in which the phase response will not show jumps of π.

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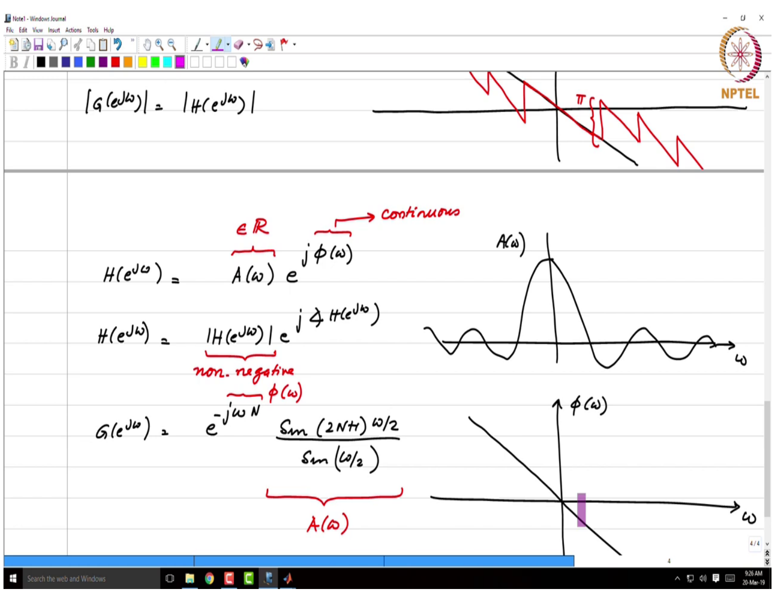

So, to see how that comes about, we will write $H(e^{j\omega}) = A(\omega)e^{j\phi}$. Whereas, previously we have been writing the frequency response as $|H(e^{j\omega})|e^{j\angle H(e^{j\omega})}$ and by definition $|H(e^{j\omega})|$ is non negative. Whereas, in this representation $A(\omega)$ is a real valued function.

And we will see that this is really $\phi(\omega)$ and for systems with rational transfer function in this representation, you can show that this will be continuous. And again considering the example we have just now seen. We have seen $G(e^{j\omega}) = e^{-j\omega N}\dfrac{\sin(2N+1)\omega/2}{\sin(\omega/2)}$.

And in the modified representation of the frequency response, this is $A(omega)$. And this happens to be $\phi(\omega)$ and the difference is $A(\omega)$ I had mentioned is now a real valued function. The main difference of course is, now this can take on both positive as well as negative values and hence in this representation, if you now plot the frequency response as a product of amplitude response times $e^{j\text{phase}}$ response where phase response is now continuous.

Now $A(\omega)$ has this shape. Main difference is it now can take on negative values as well and the corresponding phase response is just this term which is linear. Whereas, if you represent this in magnitude times $e^{j\text{phase}}$, whenever you cross a zero, you are acquiring a phase change of π. A phase change of π is a sign change, the magnitude being non negative cannot take a negative values.

Therefore, the sign change that crossing a zero causes has to be absorbed in the phase response as a jump of π. Whereas, in this representation, when you cross a zero, you acquire a sign change that sign change is absorbed in the amplitude response because it is no real value and hence the phase response is continuous without any jumps of π. Let us again continue on the theme of change of π when you cross a zero by looking at this particular example.

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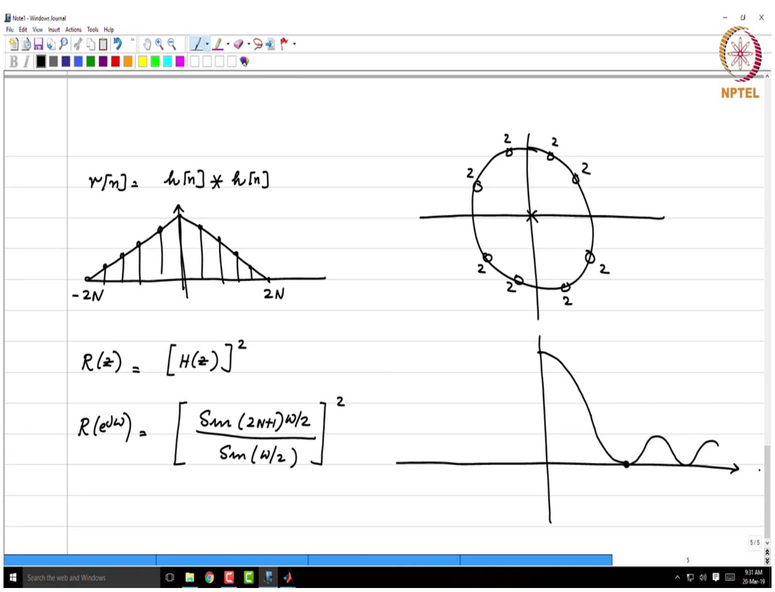

So, now, you have say $r[n]$ which is now $h[n] * h[n]$. Where $h[n]$ is the all 1's sequence between $-N$ to $+N$. Therefore, the all ones sequence between $-N$ and $+N$, if you convolve a rectangular pulse with itself, it will become a triangle. Therefore, this will be the envelope of $r[n]$. So, really if I want to plot the stem plot, it will be like this.

And this is between $-2N$ to $+2N$. And as far as $R(z)$ is concerned, so, this is nothing but $[H(z)]^2$. And the corresponding frequency response is $\left[\dfrac{\sin(2N+1)\omega/2}{\sin(\omega/2)}\right]^2$. If you now plot the pole-zero plot, of course, there will be a certain order trivial pole and then you will have zeros on the unit circle. But the difference in this case will be that, what can you say about the zeros?

Student: (Refer Time: 26:01).

So, they will be second order, yeah; repeated zeros, of second order. So, now, rather than simple zeros, you now have double zeros. Now, let us see what this means. In terms of frequency response, so, here at $\omega = 0$, you have a certain value. And then, as you approach the first zero on the unit circle for positive ω, the response drops. And then keeps dropping and then when you hit the 0, you are now crossing this double zero the rather than a single zero.

Each zero will cause phase change of π. When you cross a second order zero, the total phase change will be 2π and the phase change of 2π is now sign change. And hence the frequency response will become 0 and again go back up. So, this is how the frequency response will look. So, what this says is, this says that, you cross an n^{th} order zero, the phase change will be $n \times \pi$ and depending upon whether n is odd or even, you will either have a sign change or no sign change.

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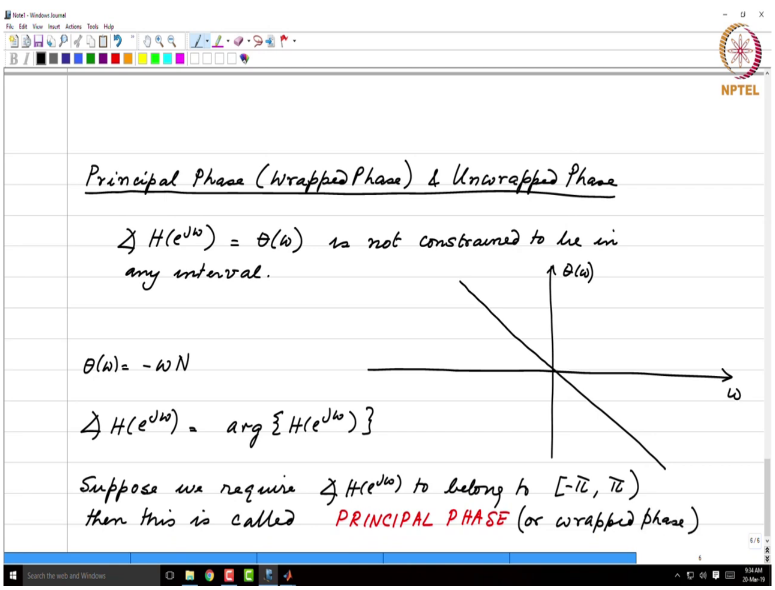

So, this is along the lines of the question that was raised. In general, the $\angle H(e^{j\omega})$ which you can denote it as $\theta(\omega)$ is not constrained to lie in any interval. And the example for this is the example that we had seen earlier which triggered the question. So, there, we saw that the phase angle was $-\omega N$. Because, it was $e^{-j\omega N}\dfrac{\sin(2N+1)\omega/2}{\sin(\omega/2)}$.

Therefore, $\theta(\omega)$ was $-\omega N$. Therefore, if you plot $\theta(\omega)$ versus ω, this is the plot you will get. And another notation for $\angle H(e^{j\omega})$ which I have mentioned before is *arg* which stands for argument, so, this is this $arg(H(e^{j\omega}))$. And this is not constrained to lie in any interval. Suppose, we restrict the range of the phase; suppose we require the phase angle to belong to this interval $-\pi$ to π, then this is called as the principal phase or other common term is wrapped phase.

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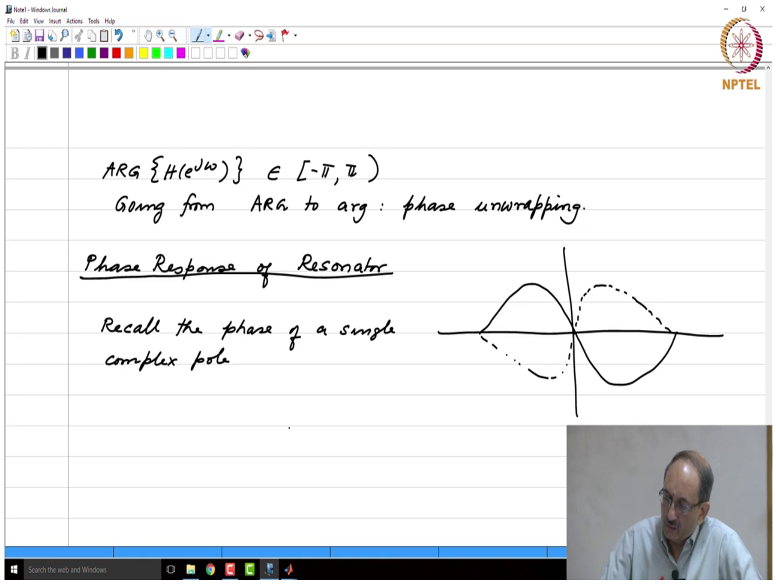

And the notation that is used is ARG and this must lie in this interval. Therefore, if you now want to plot the phase response for this particular example, you want to plot the principal phase, so, this is between $-\pi$ and π say. When it becomes more negative than $-\pi$, you add as many multiples of 2π that is needed to make the final value lie between $-\pi$ and π.

So, this is how the wrapped phase is. So, clearly in this interval which will correspond to a value here, you need to add 4π to make in between $-\pi$ and π. So, this is how the wrapped phase is. This is basically phase angle modulo 2π and the reverse process is called phase unwrapping. Therefore, given this sawtooth waveform whenever you see a jump of 2π, you will add or subtract enough multiples of 2π to make the final curve continuous.

Note that one important difference is, we have seen jumps of π before. Whereas, in this case, this is a jump of 2π whereas, the jumps we have seen before the phase response our jumps of π when you cross zero of odd order. And as I had mentioned before; going from ARG to arg; this is phase unwrapping.

And this might seem very simple; all you need to do is add or subtract enough multiples of 2π so that the resulting curve is continuous. But the problem is not as simple as this and there are research papers that have worked on this. Because you could have some ambiguities in this problem and, phase unwrapping algorithm by Jose Tripoli in 1977; I think that is one of the most widely used algorithms for phase unwrapping.

And there have been some improvements to that algorithm ever since that came out. So, this is not as simple as what it appears to be. But for our simple problems, just adding or subtracting multiples of 2π will cause phase unwrapping and the resulting phase will be continuous.

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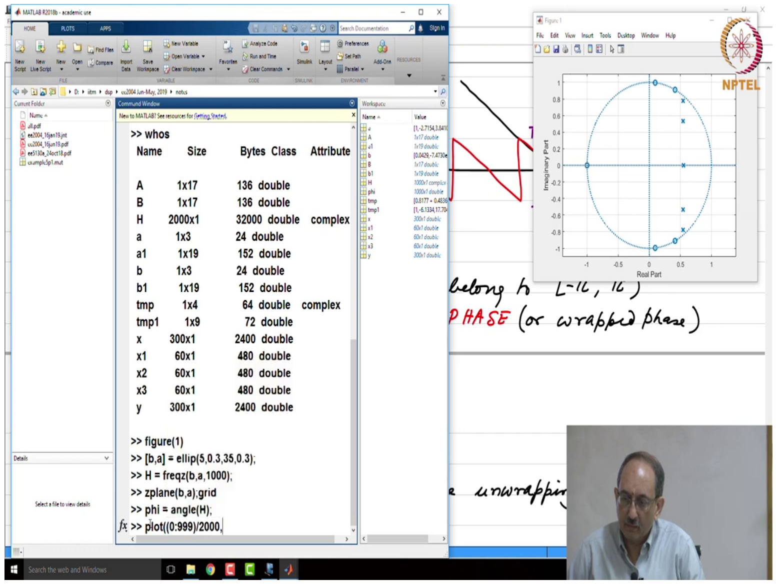

Let us see a MATLAB example to illustrate these. So, this is the elliptic filter that I had used earlier to showcase the difference between IR and FIR filters roughly having the same frequency response. But the elliptic filter was of much lower order; 5^{th} order in this case. Whereas, the FIR filter required order 32 to meet the same frequency response. So, that same filter I am using.

So, let me now compute the frequency response and I will compute this over a 1000 points from 0 to π. Now what I am going to do is, I am going to show the angle. Let me first show the pole-zero plot. So, this is this the pole-zero plot therefore, you can expect when you go along the unit circle when you hit this zero, you can expect phase jump of π. So, let us verify that.

So, here, let me plot the phase response and this is angle H; and let me plot the angle from 0 to 0.5. Because this is real valued, this is enough if you plot this from 0 to pi. So, since I have taken 1000 points.

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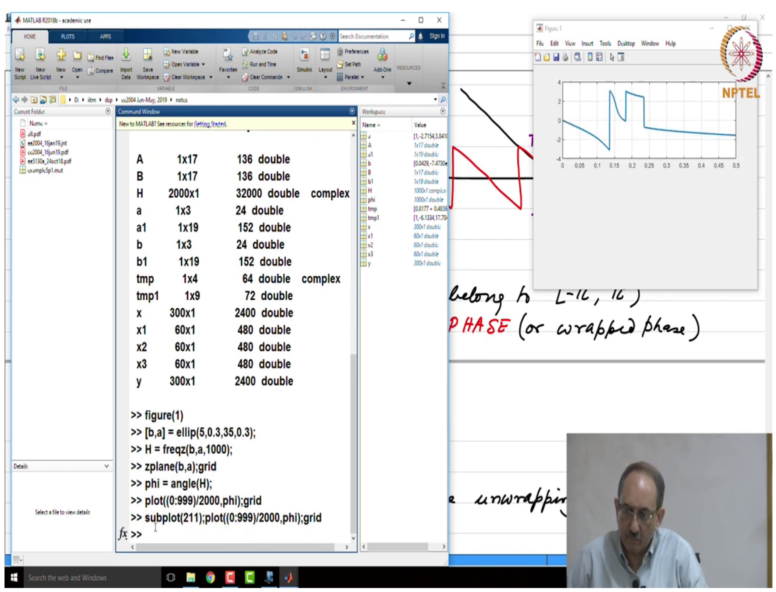

So, this is the phase response and this is the wrapped phase response. So, this phase response keeps on decreasing. At this point, it is trying to become more negative than $-\pi$. Therefore, wrapping will cause, you need to add enough multiples of 2π so that the phase always stays between $-\pi$ and π. Therefore, this jump is a jump of 2π; because, you do not want the phase response to go more negative than $-\pi$ or more positive than $+\pi$.

Therefore, at this point, you add 2π and you reach this point. Therefore, this jump is 2π and then the phase response keeps on decreasing. And now you hit the first zero; therefore, this jump is now π when you cross the first zero on the unit circle and then the phase response keeps on decreasing. Remember, I had mentioned to you that by convention MATLAB uses a jump of $+\pi$ for $\omega > 0$.

So, this is indeed $\omega > 0$. Here I have plotted f rather than ω; I have plotted $\omega/2\pi$. At this point, again you encounter the second zero on the unit circle. So, by convention, you have to jump up by π. If you jump up by π, you will exceed the $-\pi$ to $+\pi$ range. Therefore, you actually jump down that is the reason why you are jumping down here because this is the wrapped phase. So, this is the overall phase response, it is now staying between $-\pi$ and π. Let us now plot the unwrapped phase and for ease of comparison, let me plot this as plot number 1.

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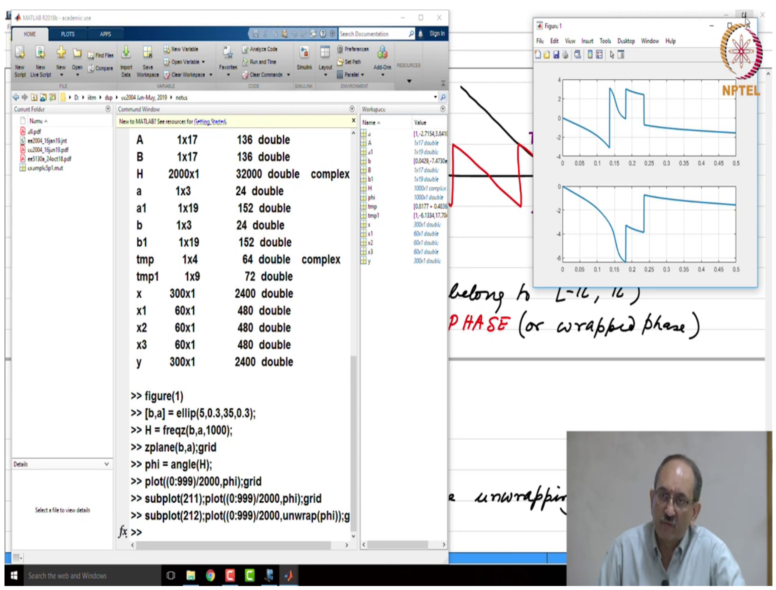

So, MATLAB has everything that you want. If you want to unwrap, just say unwrap, it will unwrap, right. So, MATLAB can do anything short of eating your lunch and writing your quizzes, MATLAB can do anything you want. MATLAB and Google together solve all of life's problems. So, this is the unwrapped phase. So, now, you see that this jump of 2π is gone because you have unwrapped.

So, now, it keeps going, now you hit the first zero. Unwrapping will not eliminate jumps of pi, it only element jumps of 2π. And you are now jumped up because this is a zero on the unit circle and we are looking at positive frequencies. And again after this jump of π, you keep going and then you hit the second zero and now again you are jumping up by convention and you keep going.

So, this is the unwrap phase, this is the wrapped phase. So, this tells you the difference; as far as the signal itself is concerned, it does not matter whether you use wrapped phase or unwrapped phase. Because, it is e^{jangle}; any angle or angle modulo 2π, it does not make a difference. And hence as per the signal is concerned, using wrap phase or unwrap phase does not matter.

However, there is one area that is called *cepstral* processing and there you have to work with the unwrapped phase. If you use the wrap phase there, you will get the wrong answer. There you take the phase angle and deal with the phase angle as a separate function, you process the phase to get some more information out of it. And in those cases, you definitely have to deal with the unwrapped phase rather than the wrapped phase.

So, there are cases where unwrapping is important, but that is outside the scope of this course. Let us look at the phase response of a resonator just to get some more feel for how phase response works. Remember, recall the phase response of single complex pole, the phase response of a zero was like this. This is the first thing that we started off with when we were looking at phase response of a single complex zero; this zero was on the real axis.

So, that is why it went to 0 at that particular point. For a single complex pole, the phase response for a pole is just the negative of this therefore, this is how the phase response will be.

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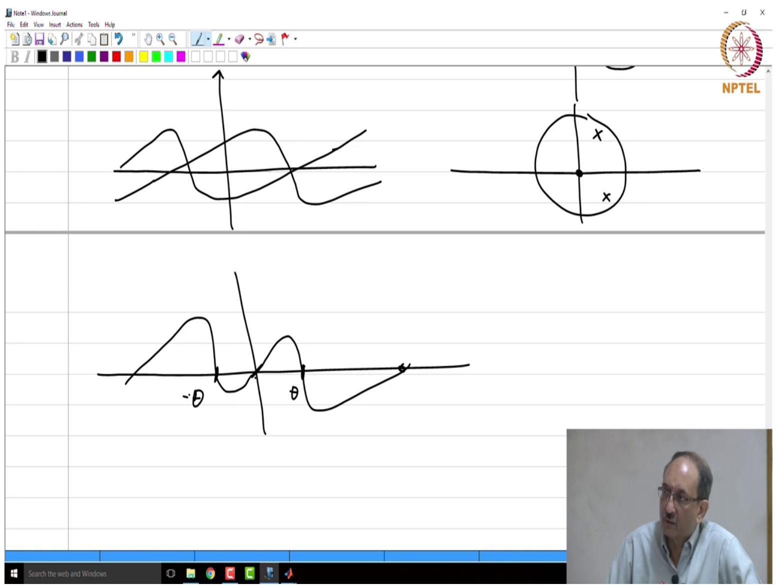

And as far as a resonator goes so, this is how the pole-zero plot of a simple resonator looked if you recall. So, at $re^{j\theta}$ and $re^{-j\theta}$, you had 2 poles. And if you looked at the phase response of each pole, you will see that the response will look like this. So, this is no different from this, only that the point at which it crosses 0 is now located at θ.

Here it is at $\omega = 0$, because in this particular example, we were looking at the pole lying at $\theta = 0$ or on the real axis; positive real axis. And if θ were not 0, all you do to the phase response is shift it to the left or to the right by whatever amount. So, this is the phase response of this first quadrant pole. The phase response for a second quadrant pole will be like this.

And the overall phase response is the addition of these individual phase responses and you will see at $\omega = 0$, each will contribute equal and opposite values. Therefore, at $\omega = 0$, you will have 0. Similarly at π and $-\pi$, you will have equal and opposite contributions; phase response will be 0. And when you cross a zero, the phase will change rapidly therefore so, this is how the overall phase response will look.

And the most rapid change will occur at $+\theta$ and $-\theta$ and this is no different from what we saw for the system that had two zeros on the unit circle.

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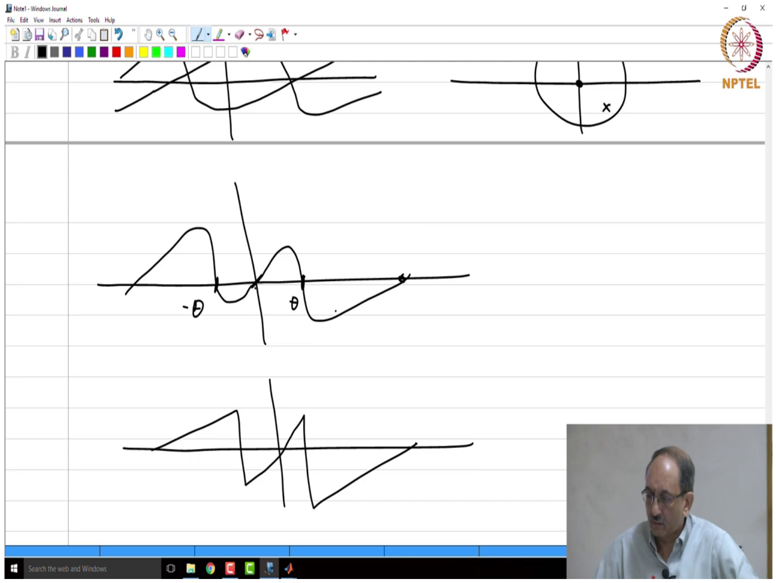

Remember, when we are looking at zeros on the unit circle, we had something like this, right. And this is no different from that. In the zeros case, this is precisely linear because zero is on the unit circle. Whereas, in the pole case, you cannot have poles on the unit circle therefore, you will have a response something like this. So, this is the phase response of a resonator, let me quickly see if I can show this in MATLAB.

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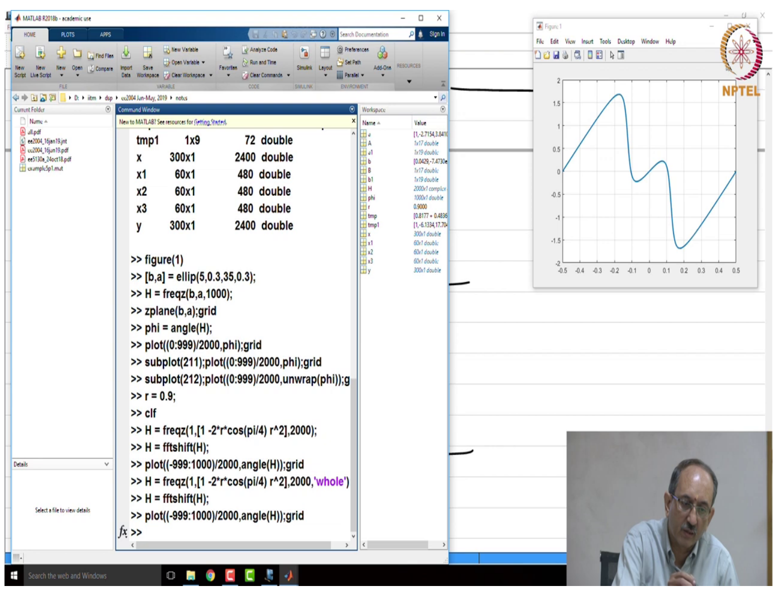

Therefore, if I have say $r = 0.9$ and so the numerator is 1, denominator is $1 - 2rcos(\ )$. Let me have $\theta = \pi/4$ and $1 - 2r\cos(\theta)z^{-1} + r^2z^{-2}$. Therefore, the other term has to be r^2 and then let me evaluate this at 2000 points, all right. And, then I will use *fftshift* of this.

And then I will plot angle between -999 to 1000 divided by 2000. I have 2000 points and then I will plot angle of H, I really should have given whole. So, this was the rough sketch I had plotted and this in on my notes in this the exact sketch. So, at $\omega = 0$, you have the phase system going to 0.

And the most rapid phase change happens when around the region where you cross the pole. At $\omega = 0$ and rather at $f = 0$ and $f = -1/2$ and $+1/2$, the phase response goes to 0 and this is how the overall phase response is for a resonator. So, this is the exact plot. What I had plotted in my notes, sketched in my note was just rough. So, this the exact plot for $r = 0.9$ and $\theta = \pi/4$. And close $r = 1$, the more rapid will this phase change be. And deeper the pole is inside the unit circle, gentler will this change around this region will be.