Digital Signal Processing
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Lecture 56:
Magnitude Response (4), Phase Response (1)

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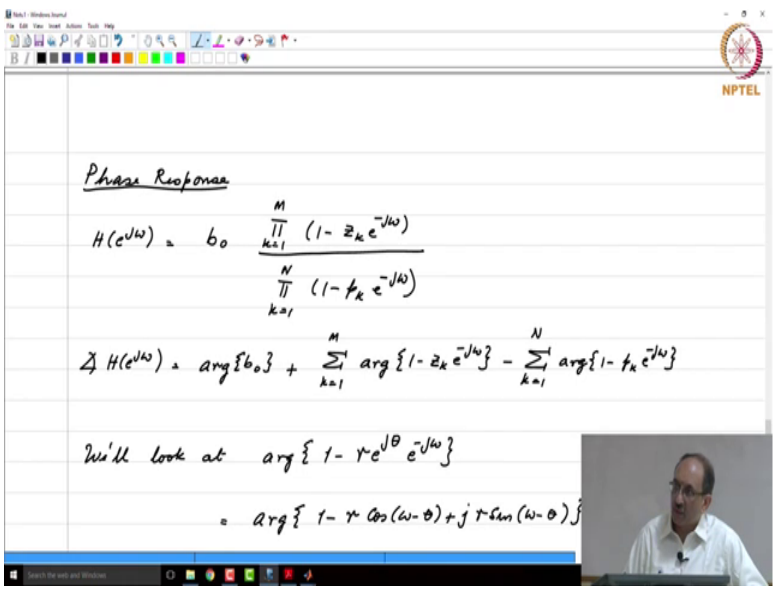

So, this is as far as Magnitude Response goes. Then we will move on to the other part of frequency response namely, Phase Response. So, we have $H(e^{j\omega})$. So, this is $b_0\dfrac{\Pi_{k=1}^{M}(1-z_k e^{-j\omega})}{\Pi_{k=1}^{N}(1-p_k e^{-j\omega})}$.

So, this the form of the frequency response for rational transfer function, where we have factored the numerator and the denominator and now we are looking at the angle; and the notation arg is used to denote the angle of a complex number; arg being the short form for argument. And you have $\arg\{b_0\} + \displaystyle\sum_{k=1}^{M} \arg\{1 - z_k e^{-j\omega}\} + \sum_{k=1}^{N} \arg\{1 - p_k e^{-j\omega}\}$.

So, this is pretty straightforward based on the formula for angle of ratio of complex numbers. So, now, let us concentrate on one typical factor and get a field for what is happening here. Therefore, we look at argument of $(1 - \ )$, let us take the zero to be at $re^{j\theta}re^{-j\omega}$ and this of course is, $\arg\{1 - r\cos(\omega - \theta) + jr\sin(\omega - \theta)\}$.

All I have done is I have just multiplied this out and replaced $e^{j(\ )}$ by $\cos(\ ) + j\sin(\ )$.

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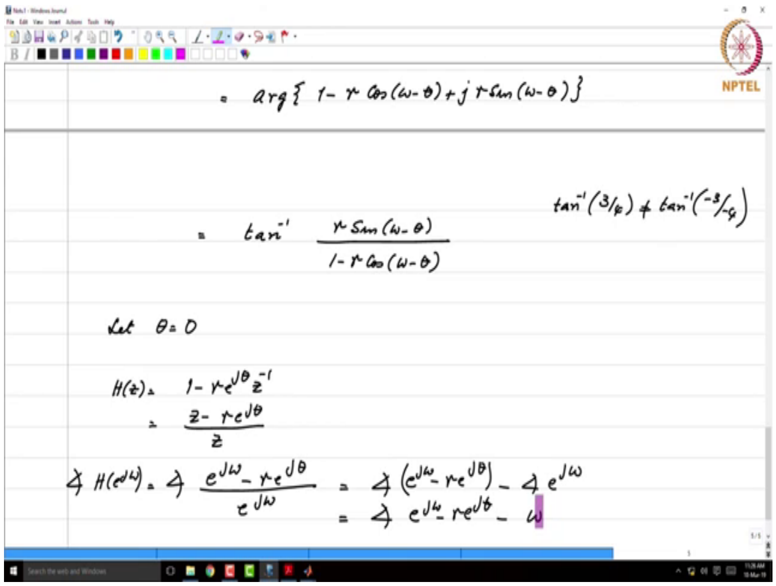

And this is nothing but $\tan^{-1} \dfrac{r\sin(\omega - \theta)}{1 - r\cos(\omega - \theta)}$ and this tan inverse is the four quadrant inverse tan. And the reason why I am mentioning that is $\tan^{-1}\left(\dfrac{3}{4}\right)$ is not the same as $\tan^{-1}\left(\dfrac{-3}{-4}\right)$. So, you are looking at the four quadrant inverse tan here.

And as theta changes remember, θ is the angle of the root. As θ changes, all that is going to happen to the phase response is shift the phase response either to the left or to the right. So, basically this is the phase response of a single complex zero. For illustrative purposes, let $\theta = 0$. Similar to the geometric interpretation of the magnitude of $a - b$ where a and b are complex numbers, that interpretation is it is the length of the vector joining these two points.

Similarly the geometric interpretation of the angle of the vector joining the two complex numbers is, if you have angle of $a-b$, it is basically the angle of the vector joining these two points. So, we will exploit that interpretation and get a feel for the phase response similar to getting a field for the magnitude response geometrically.

Therefore, if you have $H(z) = 1 - re^{j\theta}z^{-1}$, this is a single complex zero. So, this is nothing but $\dfrac{z - re^{j\theta}}{z}$ and hence $H(e^{j\omega})$ is $\dfrac{e^{j\omega} - re^{j\theta}}{e^{j\omega}}$. And then we are looking at angle of this and this in turn is $\measuredangle(e^{j\omega} - re^{j\theta}) - \measuredangle e^{j\omega}$.

So, this is $\measuredangle(e^{j\omega} - re^{j\theta})$ and this is nothing but ω because $\measuredangle e^{j\omega}$ is just ω. So, the reason why I am writing it like this is, remember this z which is a trivial pole did not contribute to the magnitude response. But sure enough it cannot be sitting in the expression not contribute to both phase and magnitude. It did not contribute to the magnitude, it is surely contributing to the phase as can be expected.

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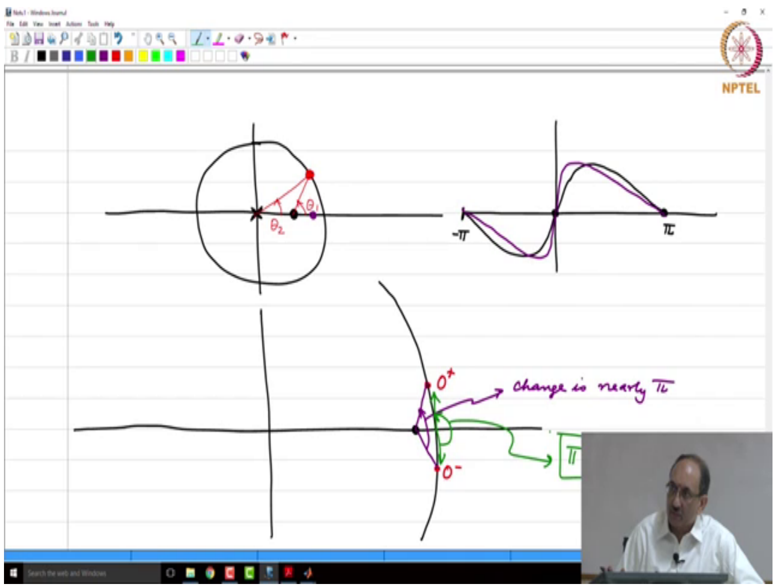

So, $1 - re^{j\theta}z^{-1}$, which is $\dfrac{z - re^{j\theta}}{z}$. So, you have a trivial pole and for illustrative purposes I am going to use $\theta = 0$. Therefore, I have a zero sitting here and this is a general point on the unit circle and now I am going to drop on the geometric interpretation to get a field for what is going on.

So, let me call this angle as θ_1; this angle to be θ_2. Now, as far as the phase response goes, you start off at this point as usual and go around the unit circle. When you are here at $\omega = 0$, then both θ_1 and θ_2 make an angle of zero and hence the difference also is 0. Therefore, the phase response starts off with a value of 0 here and now when you are at this point at $\omega = \pi$, then both vectors subtend an angle of π.

Therefore, the difference is again 0; and hence at $\omega = \pi$, the phase response is 0. When you increase the frequency from $\omega = 0$; when you go up around the unit circle clearly, θ_1 increases more rapidly than θ_2. Therefore, $\theta_1 - \theta_2$ will be positive and hence you can expect the phase response to increase and then you will reach a maximum and then become 0 again.

Because, we have fixed, we know that these two points, phase response has to be 0. The phase response is increasing when you are near the origin. And then eventually it will reach a maximum and then come back to 0. And because $\theta = 0$, this is zero on the real axis which means the simple systems impulse response is real. For such systems, magnitude response is even and the phase response is odd. Therefore, the phase response, you can complete using the odd symmetric property to be like this. So, this is the overall phase response.

Now, to get a further field for what is going on, suppose the zero is now closer than before that is compared to the earlier radius we have a larger radius now. Again at $\omega = 0$ and $\omega = \pi$, the phase response has to be 0 so, nothing changes.

But what does change is the rate at which the angle is going to increase, when you have the zero closer to the unit circle will be larger. And hence in the second case, theta 1 will rise more rapidly and hence you can expect the phase response to be something like this. So, this is the general field that we have geometrically speaking. Now, let us do this in MATLAB to get the exact phase response.

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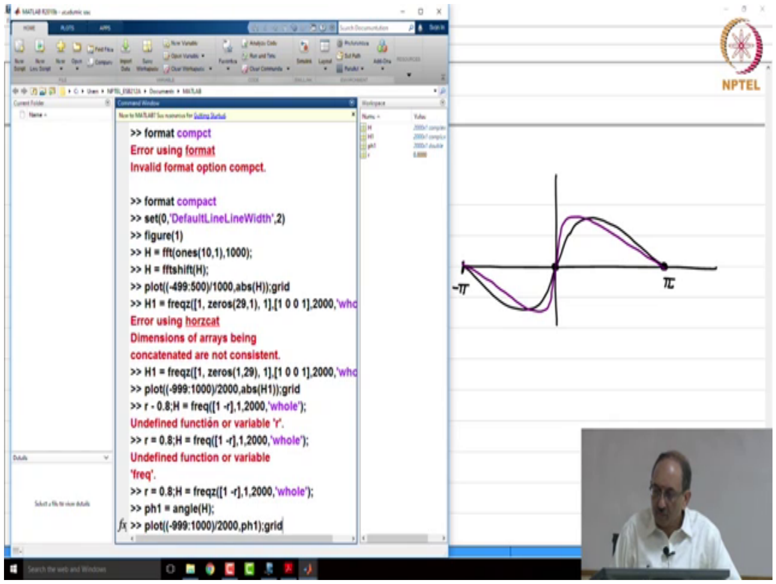

So, I have H which is the frequency response. Now, let me make $r = 0.8$; the frequency response is $[b, a]$; b is $1 - re^{j\theta}z^{-1}$. Now $\theta = 0$ therefore, this stays as it is and the denominator is 1 and I am going to evaluate this at 2000 points and I am going to evaluate it around the entire unit circle. All right.

Now, phase response ph1 is the angle of the frequency response. So, this is, remember frequency response is complex, I am using MATLAB's angle command to get the phase. Now, let me just plot this.

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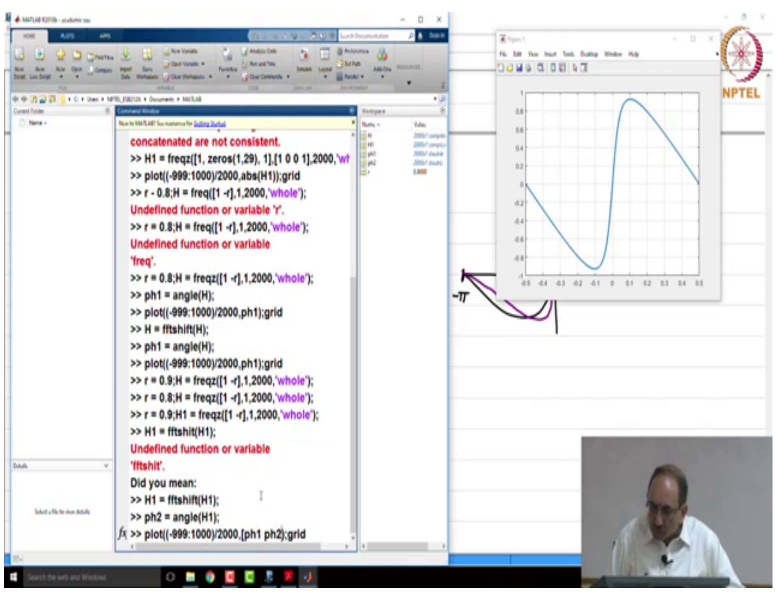

So, now it is the more familiar picture which I had drawn by hand.

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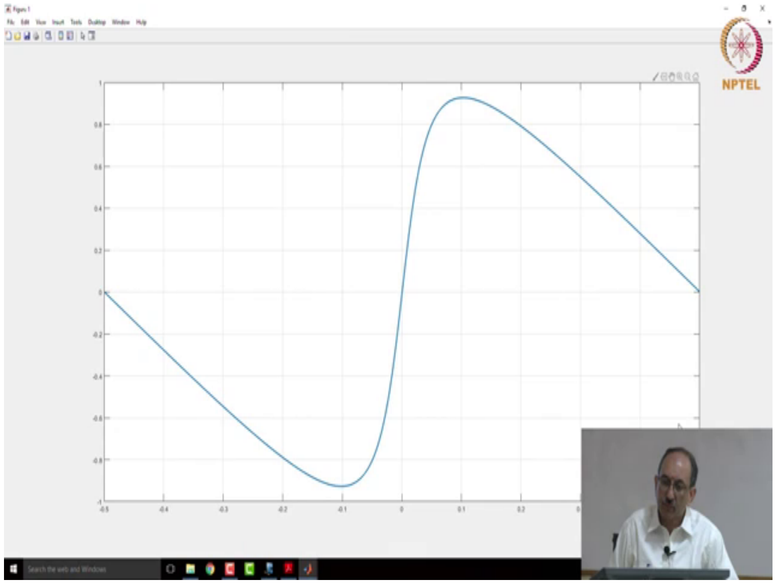

So, at $\omega = 0$ or rather $f = 0$, you have the phase response to be 0. And at $\omega = \pi$ which corresponds $f = 1/2$, again you have phase response to be 0 and it increases reaches a maximum and then falls off and this is for 0.8. Let us now make this to be 0.9. So, this is and then ph2 is angle($H1$) and then I have to plot ph2.

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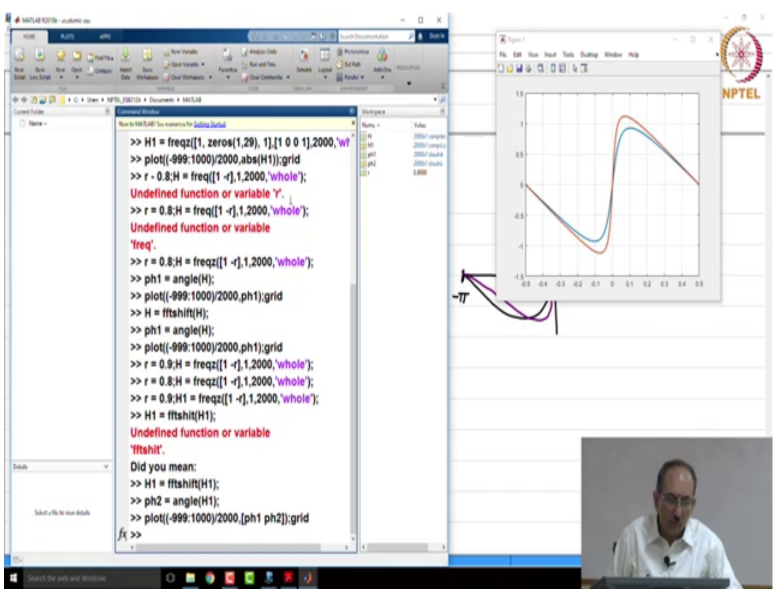

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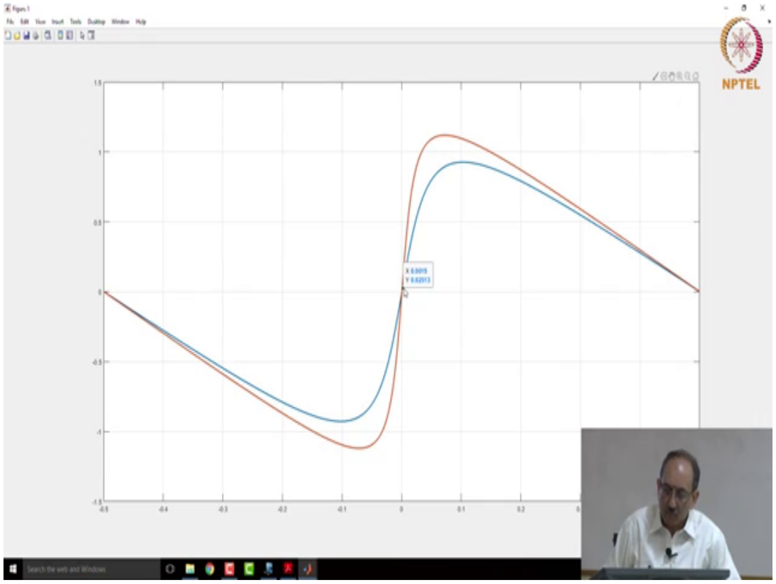

So, now you see between 0.8 and 0.9; the rapidity of the increase in the phase near the origin has increased. So, this is what we were expecting based on our intuitive explanation and MATLAB indeed confirms that this is true. So, let us carry this analysis a little more. Now, what I am trying to show here is an expanded version around $\omega = 0$. Only that, now my zero is extremely close.

So, when I am here on the unit circle, I am at $\omega = 0^-$. When I am here on the unit circle, I am at $\omega = 0^+$s and r is very very close to 1 and this is an expanded view. And you can see that at 0^-, this is the angle subtended by this zero. At 0^+, this is the angle. And when I go from 0^- to 0^+, the change is nearly π.

And in the limit, when the zero is on the unit circle: at 0^-, the vector will be a tangent at this point; and 0^+, the vector will be pointing like this and the change will be exactly π. Therefore, when the zero is on the unit circle, the change when you cross the zero will be π radians.

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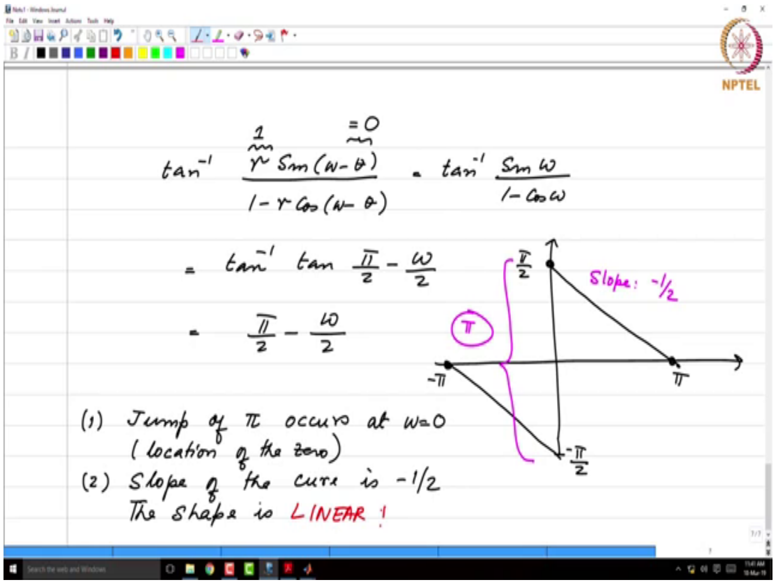

Now, algebraically what is happening is this, the phase response after all is $\tan^{-1}\dfrac{r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}$, $\theta=0$, $r=1$. Therefore, this is really $\tan^{-1}\dfrac{\sin(\omega)}{1-\cos(\omega)}$ and then this turns out to be $\tan^{-1}\ \tan\frac{\pi}{2}-\frac{\omega}{2}$. This you can verify and this turns out to be therefore, $\dfrac{\pi}{2}-\dfrac{\omega}{2}$.

And if you plot the frequency response at $\omega=0^{+}$, the value of the angle is $\pi/2$, when $\omega=\pi$, the value is 0 and in between the variation is precisely linear. And this being an odd symmetric function, this overall phase response between $-\pi$ and π is this. So, you have $\pi/2$, $-\pi/2$, ok.

And from this, the inferences you can draw are when the zero is on the unit circle, you see a phase jump of π, remember this difference is precisely π. So, this is one inference you can draw. And the location of the jump corresponds to the location of the zero. In this particular case, the zero is at $\omega=0$. Because, all that will happen is when θ changes, this curve will shift to the left or to the right, that is all. And in this case, this happens to be centered at $\omega=0$, because that is where the zero location is.

So, the jump will occur exactly at the location of the zero and the slope of the curve happens to be $-1/2$. Therefore, the points to be noted are: jump of π occurs at $\omega=0$ and it happens at $\omega=0$ because that happens to be the location of the zero, and slope of the curve is $-1/2$ and the shape is linear, all right. So, these are the inferences you can draw from the phase response.

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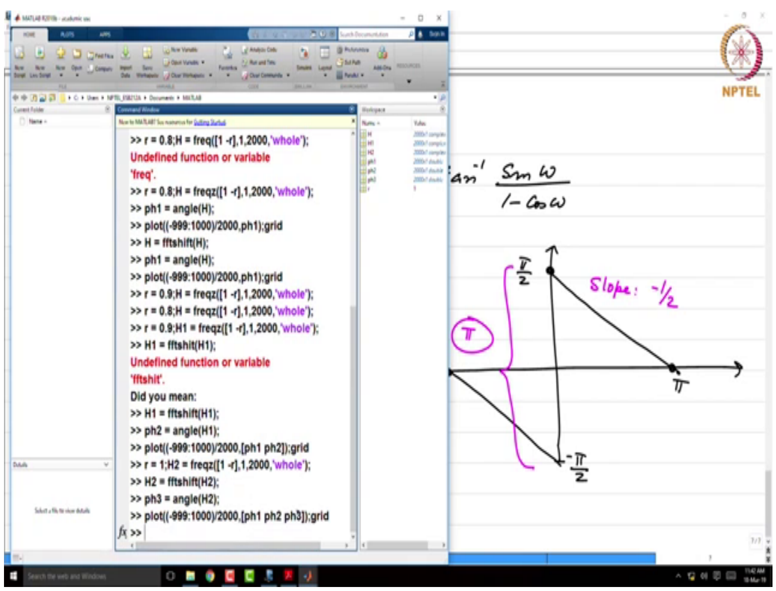

And if I now make r equals 1. Now let me add this.

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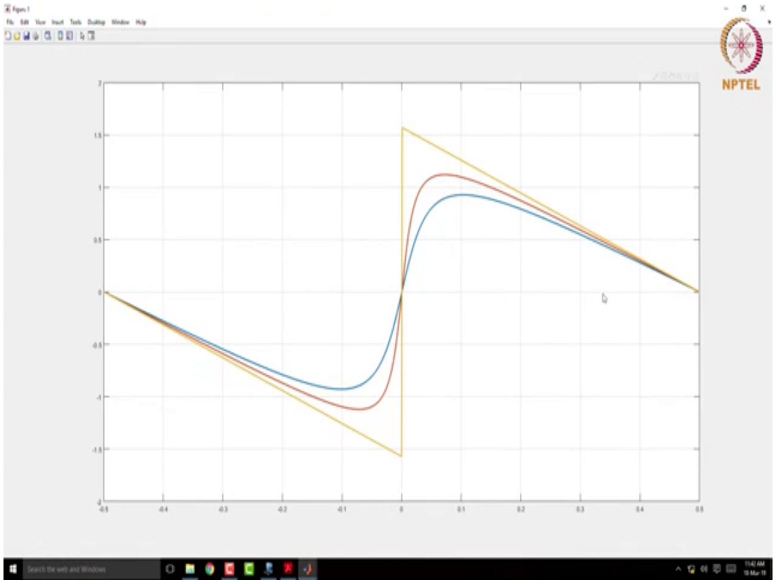

Now, you see exactly what was discussed is now being verified numerically. So, the last curve added is case when $r = 1$. So, the jump occurs at $\omega = 0$ or $f = 0$, the shape is precisely linear and you see this is roughly 1.5 something and this is -1.5, this is in practice in radians. Therefore, this jump is precisely numerically 3.4 which is π.

So, the jump occurs at $\omega = 0$; the value of the jump is π where the shape is precisely linear and the slope is $-1/2$. So, these are the inferences that you can draw when you have a zero on the unit circle. Moment you have this, you can immediately guess that any zero on the unit circle will give rise to this kind of sawtooth waveform.

And if the zero location changes, zero has to be on the unit circle for this shape to be present. And depend upon the angle, the curve will shift either to the left or to the right. So, any zero on the unit circle will give rise to such a sawtooth waveform and depending on the location of that zero, the jump will be happening at that particular location.

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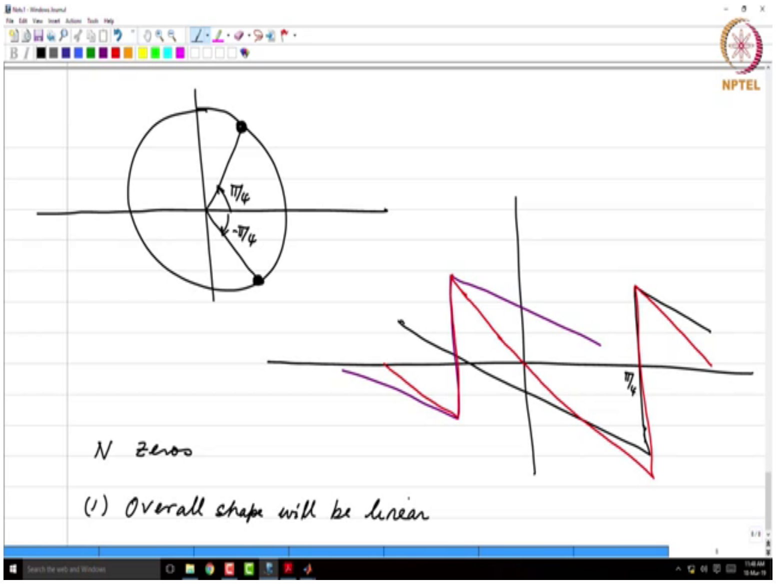

Therefore, if I have a zero here and a zero here and for illustrative purposes if I make this $\pi/4$, then this to be $-\pi/4$.

Then you can see that this zero, I will have a jump at $\pi/4$. Similarly, for this zero, I will have something like this. And the overall response will be something like this, this is the overall response.

Again the point to note is, when you are here on the unit circle, this zero will contribute one angle; this zero also will contribute another angle. But these two angles will be equal and opposite in sign. Therefore, at $\omega = 0$, the phase response will be 0. Similarly, at when $\omega = \pi$, again both the zeros will contribute equal and opposite angles and $\omega = \pi$, the phase response will go to 0.

And now you are adding two sawtooth waveforms, each having slope minus half therefore, if you add two linear curves each with slope $-1/2$, the overall slope will be -1. Therefore, in this case, the slope of the linear region will be -1 because they are the result of adding two curves each with slope $-1/2$. And the jumps will happen at $+\pi/4$ and $-\pi/4$.

And given the symmetry of the location of the zeros at $\omega = 0$ and at $\omega = \pi$, overall phase response will be 0. And in between, you will have linear response except at the location of the jumps. And in this case, the system's impulse response is real valued because you have two complex conjugate zeros and hence the phase responses are symmetric.

In general, you can have any collection of zeros on the unit circle, exactly the same behavior will be manifested. Therefore, we have N zeros on the unit circle, then overall shape will be linear because they are result of adding sawtooth waveforms that we saw earlier.

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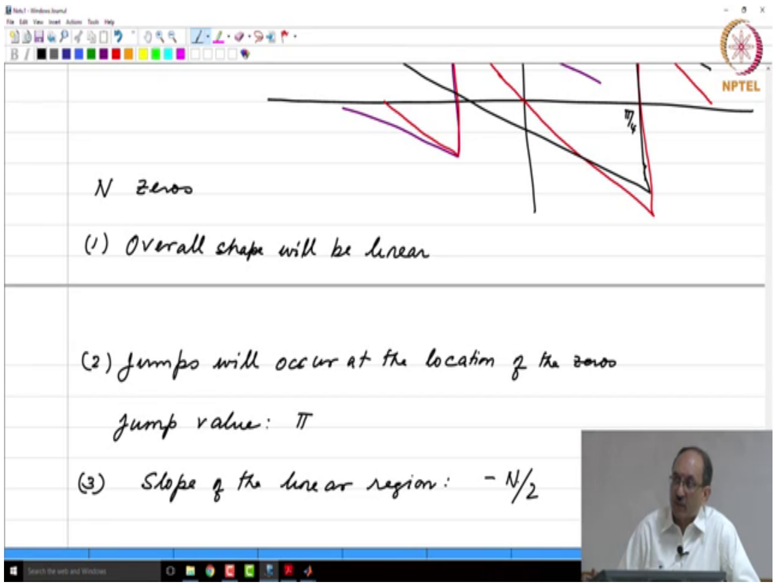

And jumps will occur at the location of the zeros and the jump value will always be π. And the slope of the linear region, remember you are adding N such sawtooth type waveforms; each sawtooth waveform has a slope of $-1/2$. If you add N such linear curves, the overall slope will be $-N/2$.

Therefore, any collection of zeros on the unit circle will lead rise to linear phase response except at the location of jumps and the jumps will always have a value of π and the slope overall slope will be $-N/2$. And if the collection of zeros happens to have complex conjugate property, then the overall phase response will be odd symmetric.

On the other hand, if there is no such relationship between the zeros, you cannot expect the phase response to be odd symmetric. Phase response will be odd symmetric only if the impulse response is real value and that depends on the location of the zeros. If every zero has a complex conjugate pair, overall phase response will be odd symmetric otherwise it will not be.