Digital Signal Processing
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Lecture 52: Magnitude Response (3) -Resonator

Let us get started, we are looking at frequency response and we saw how poles and zeros shaped the frequency response and we saw the response of a single complex pole and single complex zero. And based on the log magnitude response, we made the observation that zeros cause sharp valleys and broad peaks whereas for poles, the peaks are sharp and the valleys are broad.

And we saw crude low pass and high pass filters based on single complex pole and single complex zero. And the main inference that was drawn was a low pass filter realized using a pole had a bandwidth; for the example that we looked at that was 15 times narrower than the corresponding low pass filter realized using a single zero.

And we also made the remark that in the absence of poles, we had an FIR filter; the frequency response is completely shaped by zeros because there are no non trivial poles to work with. And using a simple example, we showed that to meet the frequency response specifications, typically it requires a much lower order pole zero filter than the corresponding all zero filter. Now, let us continue with the frequency response and look at a second order system to get some more feel for where these things go.

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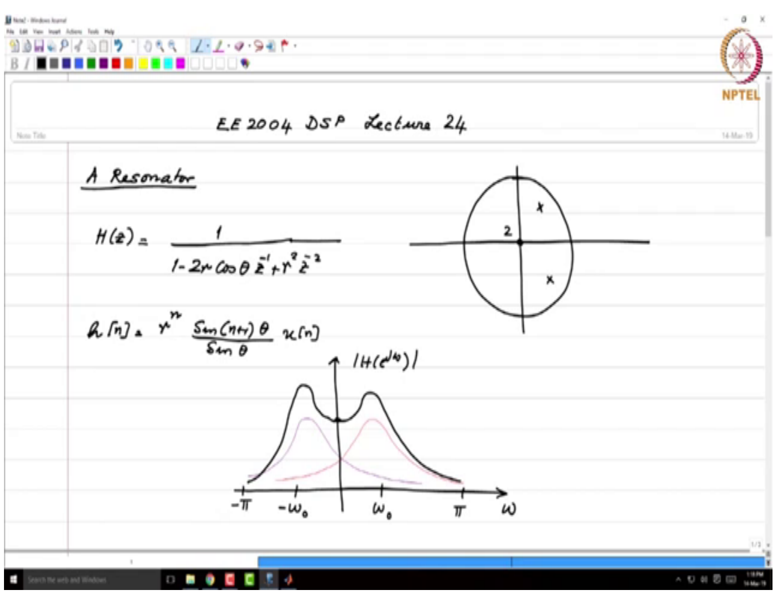

So, we will look at resonator and the frequency response of a resonator is given by this. And we know the corresponding impulse response is $r^n \dfrac{\sin(n+1)\theta}{\sin(\theta)} u[n]$. So, this is the impulse response and corresponding transfer function and if you factor the denominator, you will find that there are poles at $re^{j\theta}$ and $re^{-j\theta}$. Therefore, you have poles at $re^{j\theta}$ and $re^{-j\theta}$ and you also have a second order trivial zero. And you can very quickly get a feel for the magnitude response; you are here on the unit circle, the magnitude response may have this value.

And then from this point which is $\omega = 0$, as you traverse along the unit circle and if you approach this pole; you can expect the magnitude response to increase; peak. And then when you go past the pole, the response is going to decrease and will reach a certain value at $\omega = \pi$. And if I complete the picture from between $-\pi$ and π; this is how the frequency response looks like.

So, this is the magnitude plot and this is the response of what is called as a resonator. And what we had seen earlier; the first order system, we saw that you could realize crude low pass and high pass filters. And looking at this shape, you can see that this is a crude what kind of filter?

Student: Band pass.

Student: Band pass filter.

Band pass filter because it has maximum gain at some frequency that is not 0 or π; so this is a crude band pass filter and recall when we were looking at the response of a single complex zero or a single complex pole; the peak occurred. When the system was a single complex pole, the peak occurred precisely at the angle of the pole. So, the pole if it was at $re^{j\theta}$; the maximum response indeed occurred at $\omega = \theta$. Now, here, as you go along when you are exactly at the angle of the pole; you might expect the magnitude response to have the maximum value. So, this is the first level guess; now let us see whether this is indeed true.

Suppose, you only had this pole and no other pole or zero were present other than something being trivial. No non trivial pole or zero is present, if that were the case; then from our earlier study, we know that the peak will occur exactly at that location. Because you will be closest to the pole and hence if you just focused on that particular pole; neglecting everything else or imagining that nothing else is there, then you can expect that the frequency response will be like this.

So, this is no different from what we had seen earlier and this peak will precisely be at $\omega = \theta$; assuming θ is the pole location. Similarly, if you now consider this particular pole; then you can expect the frequency response to be like this; assuming that this was the only pole present.

But the system we are considering has both poles and if you look at this picture; you can see that this peak is affected by this tail. Similarly, this peak also is affected symmetrically by this tail and hence this peak location is not exactly at $\omega = \theta$ where θ is the angle of the pole. So, if you call this frequency omega naught and symmetrically this will be $-\omega_0$. It is important to realize that ω_0 will not be θ and the reason why this is not θ is because of the interference of the tail of the other poles response.

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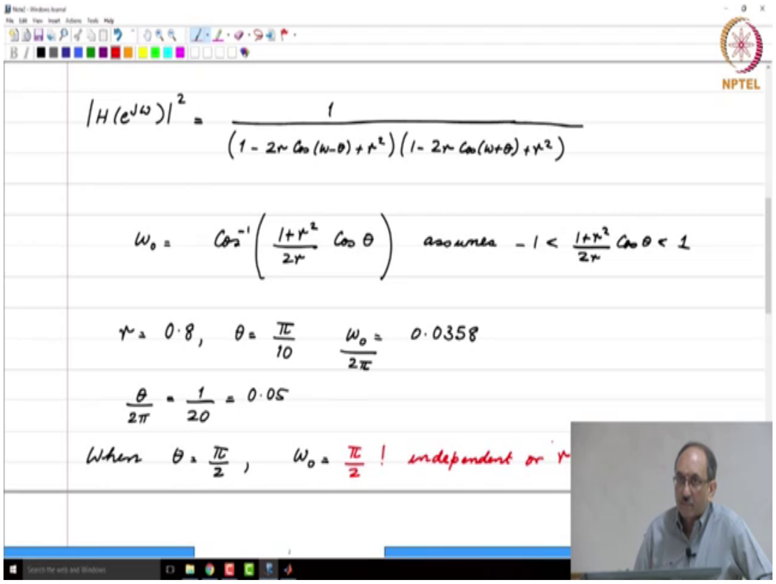

And $|H(e^{j\omega})|^2$ is nothing, but $\dfrac{1}{(\ )(\ )}$. So, $1 - 2r\cos(\omega - \theta) + r^2$ is due to the contribution of the pole at $re^{j\theta}$; we saw this expression when we are considering a single complex pole; so this term comes from the contribution of $re^{j\theta}$. Similarly, you will have $1 + 2r\cos(\omega + \theta) + r^2$ because the other pole is at $re^{-j\theta}$ and it contributes this particular term.

So, this the exact expression for the magnitude square frequency response for this second order system; it is pretty easy to see. If you have understood the magnitude square response for a single root, then this is just two such factors; one corresponding to θ, the other corresponding to $-\theta$. And now you were looking at finding the frequency at which this is a maximum and this is simple school level calculus.

So, you want to find what ω_0 is which means you have to find the maximum of this term or minimum of the denominator and the expression turns out to be this which you need to actually verify. So, this turns out to be $\cos^{-1}\left(\dfrac{1+r^2}{2r}\cos(\theta)\right)$; this is what this expression turns out to be. And then for example, if you take $r = 8$ and then $\theta = \pi/10$.

Can you tell me what ω_0 is? Right. You need to pull out your calculator, do some calculations and tell me the answer. Tell me what $\omega_0/2\pi$ is? Remember theta by 2π is $1/20$; so this is 0.05. Therefore, you would have expected this peak to be located at 0.05; ordinarily speaking, if you had not thought about the interference from the tail, but now you see that there is interference and so the peak is not precisely at the angle at which the pole is located, but shifted and it turns out to be 0.03.

Student: (Refer Time: 11:42).

Alright.

And this, you can also verify in MATLAB, you can plot the frequency response assuming these values of r and θ. And then if you plot magnitude frequency response and then locate the peak; you will find that the peak will not be at 0.05, where 0.05 is the normalized frequency; here I am talking about omega by 2π scale. This of course, assumes rather, so what does this assume? Can you guess by looking at the formula? Alright of course, r being 0; you will put the pole at the origin which will be trivial; so that is ok.

Student: r (Refer Time: 12:59) theta (Refer Time: 13:00) r not equal to 1.

$r \neq 1$; that also is in a sense understood, because this we are talking about poles we do not want that to be on the unit circle.

Student: θ has to be 1.

θ has to be?

Student: 1.

Say that again.

Student: θ has to be 1.

Not sure, I fully understand.

Student: $\frac{1+r^2}{2r}$ is (Refer Time: 13:26) equal to 1.

Student: So, for cos inverse, $\cos(\theta)$ has to be less than or equal to $2r/(1+r^2)$.

Very good. So, what you are really assuming here is, you are assuming that this is $-1 < \frac{1+r^2}{2r}\cos(\theta) < 1$. So, this you assume that this is true because as was pointed out if this argument exceeds 1, then cos inverse of something that is greater than 1 will be; cos inverse of a quantity that is greater than 1 will be?

Student: (Refer Time: 14:19).

Will purely be imaginary; here we are looking for something that is real valued and hence that is another way of demanding why this has to be satisfied. Now, pictorially what is happening as far as this condition is concerned is this; when this pole moves closer to the origin, that is when θ becomes smaller in value; then both these poles approach each other which means that these two peaks will move closer to each other.

And this condition where you require $\frac{1+r^2}{2r}\cos(\theta)$ to be between -1 and 1; if that is satisfied, then you will see a distinct peak away from the origin. If the pole angle and the radius are such that these two are; one way of seeing this is these two being very close and the angle and the radius are such that this condition is violated, then what do you expect?

Student: (Refer Time: 15:52).

Very good, if this condition is violated, then there will be only one peak at the origin. So, this will also happen when these poles approach π. So, what will happen is these two peaks will move away as the angle increases and these two will shift towards π. And again, two distinct peaks will be seen as long as that condition is satisfied, if that condition is violated, then a single peak at pi will be present. Even though you have two roots; if r and θ such that this condition is violated, you will have a single peak at $\omega = \pi$. Similarly, you will have a single peak at $\omega = 0$ depending upon what the angle is.

So, algebraically you require this to be less than 1, because you do not want cos inverse to be imaginary.

Geometrically what that translates to is; a single peak being observed rather than two peaks. And you should try this example in MATLAB and plot the frequency response and get a feel for the peak location. You can evaluate the frequency response at say 2000 points and then plot it between $-1/2$ to $1/2$; after dividing by 2π and then you will see that the frequency response peak is close to this value.

So, we saw that this is a crude band pass filter. The canonic band pass filter, what can you say about its gain at $\omega = 0$ and $\omega = \pi$; namely the lowest and the highest frequency, the canonic gain should be 0. Therefore, now let us look at an improved resonator.

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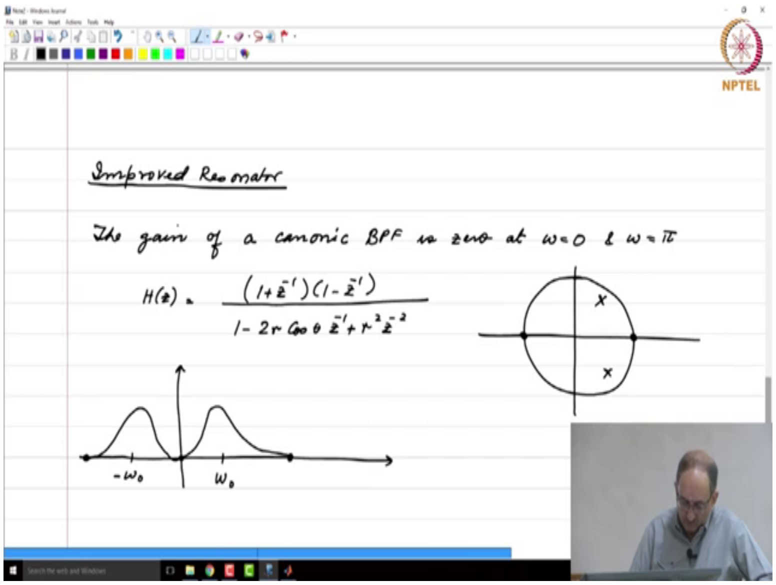

So, the point to be made here is the gain of a canonic band pass filter is 0 at $\omega = 0$ and $\omega = \pi$. Therefore, what we will do is if you look at this transfer function, there is a second order trivial zero. Let us move these zeros to which locations? At $z = ?$, $\omega = 0$ and $\omega = \pi$ corresponding to $z = 1$ and -1. Therefore, if you now have $H(z)$ with zeros at -1 and $+1$, the denominator of course stays the same which is $1 - 2r\cos(\theta)z^{-1} + r^2z^{-2}$.

And if you now look at the pole-zero plot; so these two poles are as before, but now instead of having a second order trivial zero; you have now moved those zeros to $+1$ and -1. And hence, if you look at the frequency response at $\omega = 0$, the frequency response is indeed exactly 0. Because of the zero at $z = 1$ and then when you move away from the origin; you are going to approach this pole, so the gain will go up; it will reach a peak. When you cross the pole, the frequency response will decrease in magnitude. And when you approach $\omega = \pi$, because you will you are now going to hit the other zero, the response will be precisely 0.

Therefore, the overall response roughly looks like this. Again the question arises whether the peak occurs at exactly the same location as the angle of the pole and it turns out in this case also there is peak shifting.

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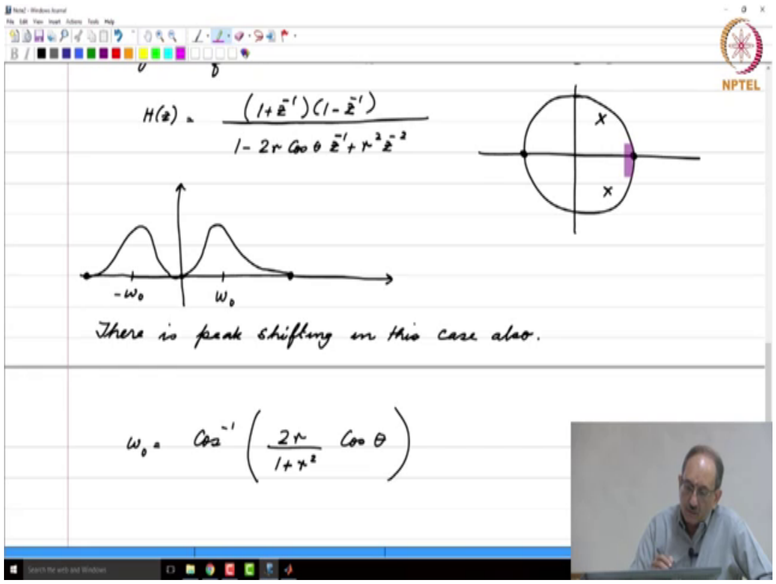

So, there is peak shifting in this case also and the expression is very similar; it is now $1 + r^2$ the denominator. So, its cos $\left(\frac{2r}{1+r^2}\cos(\theta)\right)$; now what about any condition to be satisfied? No condition needs to be satisfied. Because $2r$ is always less than or equal to $1 + r^2$; therefore, $\frac{2r}{1+r^2}$ is always less than 1.

Therefore, this product is always less than 1 and therefore, there is no question of this argument exceeding unity. And geometrically also, this picture is reinforced; earlier when this condition was violated, the formula was $\cos^{-1}\left(\frac{1+r^2}{2r}\cos(\theta)\right)$. And that had to be between -1 and $+1$ and when that was not satisfied; the peaks merged into 1 at the origin or at rather at $\omega = 0$ or at $\omega = \pi$.

In this case because of the zeross present at $\omega = 0$ and $\omega = \pi$, you will always have two peaks; there is no question of two peaks merging into one. So, this is the geometric inference or rather fall out of this condition never being violated that is the argument exceeding unity is never going to happen in this case. And the frequency response displays two distinct peaks; the merger of peaks does not occur here.

Another inference you can draw from this formula, that is here is as follows. Let us also, before we point that out; suppose you fix r for the resonator and then merely vary θ; that is the poles can come close to each other or away from each other. So, when you fix r and then vary θ; these two peaks either will approach each other or go away from each other.

When they approach each other that is when theta becomes smaller and smaller, then you see that the interference of the tail is going to become smaller or larger? When these two things individual things come closer to each other; the interference of the tail is going to become?

Student: Larger.

Larger. And hence you can expect more peak shifting. Similarly, when they move away from each other; the interference will become smaller. But you see that as these move away; when they cross $pi/2$, they start approaching each other on the other side. Therefore, as these move away from each other, the interference reduces. But after a certain point, the interference starts to.

Student: (Refer Time: 24:59).

Increase again, because they become closer on this side. What about the interference when $\theta = \pi./2$? When $\theta = \pi/2$, ω_0 is? ω_0 is also $\pi/2$ and this is independent of r. Another observation you can make; earlier what we had done was, we had fixed r and we were varying θ.

Now, let us fix θ and then vary r; for the same theta if r becomes closer to 1, then each of these individual responses become peak here; these become peakier in nature. In which case, what can you say about the interference? Does it reduce or increase?

Student: Reduces.

It reduces; therefore for the same θ if r approaches 1, peak shifting reduces because the interference is less.