Digital Signal Processing
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Lecture 51:
Magnitude Response (2)
-Response of a single complex zero/pole

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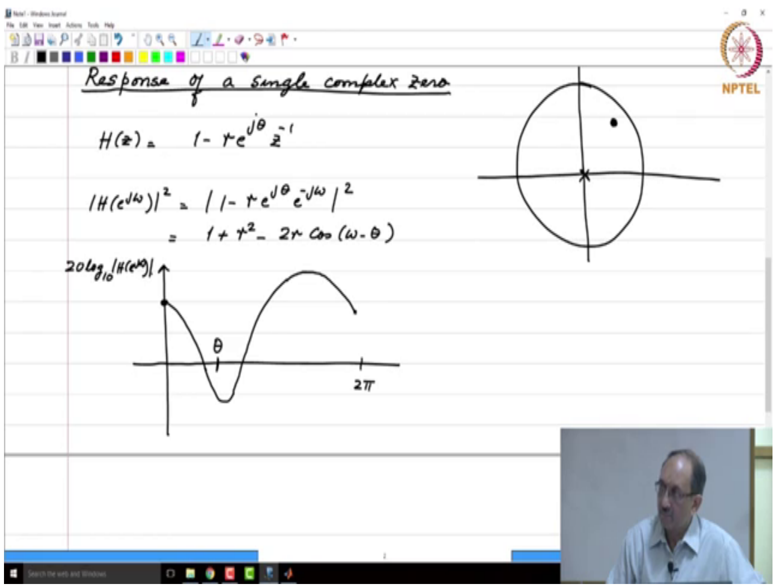

Now, let us look at more detailed of the Response to a single complex zero. So, by response we mean magnitude frequency response. So, we have $H(z) = 1 - re^{j\theta}z^{-1}$ therefore, you have a non-trivial zero at $re^{j\theta}$. And, the magnitude squared frequency response is given by $|1 - re^{j\theta}e^{-j\omega}|^2$ and this you can write it as this times its complex conjugate.

So, this can be written as $1+r^2-2r\cos(\omega-\theta)$, this is easily verified. So, this is the algebraic expression and if you plot the frequency response, note that the zero is fixed. So, r and θ are fixed known quantities, the only variable in this expression is ω and ω goes from 0 to 2π or between$-\pi$ and π. In terms of pole-zero plot, you have a trivial pole and then you have a non-trivial zero, this radius is r, the angle is θ because, the zero is at $re^{j\theta}$.

On a linear scale if you plot this, this will look like what function? Yeah, this will look like a simple cosine, because the only variable here is ω and this will look like a shifted cosine depending on the parameters. But, we will as I had mentioned we will plot all these things on a log scale and if you look at this on a log scale.

So, if you are here on the unit circle, you might have a certain magnitude response and then when you go along, you are going to approach this zero. And, hence the response will fall and when you are exactly at the same angle as the angle of be zero that is when $\omega = \theta$, you will be closest to the zero. So, you can expect the response to have a minimum value at that particular frequency.

And, hence the response will have a minimum value at $\omega = \theta$. When you go past the zero, the response is going to increase and then when you cross the zero, the response is going to increase. And, then when you are exactly at the opposite frequency value compared to the location of the zero that is when ω equals; what value will this be a maximum? $\theta + \pi$, at $\omega = \theta + \pi$, you are farthest away from the zero and hence you can expect the response to be maximum.

And, then when you complete the circuit around the unit circle, you will reach exactly the same value at 2π which corresponds to the value at $\omega = 0$. So, this is the frequency response of a single complex zero that is log magnitude. So, this is typically $20 \log_{10} |H(e^{j\omega})|$.

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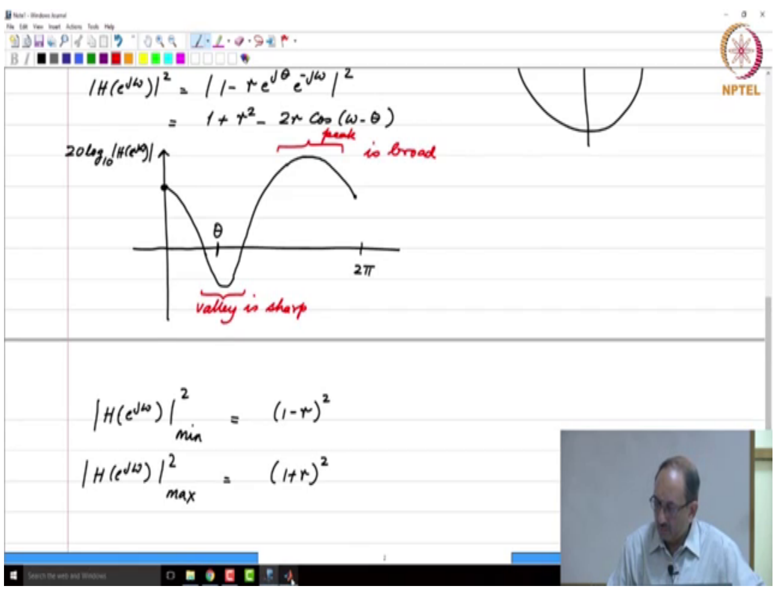

Note that $|H(e^{j\omega})|^2$ minimum; minimum will occur when $\omega = \theta$. Therefore, when $\omega = \theta$, this becomes $1 + r^2 - 2r$. And, hence the minimum value magnitude squared is $(1 - r)^2$. And the magnitude squared has a maximum when $\omega = \theta + \pi$ or when $\omega - \theta = \pi$. When $\omega - \theta = \pi$, this becomes $1 + r^2 + 2r$ therefore, this becomes $(1 + r)^2$.

So, note that on a linear scale, this will be precisely a cosine, but on a log scale, you have this and all our qualitative description of the frequency response will be based on the picture that we see on the log scale. And, you see that on the log scale, this is the valley, this is contributed by the zero. Therefore, this valley is sharp and this is the peak and the peak is broad. So, we make the general statement that zeros close to the unit circle give rise to sharp valleys.

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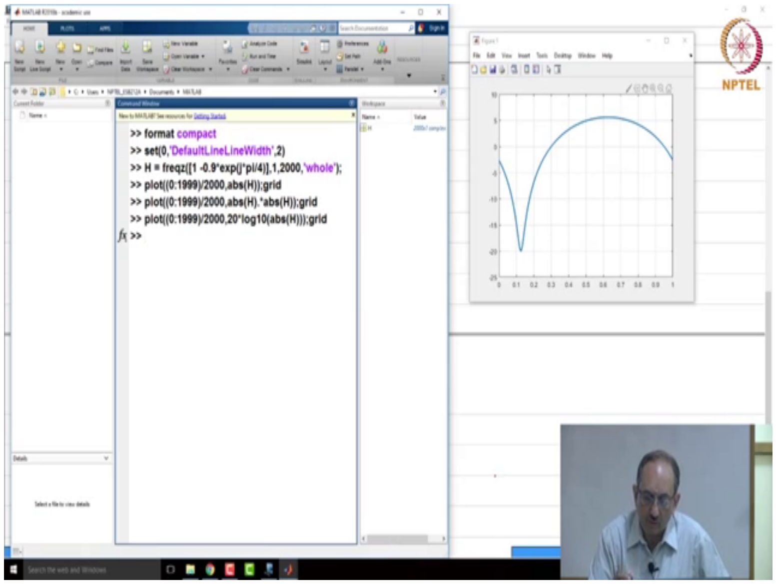

And you can easily see this in MATLAB. So, we are going to use the *freqz* function. So, H is the function that is going to have the complex frequency response. So, this is *freqz*, *freqz* takes the format (b, a, n), b is the numerator coefficient. Let me take the radius to be 0.9. So, $1 - 0.9z^{-1}$ is the numerator polynomial, because I have taken r to be 0.9 and then I can make the angle to be $\pi/4$; so, $j\pi/4$. So, this is the numerator polynomial, denominator polynomial is 1 and then I am going to evaluate this at 2000 points. In general, *freqz* assumes that b, a are real valued.

So, if you do not give any additional argument, it will evaluate the frequency response at 2000 points from 0 to π, because it will assume b and a are real valued whereas, in this case we have complex zero. And, hence we want to evaluate the frequency response from 0 to 2π. So, 0 to 2π, 2000 points are going to be used for evaluation and then I have to say *whole*. So, this is the frequency response of this particular simple minded system and now I need to plot the magnitude response. So, let me plot the magnitude response on a linear scale.

So, if I plot, since I have taken 2000 points, I want to normalize the x-axis to be between 0 to 1. We have evaluated the response from 0 to 2π. If you divide by 2π, the interval becomes 0 to 1. Therefore, I will take 2000 points and then divide by 2000 so, that the interval becomes 0 to 1. And, then I will plot the absolute value of the frequency response. Now, this will look like a cosine; really what I have done is I have plotted the magnitude, what looks like a cosine is the magnitude squared, right. Therefore, this does not look like a cosine because this is the magnitude of the transfer function. What is the cosines the magnitude? Squared. So, there is the magnitude squared which looks like exactly like a cosine.

Now, what we really want is we want the log of magnitude and we plot $20\log_{10}$ of the magnitude response. And, here you see a plot that is similar to what I had sketched roughly based on the geometric interpretation. So, the actual response is this and you can see here that the valley is sharp, the peak is broad. And, we are making these descriptions based on the log magnitude response and the angle is $\pi/4$. So, $\pi/4$ when you divide by 2π is $1/8$ and $1/8 = 0.125$ therefore, you see here that this dip occurs exactly at 0.125 and this peak will occur at $(\pi/4 + 2\pi)/(2\pi)$, all right.

Therefore, the locations of the peak and the valley are at expected places and because the radius is 0.9 and the minimum value is $(1 - 0.9)^2$ which is 0.01 and that is -20 dB. So, all these numbers are also corresponding to what they should be, no surprises here. Therefore, this is the precise plot and this is the rough plot.

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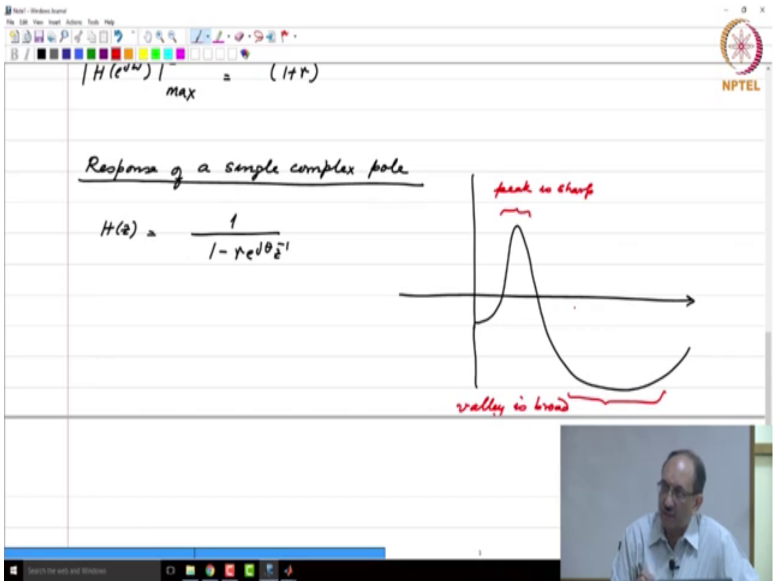

Moment you have the response of a single complex zero, response of a single complex pole is trivial. So, now you have $H(z) = \dfrac{1}{1 - re^{j\theta}z^{-1}}$ and the log magnitude plot of this will be just the negative of this. Once you have the log magnitude of the zero, the log magnitude response of the pole is just the negative of this. And, hence without repeating what was the equations that I have written earlier, let me just plot the qualitative picture.

So, this how the picture will be and what was the valley before has now become the peak. So, now this becomes the peak and is sharp and this valley is broad. And the same for the same parameters, if I plot in MATLAB the response will be just the previous plot, but for a minus sign. So, this gives you a feel for the response of a single complex pole and single complex zero. So, we saw the qualitative plot and you have also seen the exact plot.

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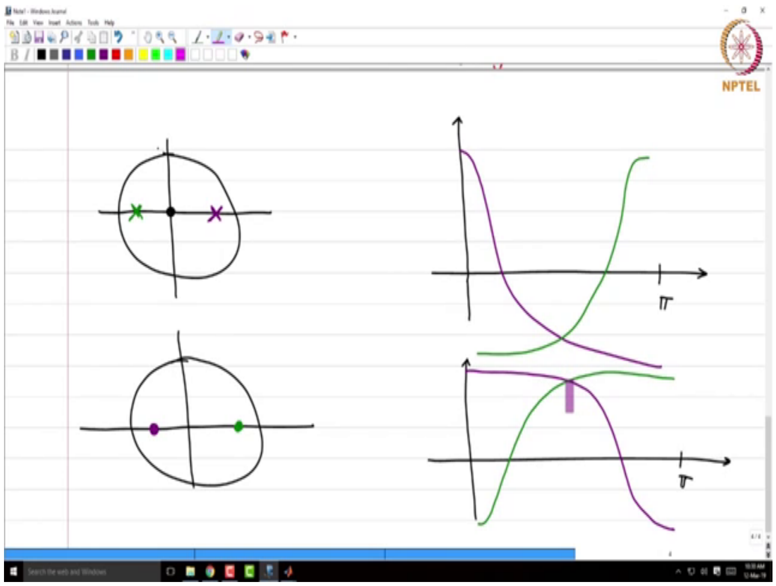

And, carrying this further, we can look at responses of crude low pass and high pass filters and we can realize crude low pass and high pass filters either using a single zero or a single pole. And, then we will see the differences in the responses; let me first draw the qualitative picture. Suppose, I have a pole here; by the way, this goes both positive and negative because we are plotting the log magnitude. So now, if you look at this response; so, this is similar to this, the only difference is here the peak was at θ because the pole is at $re^{j\theta}$.

Now, this corresponds to θ being zero and hence if theta varies; if you look at this expression here. If you look at this expression, if θ varies all that will do is it will shift this frequency response either to the left or to the right because, if you replace ω by $\omega - \theta$, you are going to shift the response. And, hence depending upon the value of θ, you will either shift to the left or to the right.

And, hence, this general picture in which we have this particular value of θ, now all we have to do is shift θ to the left to make $\theta = 0$. Because, that is what this angle corresponds to and therefore, the frequency response of this will look like this. And, this is a crude low pass filter because here the gain is maximum and then when you go along the unit circle and when you hit this point, the gain is minimum and this is a very crude low pass filter.

On the other hand, if you had the pole here, you can expect the gain to be maximum when you are at $\omega = \pi$ and you can expect the gain to be minimum when you are at $\omega = 0$. And, hence if the pole were here, you can expect the frequency response to be like this. Therefore, this is a crude high pass filter, this is a crude low pass filter and both these high pass filters have been realized using a single pole. And of course, you also have a trivial zero which does not play any role in the magnitude response.

Now, on the other hand, if you try to realize a low pass or a high pass filter using a zero; suppose you want to realize a low pass filter, at $\omega = 0$ you want the maximum gain which means you need to place the zero as far away as possible. And, hence to get a low pass filter, when you want to realize this using a zero, you have to place the zero here, but now look how the response changes; the response will be something like this, these are rough sketches.

And, similarly when you have a high pass filter, the response will be something like this. So, this follows from what we had discussed earlier. So, if you have a zero, the valley is sharp, the peak is broad. Therefore, you see the difference between these two cases.

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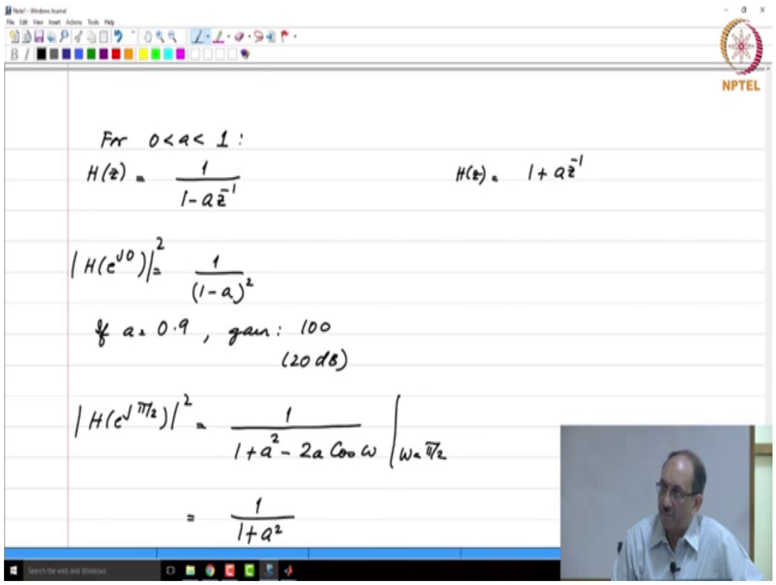

And so, these are qualitative pictures. Now, if you look at some actual numbers, we can think of $H(z)$, let us look at the low pass filter case. If you are going to realize this using a single complex pole, then the transfer function has to be of the form $\dfrac{1}{1-az^{-1}}$ where, I am assuming a is real valued and a is between 0 and 1. Therefore, the pole is at a so, that corresponds to this case here, the pole is at a and I am assuming a is between 0 and 1. So, this is a low pass filter, on the other hand if I have $H(z)$ to be $1+az^{-1}$, the zero is at $-a$ right; again I am assuming a is between 0 and 1. In both cases I have so, this is a crude low pass filter, this is a crude low pass filter and if you evaluate the frequency response, you get some feel for the numbers.

Suppose, you evaluate this at $omega = 0$, and if you evaluate the magnitude squared frequency response at $\omega = 0$. So, this will be $\dfrac{1}{(1-a)^2}$, $\omega = 0$ corresponds to $z = 1$. And, if $a = 0.9$, then maximum gain is 100 and this is 20 dB. Now, let us look at some other comparison point, let us look at $H(e^{j\pi/2})$. We are evaluating the frequency response at $\pi/2$ and remember this is $\dfrac{1}{1+a^2-2a\cos(\omega-\theta)}$, $\theta = 0$.

Because θ is angle of the root, here when $a = 0.9$, $\theta = 0$ therefore, this is what the magnitude squared is. And, when you evaluate this at $\omega = \pi/2$, you get $\dfrac{1}{a+a^2}$.

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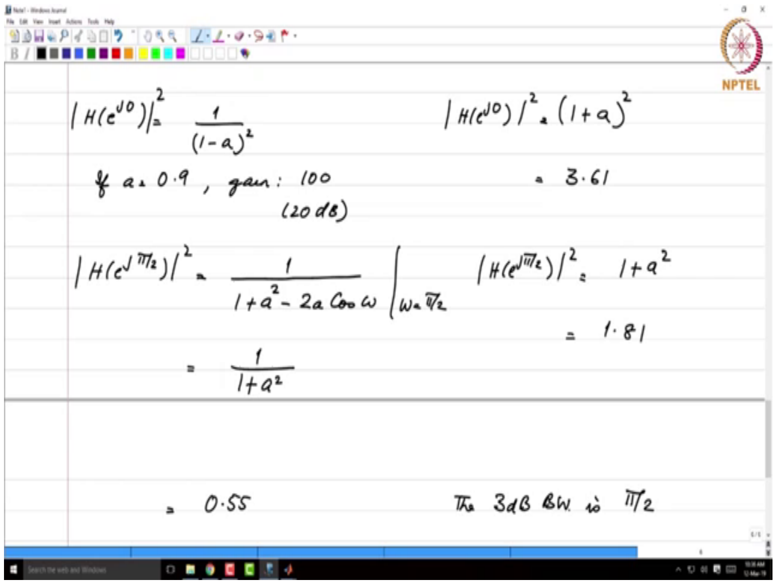

And, for $a = 0.9$, this turns out to be $\frac{1}{1+0.9^2}$; so, this is $1/1.81$ and this roughly is so, it is roughly 0.55. Now, let us look at the case here, if you look at $|H(e^{j0})|^2$, this is $(1+a)^2$ and for 0.9, this becomes 1.9^2 which is $|H(e^{j0})|^2 = 3.61$. And, $|H(e^{j\pi/2})|^2$ is $1+a^2$ as before and this turns out to be 1.81 because $a = 0.9$.

Now, if you look at these numbers for the low pass filter realize using the zero, the maximum gain is 3.6. At $\pi/2$, the gain has the magnitude squared has dropped to half the peak value. Therefore, the 3 dB bandwidth is $\pi/2$ because at $\pi/2$, the magnitude squared has dropped by half; therefore, you can infer that the 3 dB bandwidth is $\pi/2$.

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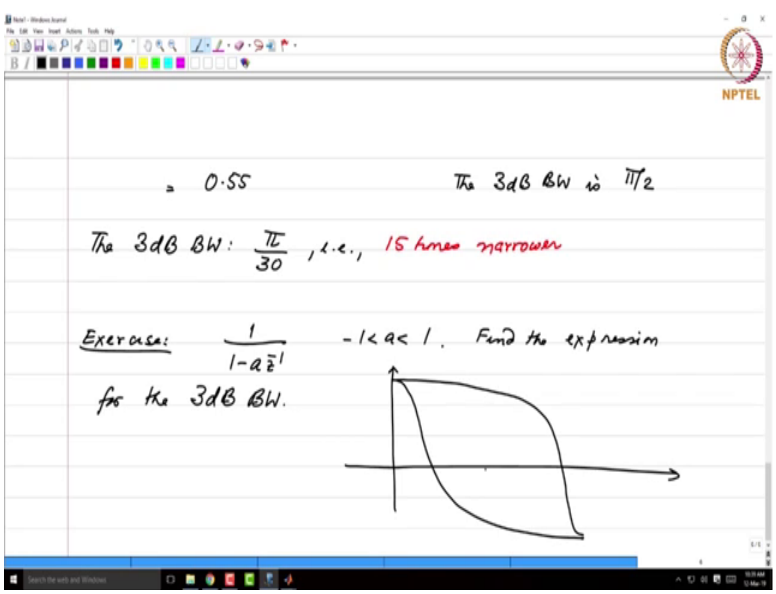

And, for the low pass filter realized using the single pole, you will find that the 3 dB bandwidth is $\pi/30$.

So, this is something you should verify later; in this case, the 3 dB bandwidth is $\pi/30$. So, this is part of this exercise you need to carry out; so, $\frac{1}{1-az^{-1}}$. So, find the expression for 3 dB bandwidth.

So, this is $H(z)$, you need to calculate $|H(e^{j\omega})|^2$ and $|H(e^{j\omega})|^2$ is an expression like this. In general, it will be $\frac{1}{1+a^2-2a\cos(\omega-\theta)}$ and then you need to find at which point the gain drops by half. Find out what the maximum gain is and relative to the maximum gain at what frequency does this drop by half.

So, you will be able to get a closed form expression and you will be able to verify that for the low pass filter case using a pole, the 3 dB bandwidth is $\pi/30$ whereas, for the same low pass filter using the zero, the 3 dB bandwidth is $\pi/2$. So, this is 15 times narrower and this is also along expected lines because, for the single complex pole, the rough frequency response is like this. Whereas, a low pass filter generated using a single complex zero, the response will be like this and clearly you can see the difference in the bandwidth.

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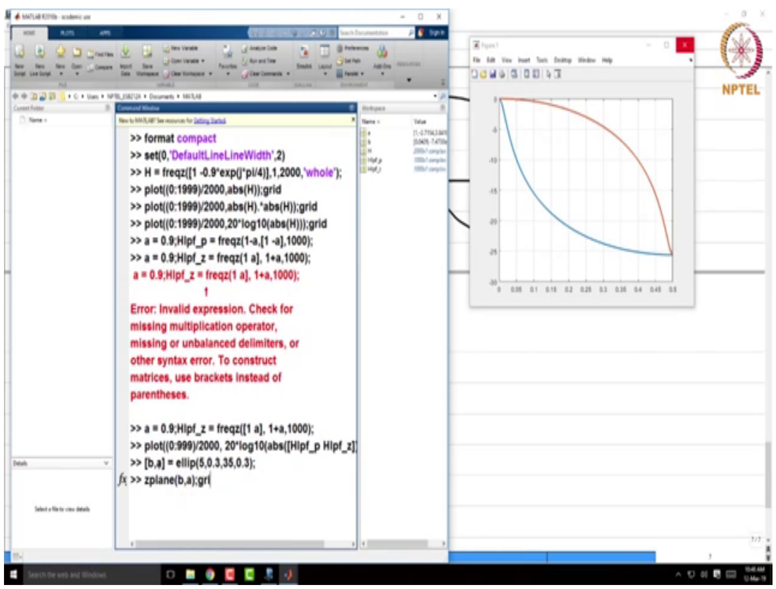

So, this example shows you that in general poles tend to give you sharp peaks and that is why when you want to shape the frequency response, if you have poles also to play with that is if you are considering systems that have both poles and 0s; then you can get very sharp transition from pass band to stop band using poles. Whereas, if you have only zeros to work with as in the FIR case, if you want similar frequency response characteristics, what you will have to do is you will have to increase the order.

So, although we have not discussed a filter design in any detail at all, just at an intuitive level, you can see that if we had both poles and zeros to play with, to get the same or desired magnitude response if you also had poles, you will end up with systems that have lower order. That is $\frac{B(z)}{A(z)}$, you will have polynomials of lower order. On the other hand if you want to have the same frequency response or roughly the same frequency response and if you had only zeros to play with that is you are working with FIR filters, then you will need much higher order to get the same rough frequency response.

And, this example kind of illustrates, gives you some feel for why this might be the case. So, we are making some generally intuitive level statement observations based on this particular example and you can do this in MATLAB, you can plot this exactly in MATLAB maybe it is worth showing this. So, I have H_{lpf} and this is realized using pole. So, this is *freaqz*, I have (b, a, n) and the transfer function is $\frac{1}{1-az^{-1}}$, let me make the numerator as $1-a$. If I make the numerator as $1-a$, then I am normalizing the gain. Because, if I have $\frac{1-a}{1-az^{-1}}$, then if I put $z = 1$, I will get $\frac{1-a}{1-a}$ and the peak gain will be 1.

Therefore so, let me make $a = 0.9$ and then numerator is $(1 - a)$, it is just a constant. So, that the peak gain is 1, denominator is $(1 - az^{-1})$, then I am going to evaluate this at a 1000 points; so, this is my low pass filter. Now, I am going to get the low pass filter, but now using zero; the numerator is now $(1 + az^{-1})$, $a = 0.9$ and the denominator because I want to normalize the gain, the denominator is just a constant and that constant happens to be $(1 + a)$.

So, I have low pass filter realized using both pole and zero, now let me plot both. So, I will plot 0 to 999, I have 1000 points and now I will divide by 2000, because I am evaluating this from 0 to π. If I normalize it by 2π, the frequency axis will be 0.5 and then let me plot $20\log_{10}$ absolute value, these are complex numbers. Therefore, this is H_{lpf} using pole and H_{lpf} using zero.

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So now, you see exactly the picture similar to what I had drawn by freehand. So, this blue curve is the low pass filter using single pole, the radius was 0.9 and the red curve is the high pass filter. And, this 3 dB bandwidth is clearly much narrower than this whereas, here the 3 dB bandwidth is $\pi/2$ roughly and you can also see that here. So, $\pi/2$ corresponds to 0.25 and you see the y-value is -3.012. So, it is 3 dB below the peak value. So, in general, poles provide a sharp peaks and hence you can expect the frequency response to be realized with the much lower order filter if you had both poles and zeros to play with as opposed to playing with only just zeros.

To illustrate this point, let me design a low pass filter, this is what is called as an elliptic filter. You will not know the details of this unless you take a course on digital filter design; I will just design the filter and then show the characteristics, the pole zero plot and the frequency response. So, this is 5^{th} order elliptic filter, never mind what these parameters are and then if you plot the pole-zero plot, this is how it looks.

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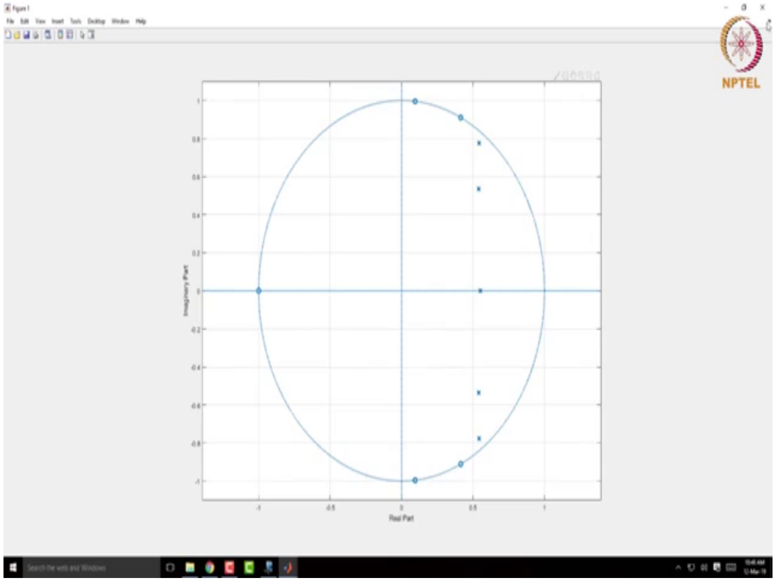

So, this is a 5^{th} order elliptic filter, these are all the poles and these are all the zeros.

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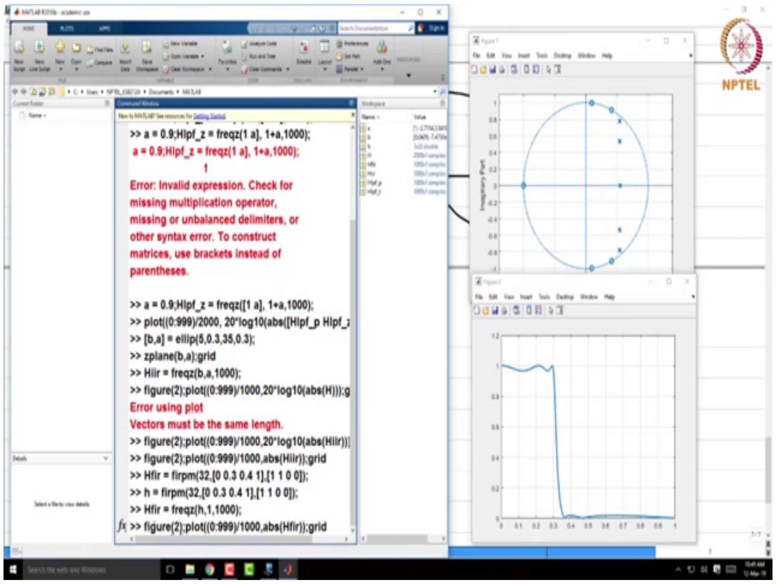

And, now let me plot the frequency response of this. So, this is clearly an IIR filter because there are un-cancel non-trivial poles therefore, this frequency response is this. So this is the frequency response and here you do not see the ripples on this scale.

So, you should see a bump due to this pole, another bump due to this and another bump due to this. Actually, you will be able to see the bumps if you plot this on a linear scale; now you are able to see the bumps in the pass band. Now, let me plot another low pass filter. So, the numerator polynomial is h itself, 1 and 1000.

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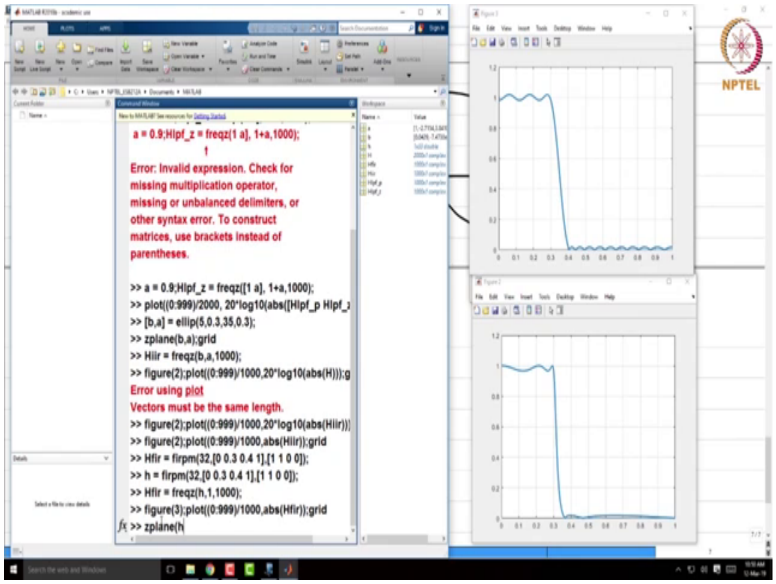

And, if I now plot this, you can see that these two are very similar, they have roughly the same pass band, transition band and stop band, but the point is this is a 32 order FIR filter. So, this is a FIR filter, this order is 32 whereas, this is just 5^{th} order. So, 5^{th} order, you are able to get the frequency response that has this pass band, this transition band and this stop band; whereas, you need a much higher order when it comes to the FIR case.

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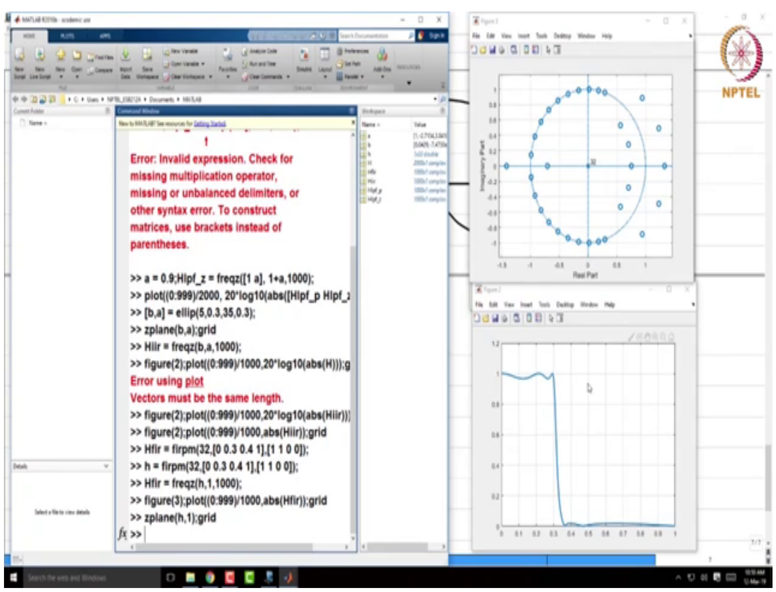

And, just one last plot, these are the zeros of this FIR filter. So, here the off unit circle zeros govern the pass band behavior, these zeros on the unit circle govern stop band.