Digital Signal Processing
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Lecture 47:
Causality & Stability, Response to Suddenly Applied Inputs, Frequency Response (1)
- Response to suddenly applied inputs

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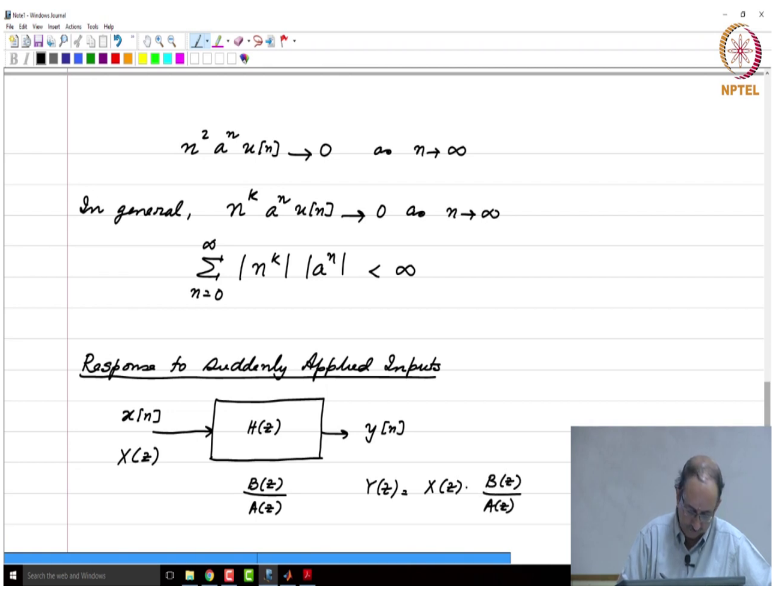

Now, let us look at Response to suddenly applied inputs. So, we have an input $x[n]$, it has Z-transform $X(z)$ is supplied to an LTI system with transfer function $H(z)$ and this $H(z)$ belongs to the rational class of transfer functions. Therefore, it can be represented as $\frac{B(z)}{A(z)}$ and hence, $y[n]$ in the time domain is the convolution of the input and the impulse response. In the transform domain $Y(z) = X(z)\frac{B(z)}{A(z)}$.

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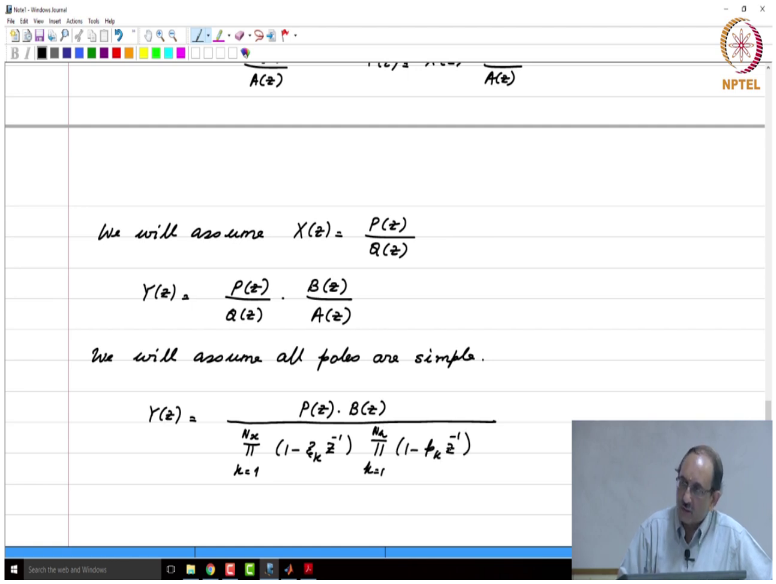

We will further assume that $X(z)$ is also a rational Z-transform therefore, $X(z)$ is of the form $\frac{P(z)}{Q(z)}$ and hence $Y(z)$ is the product of these two. So, this is $\frac{P(z)}{Q(z)} \frac{B(z)}{A(z)}$. Now, to make the discussion simple, what we will do is we will assume all the poles are simple. All poles are simple therefore, we are interested in finding the time domain response and because this belongs to the rational class, the natural thing to do would be to do partial fraction expansion.

Therefore, this is $P(z).B(z)$ and the first thing you need to do for partial fraction expansion is factor the denominator polynomial. So, this is k going from 1 to N_x. So, these are the number of poles contributed by $Q(z)$. So, this is $(1 - \xi_k z^{-1})$. So, this is the polynomial $Q(z)$ factored into its roots and we have the other part. So, this is k going from 1 to N_h, $(1 - p_k z^{-1})$.

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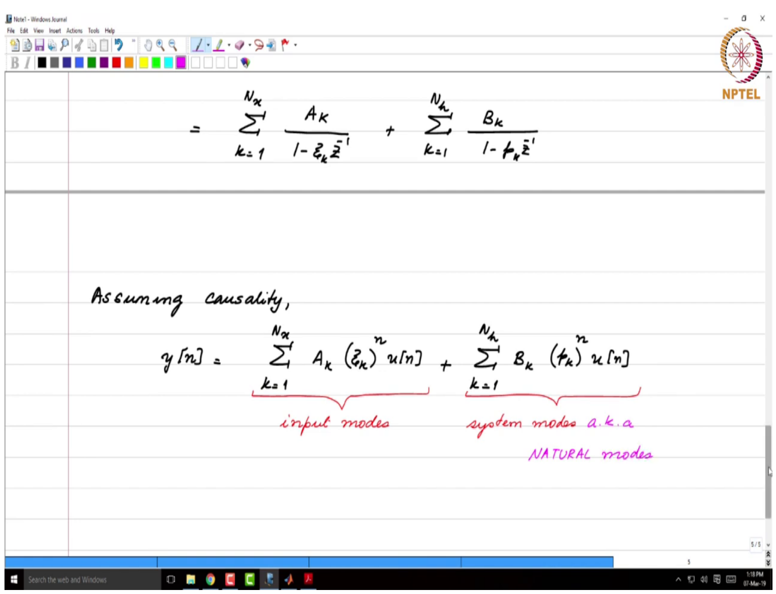

And, now we will find the residues for each of these factors. So now, let me again write the rather the partial fraction expansion in this form. So, $\displaystyle\sum_{k=1}^{N_x} \frac{A_k}{1-\xi_k z^{-1}} + \sum_{k=1}^{N_h} \frac{B_k}{1-p_k z^{-1}}$. And, we will assume causality and hence the time domain response becomes like this. So, $\displaystyle\sum_{k=1}^{N_x} A_k(\xi_k)^n u[n]$.

So, this is the inverse transform of $\displaystyle\sum_{k=1}^{N_x} \frac{A_k}{1-\xi_k z^{-1}}$, plus $\displaystyle\sum_{k=1}^{N_h} B_k(p_k)^n u[n]$. These are the two terms or two sets of terms that are present in the output. Now, if you notice, these are coming from the input, these terms are contributed by the Z-transform of the input. Therefore, these are called as the input modes. These set of exponentials are contributed by the system, so these are called the system modes and they are also known as natural modes.

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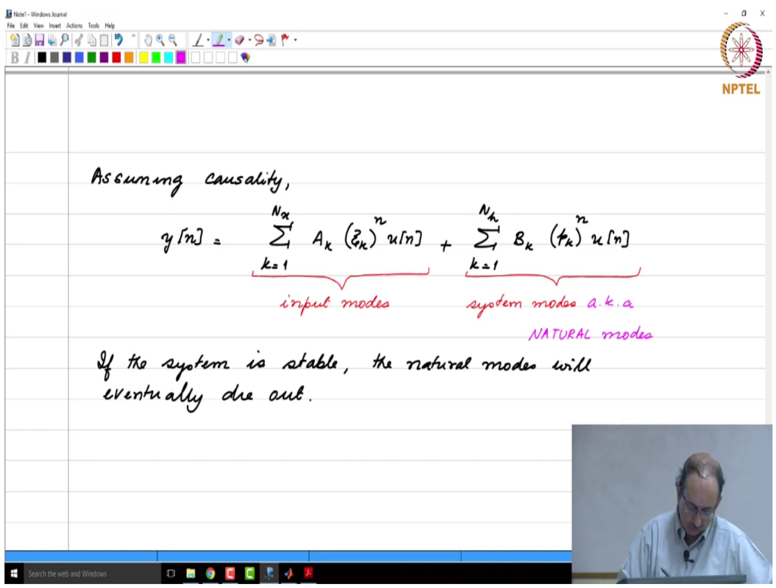

Now, if the system is stable, clearly we have assumed causality because of the form of the inverse Z-transform that we have written; then all poles will lie strictly inside the unit circle. Therefore, the natural modes will eventually die out; natural modes will eventually die out because the system is stable. Of course, if all the ξ_k are also magnitude strictly less than 1, they also eventually will die out. But, we can be guaranteed that the system mode or the natural mode, they all will die out because the system is causal and stable.

So, what this tells you is if you have a system and to that you apply an input, the output will contain two parts: one part will be the natural modes of the system which you eventually will die out for causal and stable systems and the other part corresponds to the input modes. So, suddenly excited system will always contain two parts: one part will correspond to the input modes, the other part will correspond to the natural modes of the system. Now, let us take a simple example to illustrate this.

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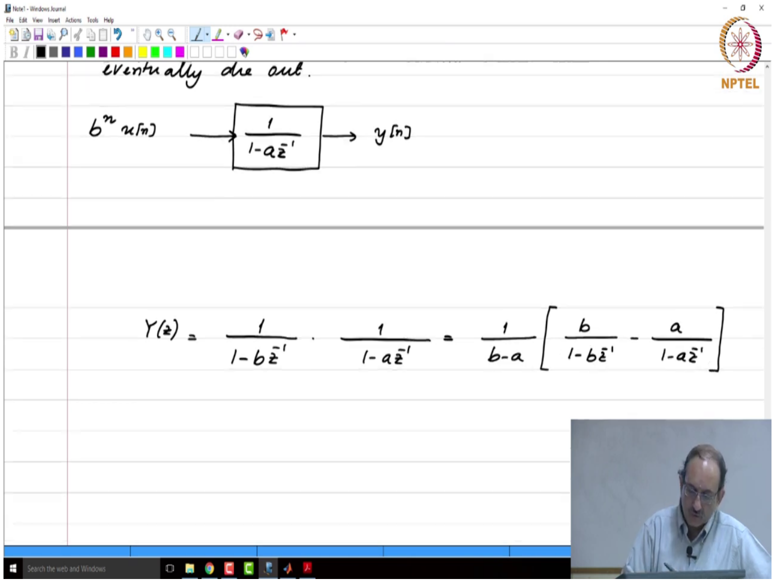

So, we have $\dfrac{1}{1-az^{-1}}$ as the transfer function of the system and then to this, we apply $b^n u[n]$. So, now let us look at what the output is, $Y(z)$ is $X(z)H(z)$, $X(z)$ is $\dfrac{1}{1-bz^{-1}}$ times $H(z)$ is $\dfrac{1}{1-az^{-1}}$. And, when you do partial fraction expansion, you get $\dfrac{1}{b-a}\left[\dfrac{b}{1-bz^{-1}} - \dfrac{a}{1-az^{-1}}\right]$; very easy to see that when you simplify this, you get back $Y(z)$.

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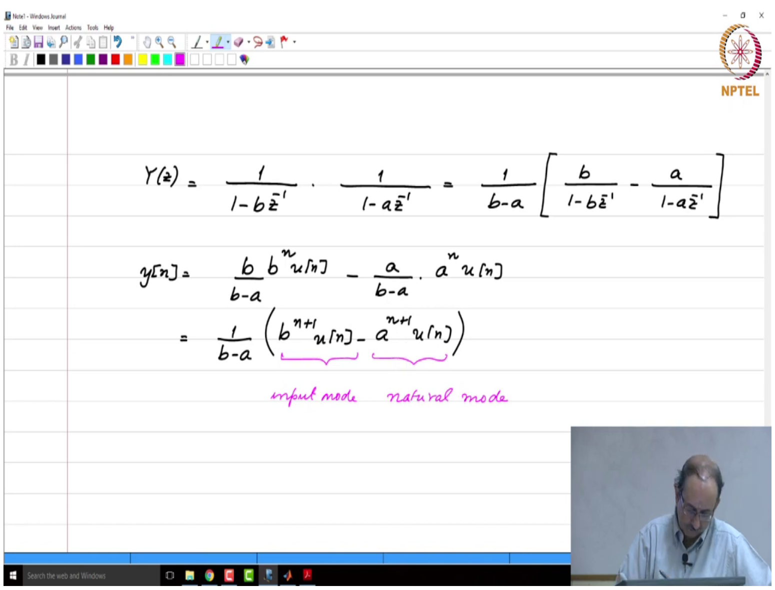

And, if you, by the way, before we get into the expression of $y[n]$; what is it that we have assumed here? Has any assumption be made? Yeah, very good, $b \neq a$. Therefore, this is b times, this after all is $\dfrac{b}{b-a} b^n u[n] - \dfrac{a}{b-a} a^n u[n]$ and this is easily seen to be $\dfrac{1}{b-a}\Big(b^{n+1}u[n] - a^{n+1}u[n]\Big)$. So, this is the output and now you are able to recognize the input mode and the natural mode.

So, this is the input mode and this is the system mode or the natural mode. Therefore, when the system whose transfer function is $\frac{1}{1-az^{-1}}$ is certainly excited by $b^n u[n]$, the output also will contain a component that corresponds to $b^n u[n]$; it will have a scale factor. In this case, the scale factor happens to be $\frac{b}{b-a}$, but you will also have the system mode and that is given by $\frac{a}{b-a}a^n u[n]$. And, this is the system mode which eventually will die out because, we have assumed the system to be casual and stable.

And, if to this particular system, instead of $b^n u[n]$, if you had applied say $\cos(\omega_0 n)u[n]$; you can go through the exercise similar to what we have done here. Again, you will be able to write the transform of $y[n]$ in the form $X(z)H(z)$ and if you did partial fraction expansion, when the input now is $\cos(\omega_0 n)u[n]$; you will again have two parts: one part will involve $\frac{1}{1-az^{-1}}$ which in turn will give rise to a component of the form $a^n u[n]$.

The other part will have a form that is $\cos(\omega_0 n)u[n]$, but what will be different is it will have an amplitude that is different from unity. If the input amplitude is 1, output amplitude will be whatever the residue is and there will also be a phase shift and there will also be this transient response. Therefore, when you apply $\cos(\omega_0 n)u[n]$ to this system, output will have a component that corresponds to $\cos(\omega_0 n)u[n]$. In addition, there will be a term that is similar to this and this is the transient response, all right.

So, you can break the output as steady state response and transient response. And, to get a better understanding, that comes from this kind of classifying the partial fraction expansion of $Y(z)$.

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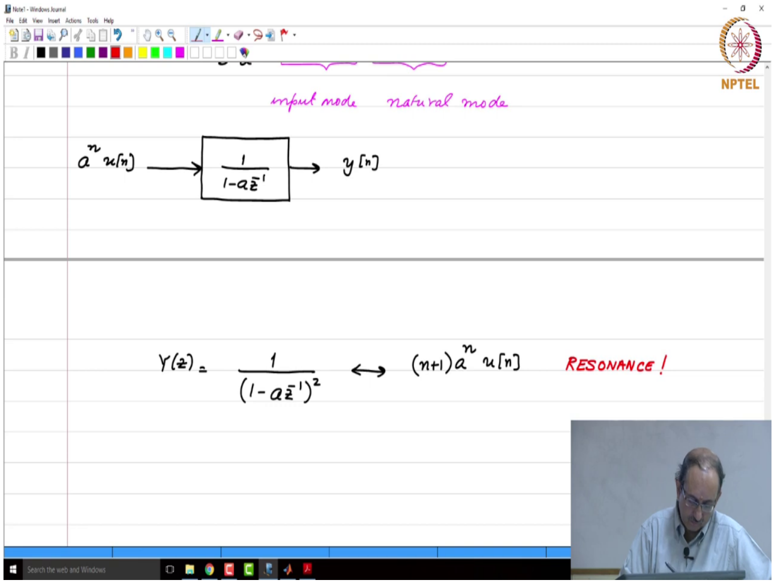

Suppose, to the same system, you now apply $a^n u[n]$ and you can easily see that $Y(z)$ is $\frac{1}{(1-az^{-1})^2}$. And, this in turn will give rise to an output of the form $(n+1)a^n u[n]$ which is just the inverse Z-transform of $\frac{1}{(1-az^{-1})^2}$. Now, if you look at this apart from recognizing the algebraic expression for what it is, does anything else strike you? Ok. Magnitude all right, that is a good observation.

Student: (Refer Time: 15:59).

Alright ok, I will come to that point in a minute; I mean you are all saying some aspects of the phenomenon that this represents. So, the input mode is $a^n u[n]$, the system mode also is $a^n u[n]$. What will happen when you excite a system at its natural modes? I thought, I heard someone say the right answer. So, this is indeed resonance. So, that is why the response initially grows, but then eventually exponential decays stronger than polynomial growth therefore, the response dies down. So, this is indeed exactly the phenomenon of resonance, which is what we expect whenever you excite a system at its natural modes.

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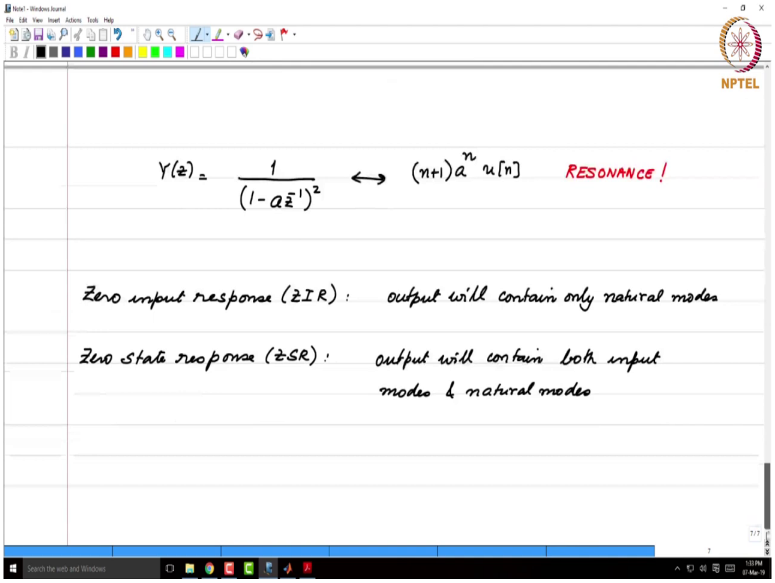

And this is a good place to point out some of the things that you had encountered before in signals and systems. I am assuming you must have been told about zero input response and zero state response, right. So, let us revisit that in this context, even in the Laplace case exactly the same arguments hold. So, if those arguments are being pointed out, what we are going to point out now is just exactly the same kind of arguments that were made then. So, zero input response so, this is typically abbreviated as ZIR.

So, what is happening here is the input is 0 and system may or may not have initial conditions. If the system has some initial conditions present, then the zero input response; what do you expect in the output what modes do you expect in the output? The output will contain only natural modes so therefore, output will contain only natural modes. So, the continuous-time case, analogy is if you had an RLC network and if there were initial capacitor voltages and inter inductor currents and if the input is made 0, all these things will decay and the output will contain only the system modes.

Now, let us consider the zero state response so, this is abbreviated as ZSR. So, all the initial conditions have been zeroed out and then you are applying an input and output will contain what? Both input modes as well as natural modes; so, will contain both input modes and natural modes.

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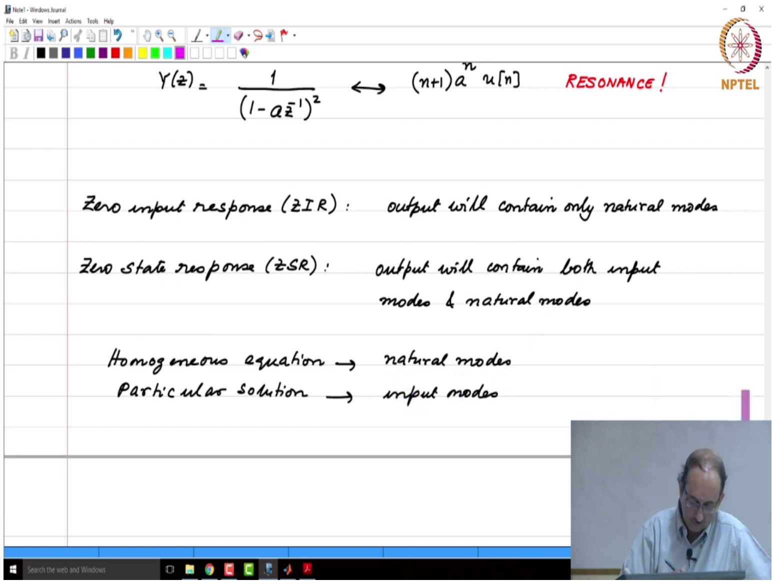

Now, when you are trying to solve differential equations in continuous-time case, the standard approach is to consider the homogenous equation, get a solution and then you will evaluate the complimentary function. So, same approach also holds for linear constant coefficient difference equations. You solve for the homogenous equation and get one solution and then you will solve for the particular solution. So, LCCDE whether D stands for differential or difference, the general approach is same. Homogeneous equation will give rise to one solution and then the particular integral in the differential equation case and particular solution in the difference equation case.

And, the homogenous equation what kind of modes do you expect? This will contain only the natural modes because, what you are going to do there is, you are going to set all the input to 0 and you will solve the homogeneous equation. Homogeneous equation will not contain any input terms and for an n^{th} order equation, there will be in general n unknown constants. And, to pin these constants down, you need auxiliary conditions which are called as initial conditions, if they are specified at the origin.

And, once you have this, then you can get the final solution as far as the homogeneous equation is concerned. In the particular solution, particular integral, there are no unknown constants in the solution and you are told that the form of the particular solution has the same form as the input. Therefore, in terms of modes, what do you expect, which modes will be present in a particular solution? Only the input modes will be present and together they form the complete solution. Now, this is the mathematicians way of solving this, this is the electrical engineering way of solving the problem.

And, if you look at this, in the zero state response, you will find input modes as well as natural modes. The part that contains the input mode will exactly be the particular solution. And the natural modes which occur both in these zero state response as well as the zero input response; if you add them up, together they will give rise to the natural modes present in the homogeneous solution, that is all. So, the natural modes, the way we look at the output given the system viewpoint, they get split into zero input response as well as zero state response.

And, the zero state response, the output containing the input modes will correspond to the particular solution. Now, let us quickly look at some typical time domain response. Yes, is there a question? Yeah, and if a mode x exists both in zero input as well as zero state, it necessarily has to be natural modes. Only these are the modes that are common to both zero input and zero state, because they arise the way the system is interconnected, which will manifest itself when you zero out the input and let the initial conditions decays out or when you supply a certain input to a system, the natural modes will manifest themselves. So, only natural modes are common between these two. Does that answer the question?

So now, let us look at typical time domain response to typical pole locations. So, this is just a recap in pictorial form of what you have already learnt.

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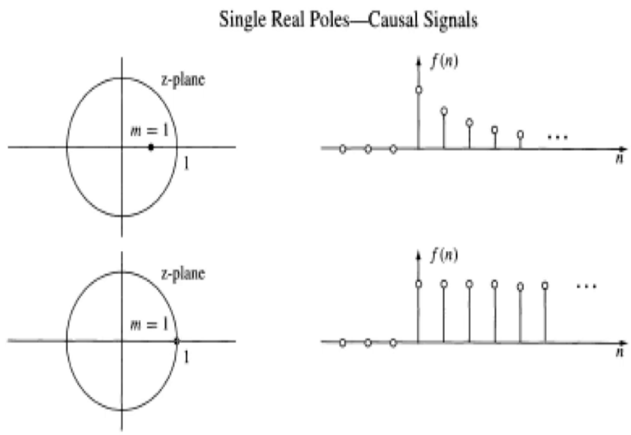

I think this is from Transforms and Applications handbook by Poularikas. So, this is a simple pole here; so, m represents the order. So, if your pole is here on the real axis, it will have the form $a^n u[n]$ and because a is less than 1 it is actually between 0 and 1. It decays down as shown here exponential decay. As the pole moves closer to the unit circle, the decay rate will become less and less and when the pole is on the unit circle there is no decay at all. So, this is $\dfrac{1}{1-z^{-1}}$ in terms of transfer function. So, this is $u[n]$ and now if the pole moves outside the unit circle, it is still of the form $a^n u[n]$, only that it is a growing exponential because $a > 1$.

Say for example, if $a = 2$, this will be $2^n u[n]$ whereas, this could be $(1/2)^n u[n]$. And, if the pole were here so, the pole is between 0 and -1. So, this could correspond to an example like $(-1/2)^n u[n]$. So, $(-1/2)^n u[n]$ is $(1/2)^n (-1)^n u[n]$. So, you have the exponential decay, but terms alternate in sign.

And if the pole is here, it is at $z = -1$ so, this is $(-1)^n u[n]$. So, $\dfrac{1}{1+z^{-1}}$, this is the inverse Z-transform. And continuing the same thing, if now the pole were outside the unit circle but on this side of the axis, you will have a growing exponential envelope. But, in addition to that, you also have the sign alternating between terms. So, this is as far as simple real pole.

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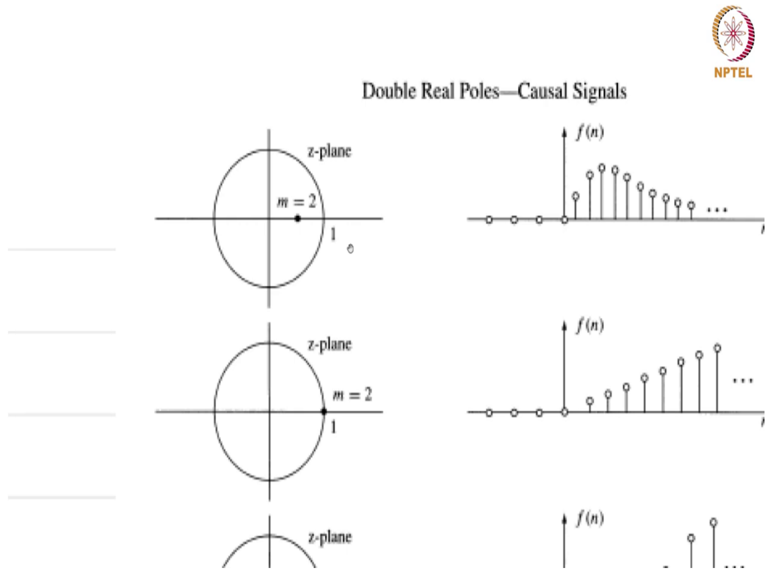

And, now, we look at double poles. So, instead of first order pole if we had a second-order pole, the output time domain expression has the form na^n. The exact expression is $(n+1)a^n u[n]$, $na^n u[n]$ initially there will be growth because, for small values the polynomial growth will win over exponential decay. But, eventually, exponential decay will be the dominating factor. Now, if you had a double pole, the output will have the form $nu[n]$ or $(n+1)u[n]$. So, which is what is shown here, actually this corresponds to $u[n]$ because at $n = 0$, you have output to be 0.

So, it is a ramp, if were if it were outside it will be $na^n u[n]$. But now magnitude of a is greater than 1 therefore, you have this growth and exactly the same kind of thing when the pole is on the negative real axis. So, the envelop is still of the form $na^n u[n]$, but you have terms alternating in sign. If you have a double order pole at $z = -1$, it is a ramp, but with alternating sign. And, this is for on the real axis outside the unit circle, the main point is the envelope is similar, but signs alternate.

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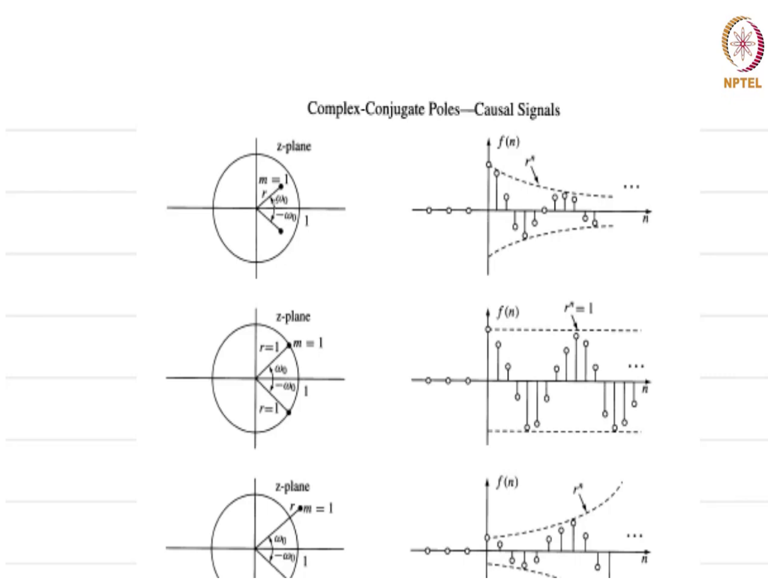

Now, the complex conjugate case; so, if you had two complex conjugate poles, radius r, angle ω_0 and $-\omega_0$, then this is of the form $r^n \cos(\omega_0 n + \theta)$. So, the envelop will decay as r^n and the alternation inside is governed by ω_0. And, if $r = 1$, you will get a pure sinusoid because there is no exponential decay. If r were greater than 1, the exponential growth is signified by r^n, where r is greater than 1.

And, if you had second order roots on the unit circle, the envelope is still linear ramp, but you also have oscillations. So, these kind of, give some feel for time domain behavior versus z-plane pole locations. So, there is nothing new, it is just a pictorial representation of all the formulas that we had seen earlier.