Digital Signal Processing

Prof. C.S. Ramalingam
Department Electrical Engineering
Indian Institute of Technology, Madras

Lecture 43:
Inverse Z-transform (3), Inverse DTFT
- Inverse DTFT

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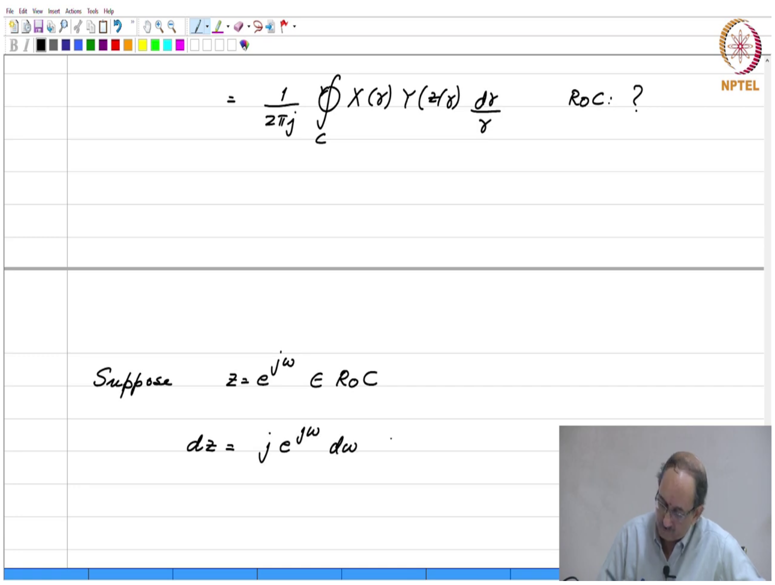

Now, let us further make use of the inversion integral and derive something else that is related. Suppose, the unit circle is part of the region of convergence. So, this means that $z = e^{j\omega}$ must belong to the RoC and hence we will make use of this fact in the inversion integral. Therefore, $dz = je^{j\omega}d\omega$.

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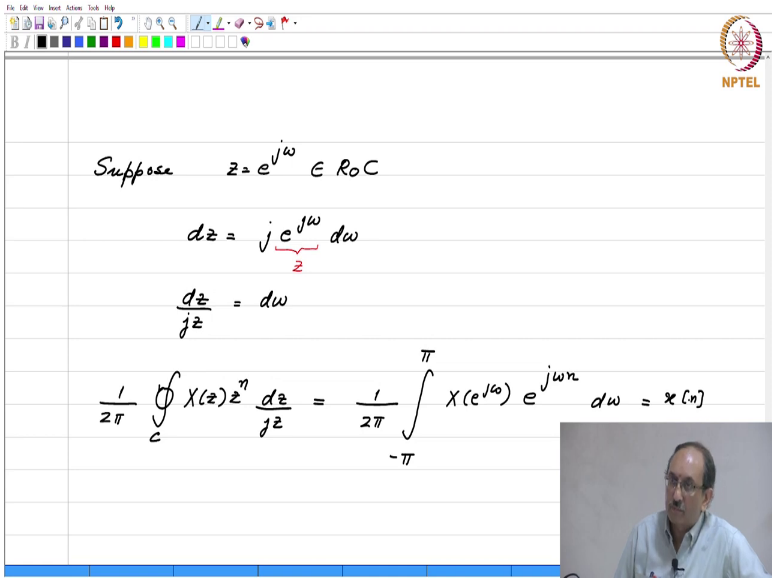

And, this after all is z and hence $\frac{dz}{jz} = d\omega$. Now, let us look at the inversion integral. Since, the unit circle is part of the region of convergence and we need to take a closed contour in the region of convergence, let us take this closed contour to be the unit circle itself. Let us take the unit circle itself to be the specific closed counter. Therefore, $\frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$.

Now, the closed contour C is the unit circle and hence this becomes $\frac{1}{2\pi}$. Remember, we are after all evaluating this along the unit circle therefore, this now becomes $X(e^{j\omega})$. z^n can be replaced by $e^{j\omega}$ and now you can actually write this as $\frac{dz}{jz}$; $\frac{dz}{jz}$ is nothing but $d\omega$. Therefore, this becomes $d\omega$.

Therefore, the contour integral really becomes an integral over ω and integral over ω if you want a closed path, the ω variable has to take the range either 0 to 2π or between $-\pi$ and π. Therefore, let us assume that this is between $-\pi$ and π. Therefore, this should give you, this after all is the inversion integral therefore, this should give you back $x[n]$.

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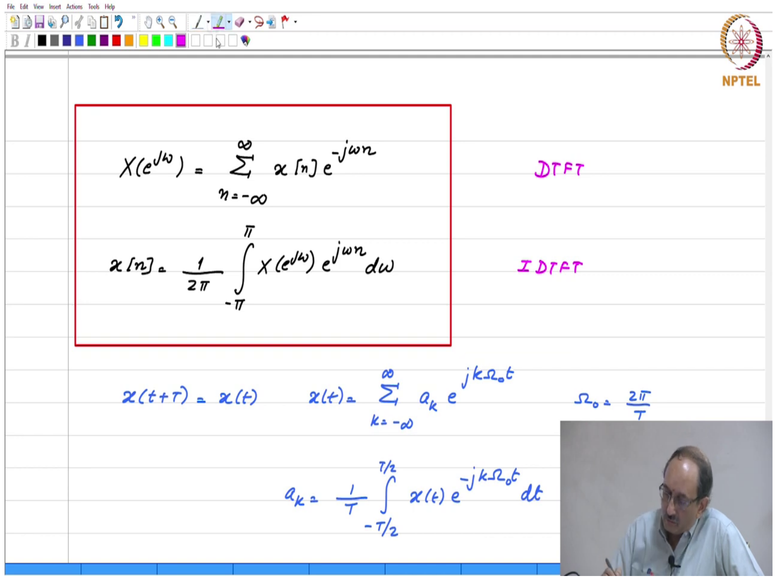

So, now recall that the DTFT was defined like this $X(e^{j\omega}) = \displaystyle\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$ and this was called as the Discrete Time Fourier Transform.

Now, we have just now seen $x[n] = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$ and this is nothing but the inverse DTFT. So, these are very important relationships. So, this is the DTFT formula and this is the inverse DTFT formula. So, in terms of Fourier analysis what you have seen in the previous course, you had seen the Continuous Time Fourier Series (CTFS) and then you saw the continuous time Fourier transform.

Now, we have the discrete time Fourier transform. This seems to be the third Fourier analysis tool that we have learnt so far and this has interpretations similar to the interpretation of the spectrum for the continuous-time case. Given a signal in continuous-time, if you want to know its frequency content, what you will do is you take the continuous time Fourier transform.

Similarly, given a sequence in discrete-time, to know it is frequency content, you will take the discrete time Fourier transform and you will get a spectrum which is a complex function of a real variable exactly similar to the continuous time Fourier transform being a complex function of a real variable. The only difference is there the real variable went from $-\infty$ to ∞, whereas here the real variable goes from 0 to 2π or $-\pi$ and π.

Given the spectrum, you can get back your time domain sequence and you can either plot this as one 3D plot or you can plot this as magnitude versus frequency and phase versus frequency, two separate 2D plots. So, exactly the same as what was happening in the continuous time Fourier transform case. There also you could have plotted the CTFT as one 3D plot, typically you will plot it as two 2D plots; the magnitude versus frequency, phase versus frequency.

If you look at this, these set of equations that we call this as the discrete time Fourier transform and this is the inverse discrete time Fourier transform and we have labelled this as our third Fourier analysis tool right, but have you seen this before? Very good, somebody said Fourier series that is exactly the right answer. So, what we call as the DTFT and IDTFT, I mean this is not really something new; if you look at the discrete time Fourier transform, the independent variable is continuous and it is a periodic function. If you have a function whose independent variable is continuous and it is periodic, then it can be expanded as a Fourier series expansion.

Therefore, what you have been calling as the discrete time Fourier transform is really this 2π periodic function being expanded in terms of Fourier series, where the Fourier series coefficients are given by $x[n]$. Just to see the connection, if $x(t+T) = x(t)$, then $x(t)$ can be expanded in terms of Fourier series, $\displaystyle\sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}$, where Ω_0 is nothing but $\dfrac{2\pi}{T}$ and the Fourier series coefficients $a_k = \dfrac{1}{T}\displaystyle\int_{-T/2}^{T/2} x(t)e^{-jk\Omega_o t}dt$. These are exactly the CTFS equations and this is the expansion and this is the coefficient.

Now, let us look at what is happening in the DTFT case. So, these are the coefficients and remember if you have a function that is periodic with period T in the continuous-time case, then the Fourier series coefficients in the frequency domain, they are spaced how much apart?

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No, the spacing is?

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And, omega naught is $2\pi/T$. So, if your function is periodic with period T in the time domain, then the spacing of the coefficients in the other domain is $2\pi/T$, right. Now, if here in the DTFT case, you have a periodic function, the periodicity is 2π. Therefore, in the other domain, the spacing has to be $2\pi/2\pi$, the spacing has to be 1 which is exactly what the time domain sequence is spaced apart; it is spaced 1 apart. Therefore, $x[n]$ is really spaced 1 apart consistent with the fact that in the other domain you have periodicity of period 2π.

The only minor difference compared to the Fourier series the way that you are used to versus what is appearing here. So, here $x[n]$ is playing the role of a_k. So, and Ω_0 is actually 1, all right. So, this is of the form $x[n]e^{j(\ )}$, Ω_0 is 1, all right. So, the variable t is playing the role of variable ω here and Ω_0 is 1. So, the only minor difference is the way you are used to seeing the Fourier series is you have $a_k e^{jk\Omega_0 t}$ whereas, in this Fourier series expansion we have $e^{-j\omega n}$. So, this is a very minor difference.

If you had you to the $-j\omega n$ here, you will have $j\omega n$ here. Similarly, in the Fourier series expansion, you can also have the definition as $a_k e^{-jk\Omega_0 t}$ in which case in the Fourier series integral a_k for the coefficients; instead of $-j$ here you will have $+j$. So, here plus here, you will have a minus whereas, if a minus here, you will have a plus and both these definitions are valid, it is just a minor modification. Therefore, if you are used to the Fourier series definition having minus here and plus here, this is exactly the Fourier series expansion of this 2π periodic function.

Therefore, this DTFT and the inverse DTFT, they are not really a new set of transforms. They are nothing but the Fourier series expansion of the 2π periodic function except that this function happens to be 2π periodic in the frequency domain and the time domain sequence can be viewed as the Fourier series coefficients of the 2π periodic function in the other domain, that is all. So, this is not a really new transform. And, moment you have the inversion integral, we can talk about certain sequences for which it is easier to compute the transform pair starting from the other domain.

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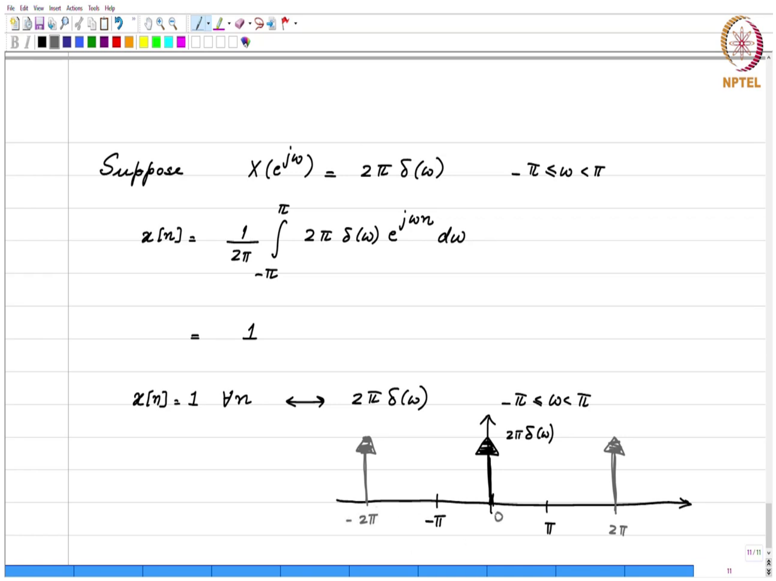

Therefore, suppose now we have $X(e^{j\omega}) = 2\pi\delta(\omega)$. Again, we have to qualify this using this. Therefore, $\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$ which is now $\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} 2\pi\delta(\omega)e^{j\omega n}d\omega$ and then if you use the shifting property this now becomes 1. Therefore, $x[n] = 1$ for all n has now transformed $2\pi\delta(\omega)$ and you have to qualify this with this as we have seen before.

So, this is the picture associated with this is. So, you now have an impulse at the origin. So, this is $2\pi\delta(\omega)$. So, this is between $-\pi$ and π and really this repeats periodically because this after all is the DTFT of a time domain sequence. Here, $x[n] = 1$ for all n is our DC sequence; we are used to calling this as the DC sequence. Therefore, you have periodic repetition. So, this occurs at 2π, this occurs at -2π, this occurred at $\omega = 0$.

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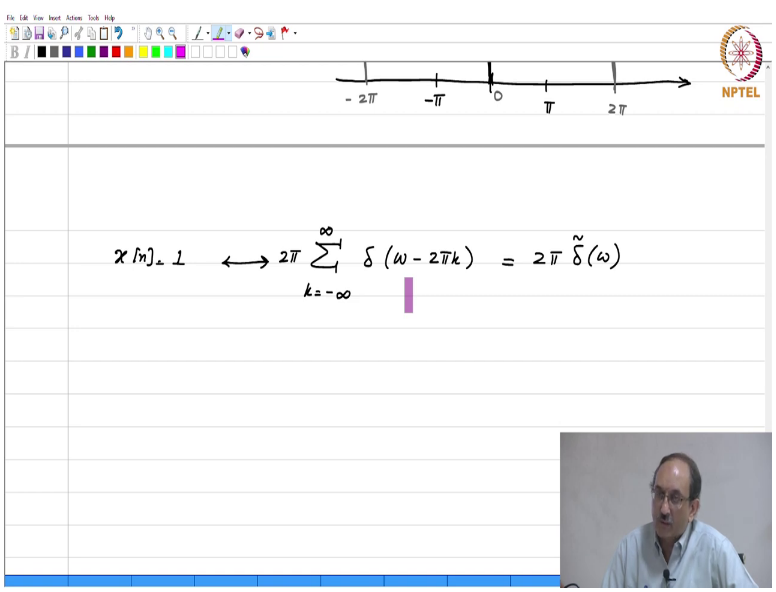

Therefore, $x[n] = 1$ has transform that is an impulse but this impulse is periodic and if you want to not use the restriction between $-\pi$ and π, you have to replace the impulse by its periodic version. Therefore, you have to replace some $\delta(\omega)$ by $\sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$. So, it is a train of impulses because it is periodic and this is non-standard notation you may not find it in too many textbooks instead of writing delta so, this is 2π.

So, instead of writing it like this as an impulse train or if you write it as a single impulse and then give this restriction, you can instead use the tilde notation whereby the tilde we mean that this is actually periodic; for sequences, some textbooks use $\tilde{x}[n]$ to denote periodic sequences. So, we are borrowing that notation and if you write $\tilde{\delta}(\omega)$, then you know that it is periodic. Now, you need not specify it this is between $-\pi$ and $+\pi$.

So, this is one sequence which is an example of a sequence having DTFT but this sequence is not absolutely summable, $x[n]$ is not absolutely summable. So, this is an example of sequence that has DTFT, but not absolutely summable. Therefore, absolute summability is a sufficient, but not necessary condition for the existence of the DTFT. If the Z-transform contains the unit circle of the region of convergence, we saw that those sequences are absolutely summable that they belong to the class of l_1.

Therefore, if the sequence is absolutely summable, for sure the DTFT exists but absolute summability is sufficient but not necessary condition for existence of the DTFT and this is the first of several examples that we will see that have DTFT that are not absolutely summable. Again, $x[n] = 1$ for all n does not possess Z-transform.

So, this is the analog of $x(t) = 1$ having CTFT $2\pi\delta(\Omega)$ but $x(t) = 1$ for all t does not have bilateral Laplace. So, you are now seeing the counterparts here in the discrete-time case.