Digital Signal Processing
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Lecture 42: Inverse Z-transform (3), Inverse DTFT -Contour Integral Method (cont'd)

Let us get started. We are looking at the Contour Integral Method for evaluating the inverse transform. We just got introduced to it we saw what the formula was, let us briefly recap the inversion integral.

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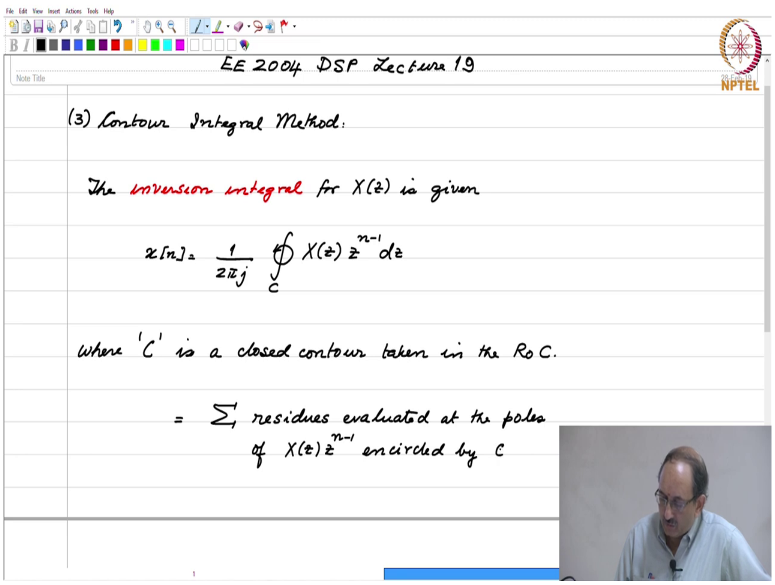

So, it is given by $x[n] = \dfrac{1}{2\pi j}\oint_C X(z)z^{n-1}dz$ where, C is a closed contour taken in the region of convergence. And, this is evaluated using the Cauchy Residue Theorem. So, this is sum of the residues evaluated at the poles of $X(z)z^{n-1}$ that are encircled by C. So, this is the residue formula.

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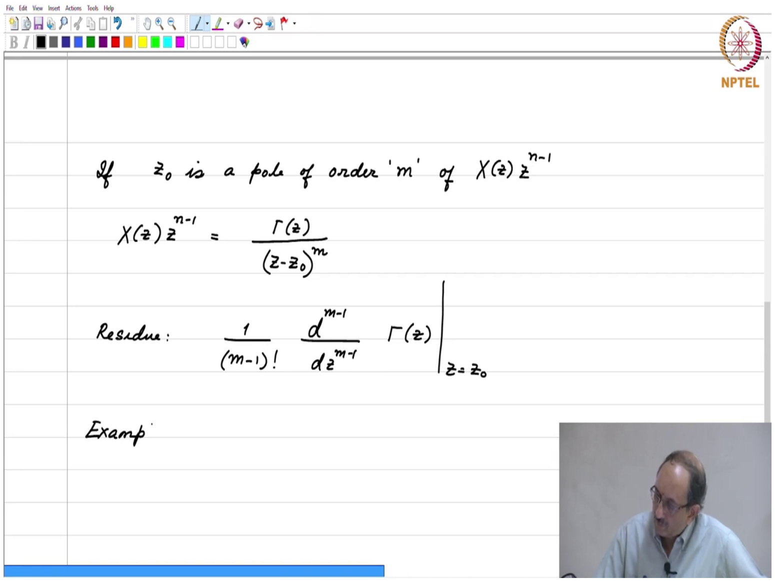

And, if z_0 is a pole of order m; so this is of order m of $X(z)z^{n-1}$. So, what we can do is, we can explicitly factor that m^{th} order pole outside. Therefore, $X(z)z^{n-1}$ can be written as $\frac{\Gamma(z)}{(z-z_0)^m}$. So, all I have done here is, I have taken $X(z)z^{n-1}$ and explicitly factored out the m^{th} order pole at $z = z_0$. The reason we want to do this is we need to evaluate the residues at every pole, and an m^{th} order pole at $z = z_0$ is the general case. And the residue for in this particular case is given by $\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\Gamma(z)$ evaluated at $z = z_0$.

So, as you have seen in the case of partial fraction expansion, we have to calculate the residue at a pole you need to cancel that pole out, by multiplying by $z - z_0$ if z_0 were a simple pole and then evaluate the rest of the expression at the location of the pole. So, this is exactly what we have done here.

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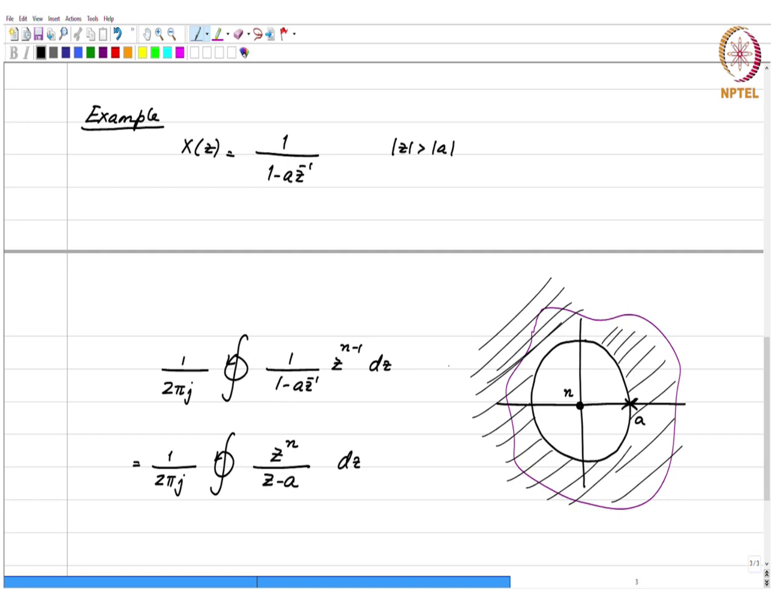

And as an example, we will take our good old friend $\frac{1}{1-az^{-1}}$, $|z|>|a|$. And you know the answer is $a^n u[n]$; we will get the same answer using the contour integral. Therefore, let us now consider this, its $\frac{1}{2\pi j}(\ )$. So, $X(z)z^{n-1}$, so this is $\frac{1}{2\pi j}\oint_C \frac{1}{1-az^{-1}}z^{n-1}dz$. And this of course is 1 by 2 pi j, this is $\frac{z}{z-a}z^{n-1}$, therefore this becomes $\frac{1}{2\pi j}\oint_C \frac{z^n}{z-a}dz$.

And if you look at the pole-zero plot, the circle that is drawn is $|z|=|a|$. And you have a pole at $z=a$ and for simplicity I have a shown the pole to be real and positive. And there is an n^{th} order trivial zero. And the region of convergence is outside this circle. Therefore, the contour C is a closed contour which is in this region, because the region of convergence is this. By definition, the closed contour C has to be taken in the region of convergence.

Now if you want to evaluate the contour integral, you need to evaluate the residues at the poles. There is only one pole here namely at $z=a$. Therefore, all you need to do is you need to calculate the residue at the location of the pole.

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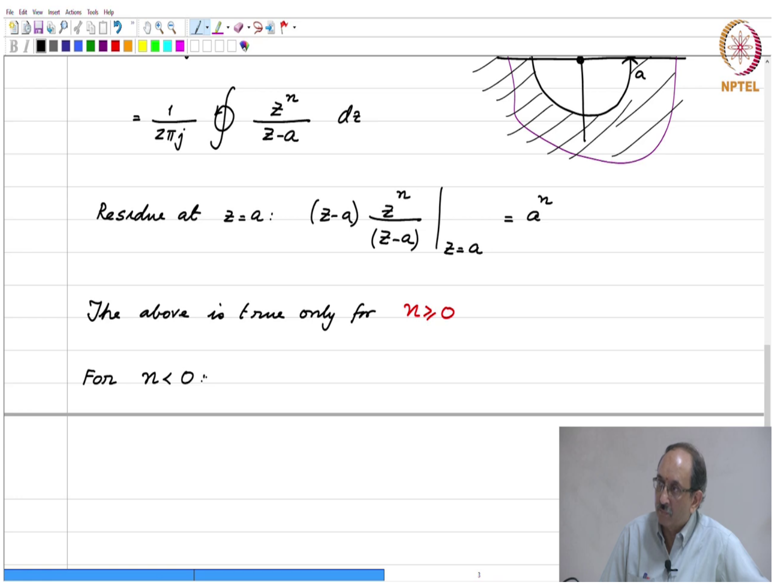

Therefore, residue at $z=a$ is given by, you need to, you need to multiply by $(z-a)$ so you cancel the pole therefore this is $(z-a)\frac{z^n}{(z-a)}$. And, then this needs to be evaluated at $z=a$ and you get back a^n which is what you expect.

So, we expected this to be the answer and contour integral sure enough gives us this. So are we done? In terms of finding the inverse Z-transform using the contour integral.

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Say that again.

Student: (Refer Time: 09:26).

Therefore; it has to be right sided, yes. So, can you develop on the answer and then complete it to point out what is left? So, what is it that we have assumed here?

Student: (Refer Time: 09:49).

So, if you stop here, you are not done because what we have assumed here is we have assumed n to be.

Student: (Refer Time: 10:00).

Greater than or equal to 0, all right. So, this is the answer for $n \geq 0$, which is a^n which is a long expected lines. What we have not yet shown is we have not shown that this gives you 0 for $n < 0$. Only then is this complete.

Now, let us do that. We will point out that the above is true only for $n \geq 0$. Now, we will consider the case when $n < 0$.

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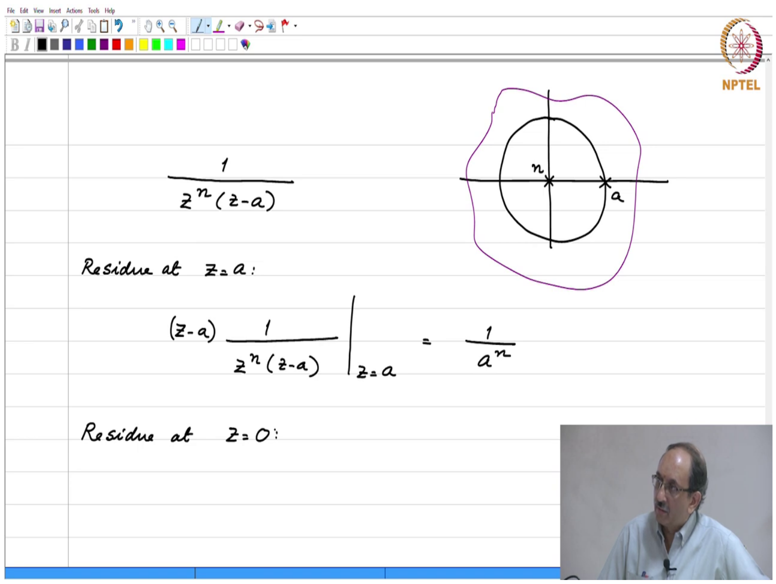

When $n < 0$, you can write this as $\dfrac{1}{z^n(z-a)}$; where now n will be, we will consider values of n that are?

Student: (Refer Time: 11:21).

Positive, because we have written it in this form, ok. And, now this changes the pole-zero plot, whereas before you had only one pole, namely at $z = a$ so that is still there. But, what we now have is, we now have an n^{th} order pole at 0. So, we have an n^{th} order trivial pole, ok. Therefore, if you now take this contour and look at it, it is now encircling not just the pole at $z = a$, but also the trivial n^{th} order pole. So, when you compute the residue, you have to compute the residue at $z = a$ and also at the n^{th} order trivial pole, ok. So, residue at $z = a$ is easy, because you need to multiply by $(z - a)$ and then evaluate it at $z = a$. And you get $\dfrac{1}{a^n}$.

Residue at $z = 0$ and now this is no longer a simple pole, its a multiple pole. Therefore, now you have to apply the formula that was given.

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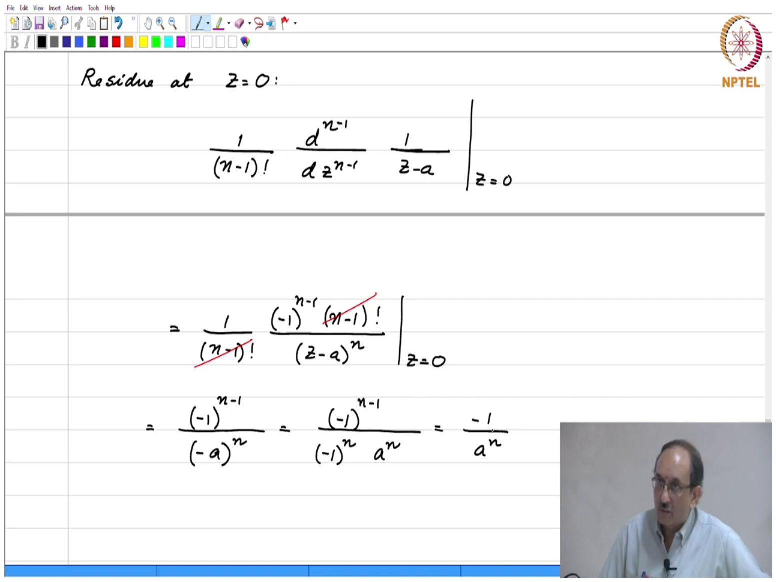

So, the formula that was given was $\dfrac{1}{(n-1)!}\dfrac{d^{n-1}}{dz^{n-1}}\dfrac{1}{(z-a)}$. And then you need to evaluate the $(n-1)^{th}$ order derivative at $z = a$. So, this will be.

Student: (Refer Time: 14:18).

You are right; $z = 0$ because we are evaluating it at the origin, right. So, $\dfrac{1}{(n-1)!}(\ )$. And now you need to find out the $(n-1)^{th}$ derivative of $\dfrac{1}{(z-a)}$. So, this is nothing but $\dfrac{1}{(n-1)!}\dfrac{(-1)^{n-1}(n-1)!}{(z-a)^n}$ evaluated at $z = 0$. So, this is from calculus if you recall this the $(n-1)^{th}$ order derivative of $\dfrac{1}{(z-a)}$, all right.

So, nice cancellations are happening here. So, this cancels out. And then you need to evaluate this at $z = 0$. Therefore, you have $\dfrac{(-1)^{n-1}}{(-a)^n}$. So, this is $\dfrac{(-1)^{n-1}}{(-1)^n a^n}$, so which is $\dfrac{-1}{a^n}$.

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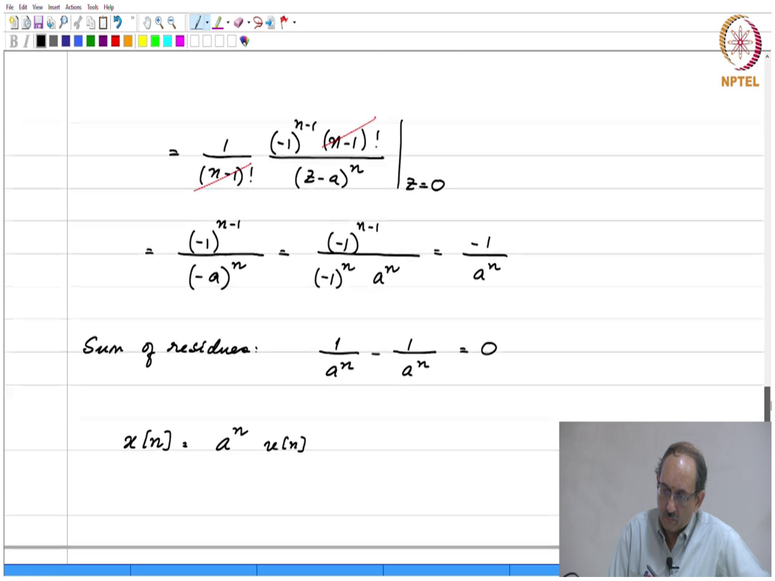

And, sum of residues is $\frac{1}{a^n} - \frac{1}{a^n}$ and you get 0. So now, we have indeed shown the final answer is $a^n u[n]$. We saw it was a^n for n$n \geq 0$, and when $n < 0$, the contour integral gives us residues which add up to 0 for any negative value of n. And, hence $x[n]$ is indeed $a^n u[n]$. For rational transfer function; question.

Student: (Refer Time: 16:48).

So if you look at, that is a good question. So, if you look at this circle, it is taken in the region of convergence, all right. So, when you have this picture for negative values of n, you see that it encircles this;, n^{th} order trivial and the pole at this one. Now to see the difference, if you now take exactly the same $X(z)$ but now you consider $|z| < |a|$ and if you work the problem out, you will now take these contour C which always has to be in the region of convergence. Therefore, this contour C will lie inside the circle $|z| = |a|$. And then you will see that in that case, it will not encircle the pole at $z = a$.

So, it is a very simple exercise. In fact, $\frac{1}{1 - az^{-1}}$, $|z| < |a|$, you need to work it out using exactly the same method contour integral, and then you need to show it is $-a^n u[-n-1]$. There you will see the difference contributed by the contour C lying inside the circle $|z| = |a|$, all right. Does that answer the question? Ok.

So, we now have $x[n]$ to be indeed $a^n u[n]$. So, the contour integral because of the Cauchy residue formula that is given here: for rational functions, I mean this is as easy to evaluate as anything else. Therefore, you are not really doing any contour integration, you are using the residue formula and the residues are easy to find when $X(z)z^{n-1}$ is rational.

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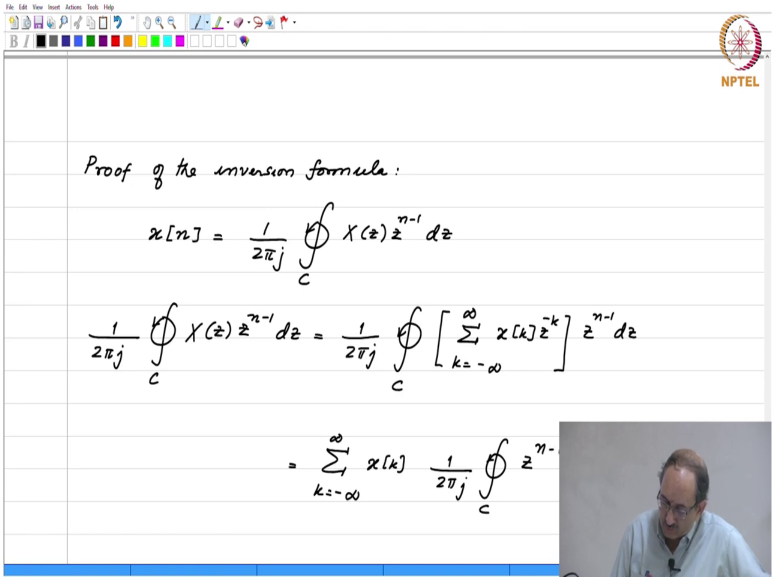

Now, what we will do is, we will show that the inversion integral is indeed correct in the sense that it gives you $x[n]$ back. So, this is what we want to prove.

So, what we will do is we will start off with this and then we will replace $X(z)$ by the Z-transform. So, this is nothing but $\dfrac{1}{2\pi j}\oint_C (\,)$. Now, instead of $X(z)$, we will write this as $\dfrac{1}{2\pi j}\oint_C \Big[\sum_{k=-\infty}^{\infty} x[k]z^{-k}\Big] z^{n-1}dz$. Note that, here for the dummy index of summation we have used k, you do not want to use n because n is already appearing here. So, that is why I have used k.

Now we will happily interchange these two limiting processes, namely the integral and the infinite summation. Therefore, this becomes $\sum_{k=-\infty}^{\infty} x[k]\ \dfrac{1}{2\pi j}\oint_C z^{n-1-k}dz$.

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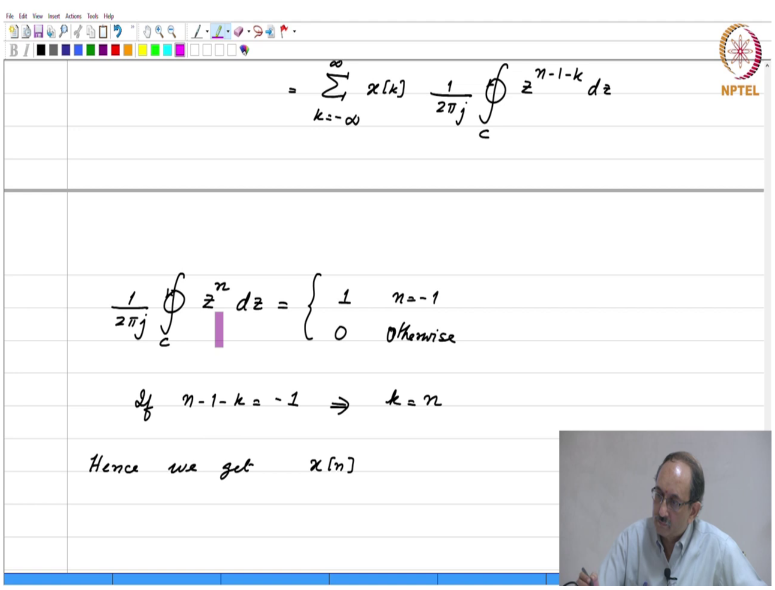

$\frac{1}{2\pi j}\oint_C z^n dz$ equals; I really cannot.

Student: (Refer Time: 22:25).

Very good so, so this is 1 when $n = 1$ and.

Student: (Refer Time: 22:33).

0 otherwise, ok. I thought you people did not have a course on complex variables.

Student: (Refer Time: 22:40).

So, only some of you know this, all right. Therefore, using this result, if you apply this to this integral here, clearly the integral will be 1 if $n-1-k=1$. So this implies, this integral will exist only when $k=1$, otherwise the integral will be 0. And hence if you look at this expression here, for every value of k, it is going to be multiplied by a term that will be 0 whenever $k \neq n$. Only when $k=n$, will this term be 1. For all other values, this term will be 0 and I have, therefore all these terms will vanish. Therefore, the only surviving term will correspond to $k=n$ and therefore. Hence we get $x[n]$ is the final answer; every other term in this summation is 0.

Therefore, this indeed gives us $x[n]$ as expected. So, in the proof of the inversion integral, the key step is knowing this result, which follows from a complex variable theory.

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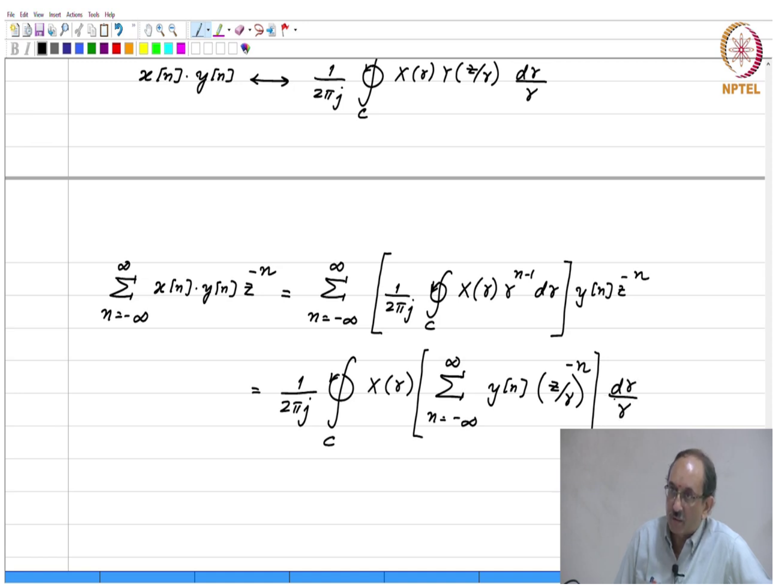

When we did the Z-transform properties, one of the properties was: if you multiply in the time domain, you convolve in the other domain except that in the Z-transform case, the convolution in the other domain is complex convolution. And the Z-transform of $x[n].y[n]$ was $\frac{1}{2\pi j}\oint_C X(\gamma)Y(z/\gamma)\frac{d\gamma}{\gamma}$. This was the formula that was given. And it was mentioned at that point in time that this will make sense once we have the inversion integral formula with us. Now that we have the inversion integral formula with us, let us try to see if this is true.

So, we are looking at the Z-transform of $x[n].y[n]$. Therefore, we sum up overall n, $\sum_{n=-\infty}^{\infty} x[n].y[n]z^{-n}$, this is after all the basic definition. So, this is sum over all n. And for $x[n]$, I will replace $x[n]$ by its inversion integral, therefore this is $\frac{1}{2\pi j}\oint_C X(\gamma)\gamma^{n-1}d\gamma$. Again, you do not want to use z here because z already is part of the expression here. So, this is $y[n].z^{-n}$.

And now we will do what we always do. We will interchange these two things. So, this is $\frac{1}{2\pi j}\oint_C$, and this is $X(\gamma)$. Now, what is remaining is $\sum_{n=-\infty}^{\infty} y[n]z^{-n}\gamma^{n-1}$. And, now we have $z^{-n}\gamma^{n-1}$, therefore this can be written as $(z/\gamma)^{-n}$, all right. We still have to account for one power of γ. So, now, this can be written as $\frac{d\gamma}{\gamma}$, because you need γ^{n-1}. What falls out of this is γ^n, therefore this is this.

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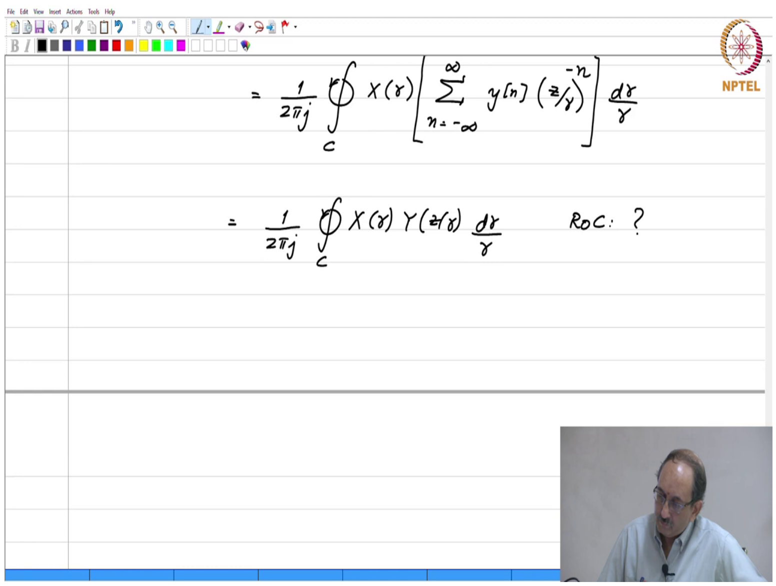

And, now you can immediately see that this is indeed $\frac{1}{2\pi j}\oint_C X(\gamma)Y(z/\gamma)\frac{d\gamma}{\gamma}$.

The only remaining thing that we have not yet shown is the RoC. So, I want you to take a stab at what the RoC should be. Remember, whenever you state the formula, you should not only state the algebraic expression, but also you need to give the corresponding RoC. And the hint is the RoC should be such that both $X(\gamma)$ and $Y(z/\gamma)$ are part of the RoC, z is a parameter here. Therefore, the multiplication in the time domain giving rise to this expression. Now you have actually shown that it is indeed true once we have the inversion integral formula with us.