Digital Signal Processing

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Lecture 40:
Inverse Z-transform (2)
- Power series method

So, we started looking at the Inverse Z-transform ideas and the first thing that we did was use Partial Fraction Expansion and this is a natural way to compute Inverse Z-transform for functions that are rational.

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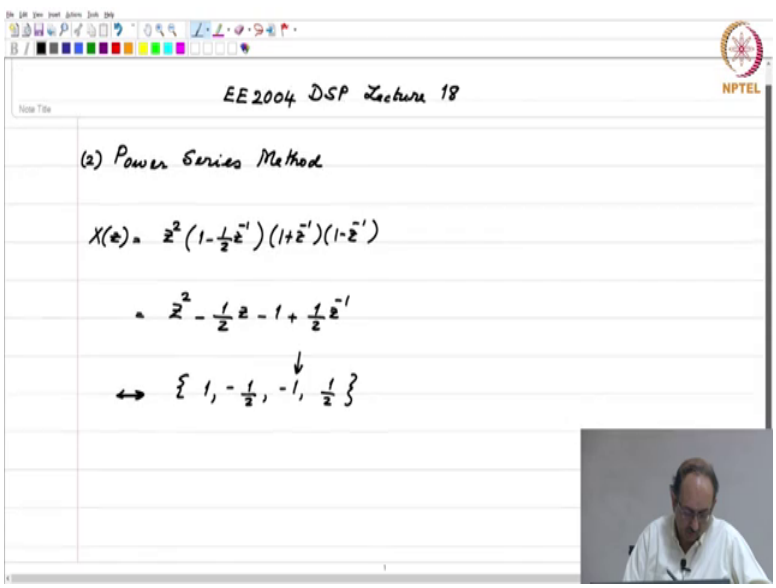

We will look at the second method; power series method is the next approach that we will use. Suppose $X(z)$ was something like this, $z^2\big(1-\frac{1}{2}z^{-1}\big)(1+z^{-1})(1-z^{-1})$, then you can expand this and if you did this. So, this will turn out to be $z^2 - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1}$, this is pretty straightforward for you to verify. And this is in the form of the Z-transform expression that is, this is in terms of the power series; therefore, you can easily identify the individual terms. So, this would be $\left\{1, -\frac{1}{2}, -1, \frac{1}{2}\right\}$ with the $n = 0$ term being shown by this arrow, straightforward application of the direct definition.

Note that, this has no counterpart in the Laplace transform case.

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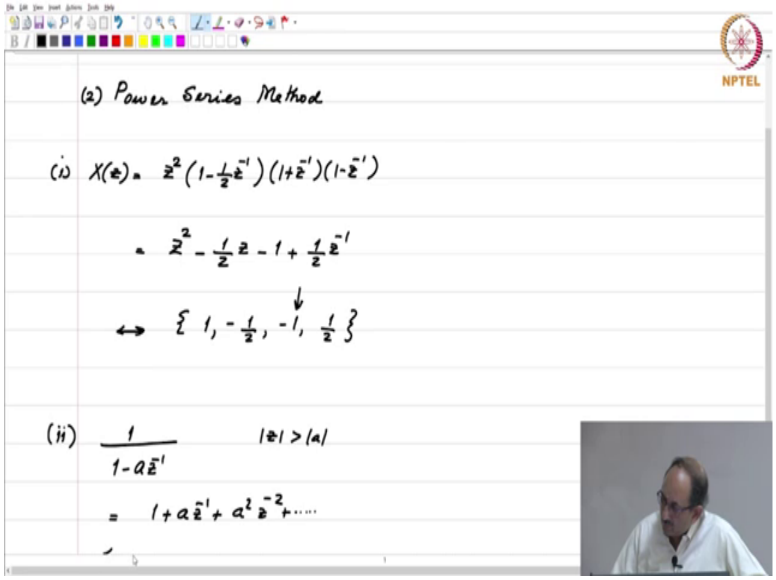

And, we will look at this $\dfrac{1}{1-az^{-1}}$, and this of course is $1 + az^{-1} + a^2z^{-2} + \ldots$ and so on; again when we computed this Z-transform of $a^n u[n]$, we started off with something like this and we got this closed form expression.

We are now retracing our steps back just to point out the relevance of the power series method in this context and this of course, is this.

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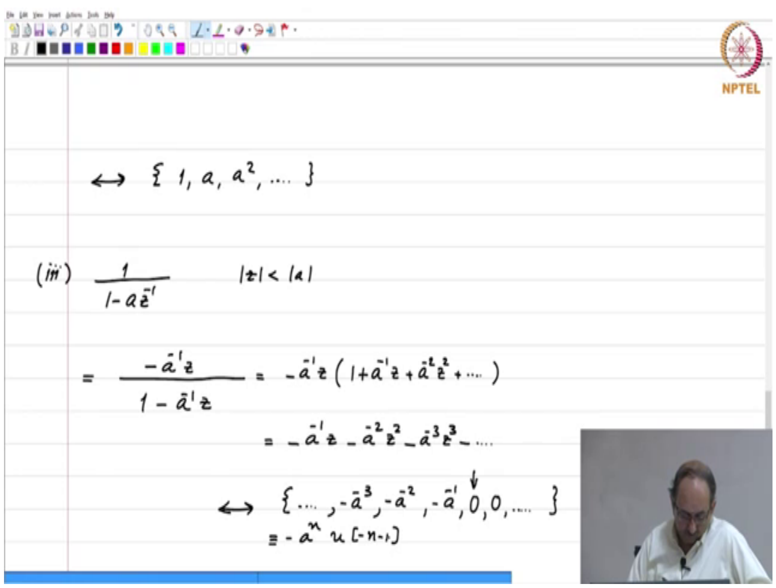

And, this you can immediately identify the terms of the series to be $\{1, a, a^2, \ldots\}$ and so on. Now, let us continue the power series to the related example. So, this is $\dfrac{1}{1-az^{-1}}$.

Now, the main difference is, it is $|z| < |a|$; therefore, when you take the inverse Z-transform you expect an anti-causal sequence in the time domain. And, an anti-causal sequence in the time domain has positive powers of z in the Z-transform; therefore, when you expand this in terms of power series, you have to expand this in terms of positive powers of z, all right.

And, to see that you multiply by $-az^{-1}$ and if you did this, if the denominator if you multiply by $-az^{-1}$; for this factor, you will get 1 and we will get again $-a^{-1}z$. So, this can be written as $-a^{-1}z(\ )$ and $\dfrac{1}{1-a^{-1}z}$ can be expanded in terms of power series.

So, this is nothing but $1+a^{-1}z+a^{-2}z^2+\ldots$ and so on; and therefore, this is $-a^{-1}z-a^{-2}z^2-a^{-3}z^3-\ldots$ and so on, you see the pattern here. And this of course, if you identify term by term, the inverse Z-transform since this has only positive powers of z. If this were the origin, all these terms are 0; and here, you have $-a^{-1}$, $-a^{-2}$, $-a^{-3}$ and so on.

And, clearly you recognize this as $-a^n u[-n-1]$; and to throw more light on this, if you looked at the previous example $\dfrac{1}{1-az^{-1}}$; what you are really doing is, you are doing long division.

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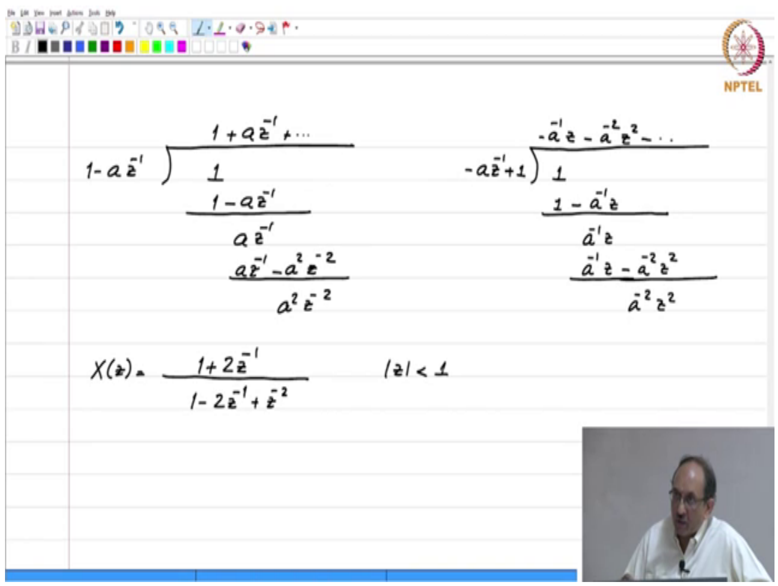

Therefore, you have 1 and $1-az^{-1}$ here all right, and then so this is a throwback to your high school days.

So, this is just plain long division; therefore, if you divide this you get this. So, you have az^{-1} and you keep going. So, you have $az^{-1}-a^2z^{-2}$ and again you have a^2z^{-2} and you see the pattern here. So, you can think of this power series that you got as just plain long division.

And, now suppose you have $\dfrac{1}{1-az^{-1}}$, but you wanted the anti-causal sequence. So, region of convergence is $|z|<|a|$ and you want the anti-causal sequence, you need to again get a power series; but this time in terms of powers of z, we already saw that the way we did that here, we saw that here.

Now, we will again get this by long division. So, this is $\dfrac{1}{1-az^{-1}}$; but now the, we will write this by this manner, $-az^{-1}+1$, all right. Therefore, if you now divide, this will be $-a^{-1}z$. So, you get 1, right $-a^{-1}z$ and just to see the pattern. So, this is $-a^{-1}z$ and to get this, the next term has to be $-a^{-2}z^2$, all right.

So, this will be $a^{-1}z - a^{-2}z^2$ and you see the pattern here. Again you see that, you get an answer that you expect for the anti-causal sequence. The main point here is, when you are doing now long division; you are arranging this term like this, rather than $1 - az^{-1}$; which was what you are normally used to, if you did this, you will get in terms of negative powers of z.

Whereas in this case, because the RoC is $|z| < |a|$, you want a power series expansion in positive powers of z and which is why you need to write it like this. Let us take one more example, suppose I have $X(z)$ to be $1 + 2z^{-1}$; and the denominator I have $1 - 2z^{-1} + z^{-2}$. And, you can see that the denominator is really $(1 - z^{-1})^2$; therefore, there is a second order pole at $z = 1$. Now let me state that, I want the RoC to be $|z| < 1$.

Therefore, you can now expect an anti-causal sequence. Let us again do long division.

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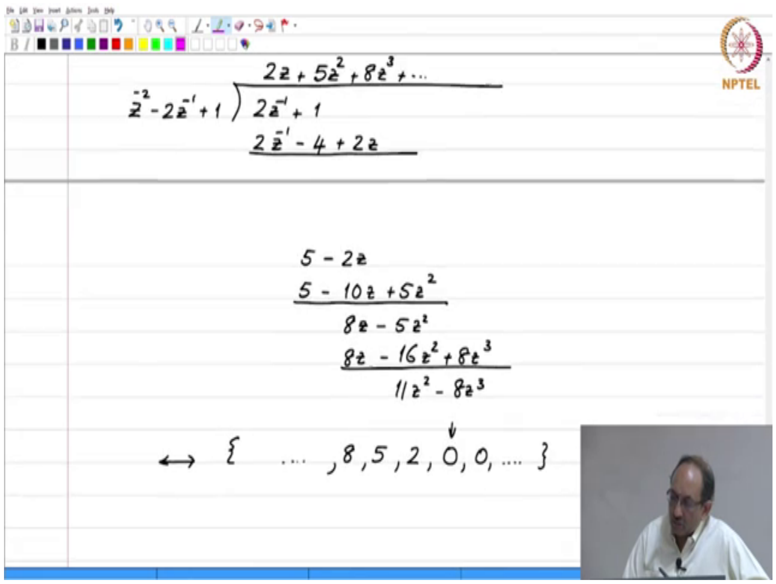

So, now for long division, if you want positive powers of z, you should write the this term to be, $z^{-2} - 2z^{-1} + 1$. This is how you should write this, because you want the answer to contain, you want the quotient to contain positive powers of z.

And, now this term should be written like this, $2z^{-1} + 1$; therefore, this now becomes $2z$, all right. So, this becomes $2z^{-1} - 4 + 2z$. And now if you subtract, you get $5 - 2z$, therefore, this becomes $5z^2$; and hence this becomes $5 - 10z + 5z^2$. And if you subtract, this becomes $12z - 5z^2$, let me write down, sorry, this is $8z$.

Let me write down one more terms, so this becomes $8z^3$; therefore, this becomes $8z - 16z^2 + 8z^3$ and this becomes $11z^2 - 8z^3$ and so on. And, hence you are able to see that the inverse Z-transform is an anti-causal sequence; therefore, let me mark down the $n = 0$ index term. So, for all positive indices, the inverse Z-transform is 0; and the $n = -1$ term is 2 then you have 5, 8 and so on.

So, this is how the time domain sequences in terms of using long division, given that the RoC is $|z| < 1$; therefore, you get a sequence that is left sided. And the main point to note here is, if you are going to use long division, you have to arrange the denominator like this; the numerator also like this; so, that when you are, when you carry out a long division, you gets positive powers of z, all right.

Just as a sanity check, we can also compute the inverse Z-transform using a different method based on what you have already seen.

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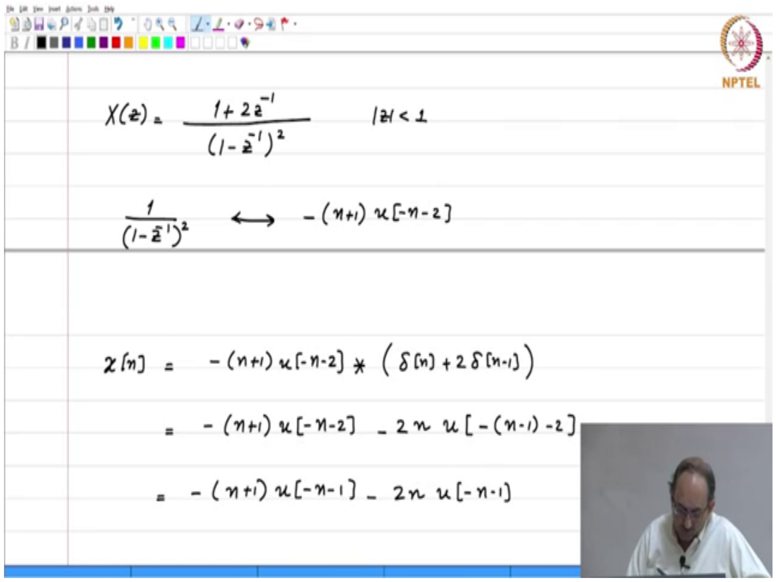

So, recall that the given $X(z)$ is $1+2z^{-1}$, the denominator after all is $(1-z^{-1})^2$ and the RoC is $|z| < 1$, because we want the sequence to be anti-causal.

Now, $\dfrac{1}{(1-z^{-1})^2}$, it is given that this RoC is $|z| < 1$; this was one of the things I had asked you to work out. I had worked out in class, second order pole $|z| > |a|$, all right. So, if you had tried this exercise already, you would find that this answer is $-(n+1)u[-n-2]$; this is what the inverse Z-transform of $\dfrac{1}{(1-z^{-1})^2}$ is, given that $|z| < 1$, all right.

So, now $x[n]$ can be obtained as. So, this is a product of, you can think of this as a product of two polynomials; one polynomial is $\dfrac{1}{(1-z^{-1})^2}$, the other polynomial is a numerator. Therefore, if you multiply in the z-domain, in the time domain you have to convolve; and hence this is nothing, but $-(n+1)u[-n-1]$ convolved with the inverse Z-transform of the numerator. The inverse Z-transform of the numerator is nothing, but $\delta[n] + 2\delta[n-1]$, ok.

Therefore, the first term, when you use the distributive property and convolve, so this is $-(n+1)u[-n-\ ]$. So, this should be $-(n+1)u[-n-2]$. So, this is $-(n+1)u[-n-2] + 2\delta[n-1] * (-(n+1)u[-n-2])$; therefore, you have minus. Wherever, because you are convolving it with $\delta[n-1]$, is going to shift it by one sample to the right; therefore, wherever n is there you have to replace n by $n-1$.

Therefore, this becomes $2n[-(n-1)-2]$. So, this is $-(n+1)$ and this can be written as $u[-n-1]$; because remember this term, this starts at $n = -2$, because of $u[-n-2]$. If you now replace $-n-2$ with $-n-1$, this will start at $n = -1$; but if you put $n = -1$ because of this term, this whole thing goes to 0.

Therefore, these two are indeed the same, $-2n\ u[-n-1]$.

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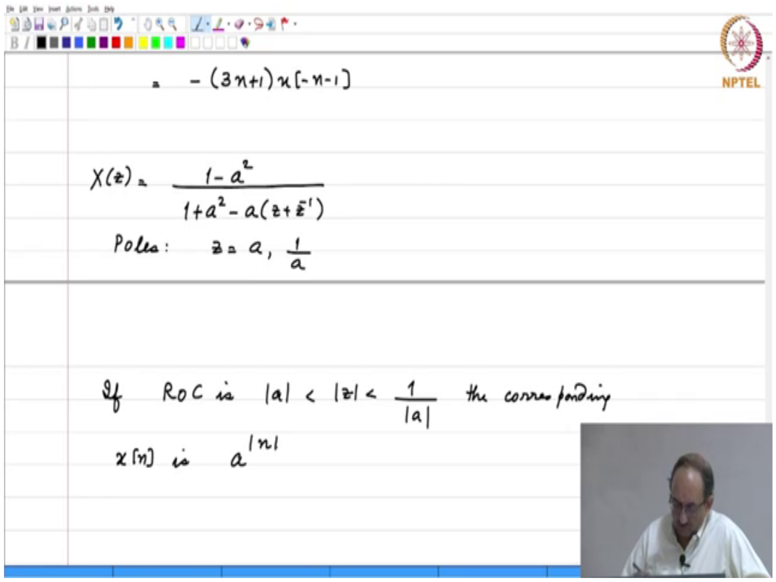

And, hence this is nothing, but now you can combine these two terms. So, this is $-(3n+1)u[-n-1]$. So, this is the inverse Z-transform. And, just to make sure what we did earlier is correct, so if you put $n = -1$, so this is 2.

If you put $n = -1$, this becomes -1 there is a minus sign outside. So, this becomes 2. And if you put $n = -2$, so this becomes 5; put $n = -3$ this becomes 8; therefore, the first three terms are 2, 5 and 8. So, which is exactly what we got here and again the sequence starts at $n = -1$. So, we have verified the long division method by another approach based on what we had known earlier.

Now, let us look at this particular example, $X(z) = \dfrac{1-a^2}{1+a^2-a(z+z^{-1})}$. If you actually expand this and simplify, you will find that the poles are at $z = a$ and $1/a$; this is a simple exercise I want you to verify later.

Therefore, there are now three possible RoCs, it can be $|z| < |a|$ or it can be $|a| < |z| < 1/|a|$, the final possibility is $|z| > 1/|a|$; therefore, three different RoCs are possible. If RoC is this, $|a| < |z| < 1/|a|$; what can you say about the nature of the sequence? It will be two sided, all right. The corresponding $x[n]$, the closed form expression turns out to be this, $a^{|n|}$, all right.

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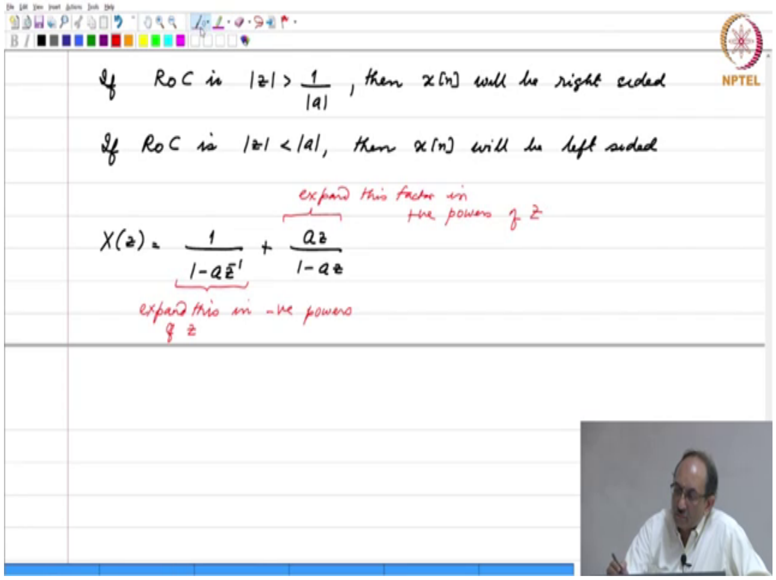

If RoC is $|z| > 1/|a|$, then $x[n]$ will be right sided. If RoC is $|z| < |a|$, then $x[n]$ will be left sided. Now remember the context in which we are looking at this is in terms of power series expansion. Therefore, given this $X(z)$, suppose I tell you the RoC is $|z| < |a|$, then you will expand this in terms of positive powers of z to get a left sided sequence. And if I tell you the RoC is $|z| > 1/|a|$, you will expand this in terms of negative powers of z and you will get a right sided sequence.

The point is using long division, you cannot get a two sided sequence. If you start off with $X(z)$ as is given here, using long division you cannot get a two sided sequence; that is the point to be made in this context. Suppose, you want a two sided sequence and you still want to use long division all right, then the way to do that is as follows; you have to write $X(z) = \dfrac{1}{1 - az^{-1}} - \dfrac{az}{1 - az}$, all right.

So, we have to verify that this indeed is the original $X(z)$. So, simple algebra, verify that this is correct ok, I am sorry this is $X(z) = \dfrac{1}{1 - az^{-1}} + \dfrac{az}{1 - az}$. And now remember our goal is to get a two sided answer. So, what we will do is, we will expand this in negative powers of z; we will expand this factor in positive powers of z.

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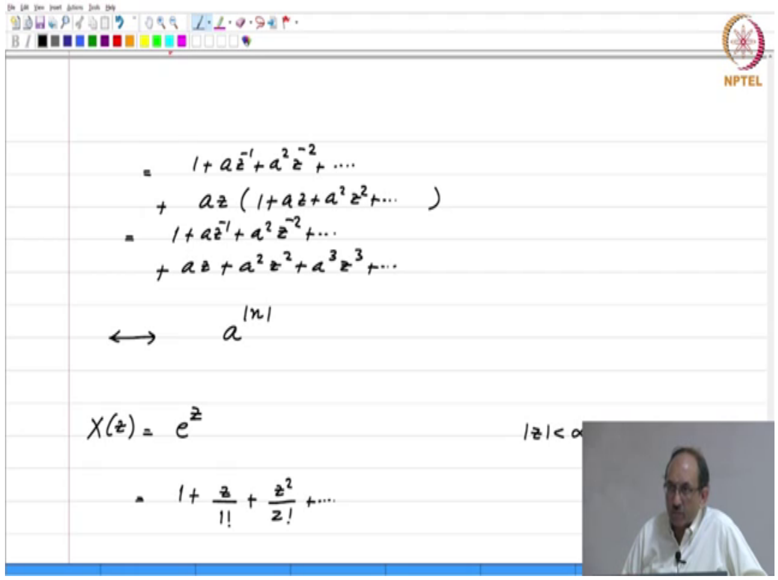

Suppose we did that. So, this is nothing, but $1 + az^{-1} + a^2z^{-2}$ and so on.

So, this is the expansion of the first term in terms of negative powers of z; and as far as the second term goes this is, $az(1 + az + a^2z^2 + \ldots)$ and so on. So, this turns out to be $1 + az^{-1} + a^2z^{-2} + \ldots$ which is the first term as it is; and the second factor is $az + a^2z^2 + a^3z^3 + \ldots$ and so on. And, if you computed the inverse transform, you recognize this is nothing but $a^{|n|}$, right. So, easily seen that this is indeed $a^{|n|}$.

One of the things about this power series expansion for example, if you do long division; let us go back and look at this particular example, just by looking at this, it was not immediately cleared that this is indeed $-(3n+1)u[-n-1]$; that is you could not recognize the closed form expression just by looking at this immediately.

So, this is a simple example maybe you can, if you try to fit a closed form expression; maybe you can guess it is after all $(-3n-1)u[-n-1]$. But if you had a more complicated expression and you had terms, it may not be immediately clear what the closed form expression is. So, using long division, you will be able to get a few terms; but you may not be able to get the close form expression for all n.

So, that is also what is expected when it is given that, find the inverse Z-transform using long division, all you are expected is compute a few terms three or four terms. Only in very simple cases will the general expression be, obvious looking at the few terms; otherwise it would not be. And the other point about long division is, it only give you either left side or right sided; if you start off from $X(z)$ and proceed as it is, here is an example where you can get the intended two sided sequence by proceeding along these lines.

The other example is, suppose $X(z) = e^z$. So, for what a range of z is this valid? So, this after all is $1 + \frac{z}{1!} + \frac{z^2}{2!} + \ldots$ and so on. So, this is a power series and the region of convergence for this power series. So, this is for, this is true for $|z| < \infty$.

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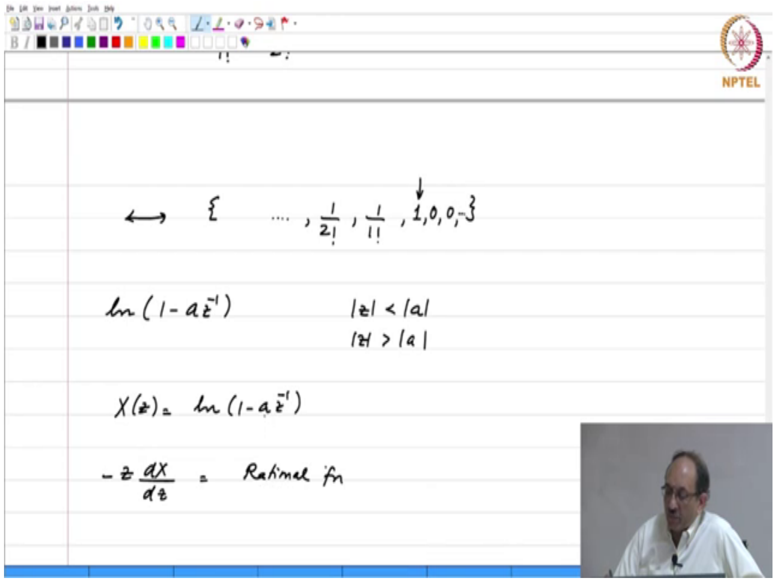

And once you have this, immediately you can guess what the answer is going to be.

So, this is a left sided sequence, because the Z-transform contains only positive powers of z. So, this is the $n = 0$ index term and $\frac{1}{1!}, \frac{1}{2!}$ and so on. So, this is inverse Z-transform. So, in terms of power series, what does this example tell you? What is the inference, what is one thing that immediately you should strike you, once you see $X(z) = e^z$ and you have computed the inverse Z-transform, ok.

But the previous one also has power series. So, here in this particular example it, so happens in the RoC is the entire z-plane, finite z-plane, right. Now I would not, I mean are you able to see the one striking difference between the examples that we have seen so far in this; that is the main question. So, Roc, if it is a finite duration sequence, the RoC the entire z-plane except possibly for 0 and or ∞, all right ok.

Student: (Refer Slide Time: 31:12).

That is the main point. So, this is not a rational function. So, what this tells you is this method is applicable to functions that are both rational and not rational; whereas, partial fraction expansion, the way we have seen is applicable only for rational transfer functions. In terms of partial fraction expansion, the more general case you can, if you are keen you can look up the Mittag Leffler expansion, all right; but you do not need those kinds of methods. We are looking at, a simple rational functions and their inverse Z-transforms. And question.

Student: $|z|$ is greater than 0, in $X(z) = e^z$.

So, $X(z) = e^z$, ok.

Student: $|z| > 0$ instead of $|z| < 0$.

So, this is also true for $z = 0$, correct. So, this indeed converges for $z = 0$, right. So, 0 being part of the RoC is not an issue.

Student: Like previously defined as $|z|$ is greater than the value than, it will be right sided (Refer Slide

Time: 32:45) $|z|$ is greater than (Refer Slide Time: 32:50)

Student: We take it as a right sided.

Yes and if $|z|$ were, if it were inside of a certain circle, then it is left sided, the counterpart of that. So, you can think of this as being inside of a certain circle only that this circle has infinite radius, and 0 can or may or may not be part of the RoC in such cases. In this case, 0 happens to be part of the RoC. So, where is your question coming from, in terms of $z = 0$ being part of the RoC, what this is that you saying I am not completely sure I understand your statement.

Yeah, but then the point is whether 0 is part of the RoC, right. So, you can think of this as being outside of circle with radius 0, but then are you excluding 0, here 0 is included. So, you do not have to say it is outside of a certain circle with radius 0. Because if you made that statement, then you are excluding $z = 0$; whereas, here $z = 0$ is indeed part of the RoC. So, that is why I had written it like this.

So, does that answer the question, ok. So, for the same e^z, you are asking about a different RoC; but here for this power series, the RoC is the entire z-plane. So, there is no question of having a different RoC given this expansion; whereas, if you had a function like $X(z) = \frac{1}{1 - az^{-1}}$, you could have two different regions, here that possibility does not arise.

Student: There is no other.

There is no other RoC for this given e^z. So, you can expand the function around $z = 0$, alright; and then typically we are, the RoC is bounded by poles. So, if for this particular, if for the function that you have taken; if you expand it using the Taylor series around the origin and then the series is valid only for a certain circle. Then that is what the, you get the inverse Z-transform. You can think of this as $\delta[n] + \frac{1}{1!}\delta[n+1]$ and so on, all right.

So remember, when you are talking about the inverse Z-transform, it is not that you evaluate $X(z)$ at a certain value of z and then look at the inverse Z-transform pertaining to that value. The inverse Z-transform is I mean, the power series expansion and you identify all the terms. So, even though it is true that if you put $z = 0$, you get 1, when it comes to inverse Z-transform you cannot evaluate $X(z)$ at one particular point and then being concerned about it is inverse Z-transform, correct. You have to evaluate this for all z and then consider the general inverse Z-transform.

And as a simple application of this, you can consider something like $\ln(1 + az^{-1})$; and then you can, I want you to look at these possibilities. And there is the power series expansion of log, $\ln(1 + x)$ has a power series expansion between -1 and 1, one of them is included; the other is not. We have $\ln(1 + x)$, 1 is included, -1 is not; if we had $\ln(1 - x)$, then 1 is not included and -1 is; and $\ln(1+x)$ and $\ln(1-x)$, these have power series expansion. So, using that expansion, try to find the inverse Z-transform.

So, that is one approach for this, and you should verify that answer with the differentiation property, all right. So, if $X(z)$ were $\ln(1 + az^{-1})$ and then $-z\frac{dX}{dz}$, right. So, this will be a rational function for which we know the inverse Z-transform; and then you have to relate this with $-z\frac{dX}{dz}$ after all is the transform of the sequence $n.x[n]$. Therefore, if you find the inverse Z-transform of the derivative, that will be the same as the sequence of Z-transform corresponds to $n.x[n]$, from that you can deduce what $x[n]$ is.

So, if you take this particular example, you can do it by two different ways; either by power series or by the derivative property and again you should verify that they are consistent with each other.