Digital Signal Processing
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Lecture 39:
Inverse Z-Transform (1)
Partial fraction method:
(a) $M < N$, simple roots,
(b) $M > N$, simple roots,
(c) $M < N$, multiple roots

So, we are now going to look at the next topic, namely computing the Inverse Z-transform.

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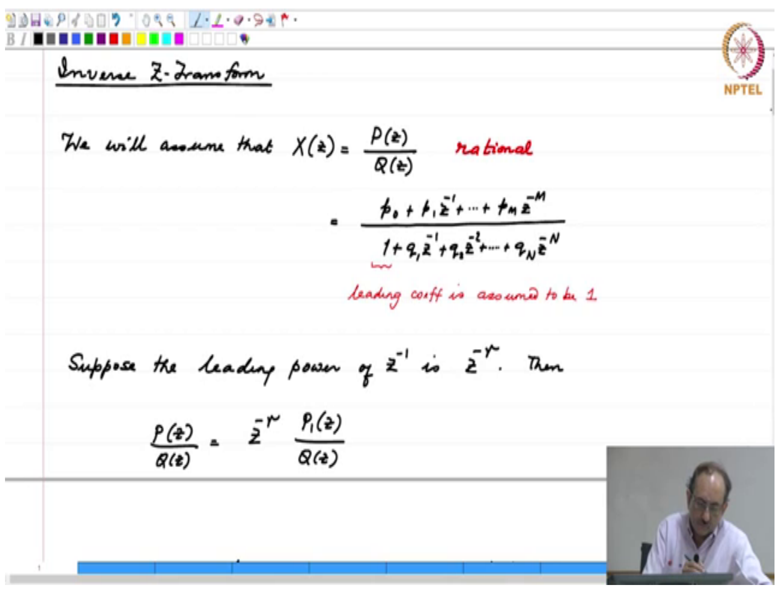

So, as mentioned before, we will consign ourselves to the class of rational transfer functions. So, we will assume that $X(z)$ is of the form $\dfrac{P(z)}{Q(z)}$. So, which means that this is rational and in particular, we will assume that these are the form $\dfrac{p_0 + p_1 z^{-1} + \ldots + p_M z^{-M}}{1 + q_1 z^{-1} + q_2 z^{-1} + \ldots + q_N z^{-N}}$ and the numerator, we will assume to be of this form $p_0 + p_1 z^{-1} + \ldots + p_M z^{-M}$, ratio of polynomials.

Now just a couple of observations about this, here we have assumed that this leading coefficient of the denominator is 1. This is not a restriction because if the leading coefficient were not 1, we can always normalize it so that the leading coefficient becomes 1. So, this is perfectly fine, it is not restrictive. The numerator is assumed to be of the form $p_0 + p_1 z^{-1}$ and so on.

Suppose the first r coefficients are zero, alright. Rather, suppose the leading power of z^{-1} is z^{-r}. Suppose, if this were the case then, you can always factor out z^{-r}. Therefore, $\frac{P(z)}{Q(z)}$ can be written as $z^{-r}\frac{P_1(z)}{Q_1(z)}$, by $Q(z)$ rather; $z^{-r}\frac{P_1(z)}{Q(z)}$. And, now $\frac{P(z)}{Q(z)}$ which was the original Z-transform and $\frac{P_1(z)}{Q(z)}$, where $P_1(z)$ is obtained by factoring out the first non-zero power of z, z^{-r}, alright. So, these when you consider the inverse Z-transform differ only by what?

Student: r.

By an r sample delay, yeah time shift. So, differ only by an r sample delay. Therefore, once you find the inverse transform of $\frac{P_1(z)}{Q(z)}$, you can always find the Z-transform of $\frac{P(z)}{Q(z)}$. So, the point that has been made so far is that assuming $\frac{P(z)}{Q(z)}$ to be in this form is not restrictive.

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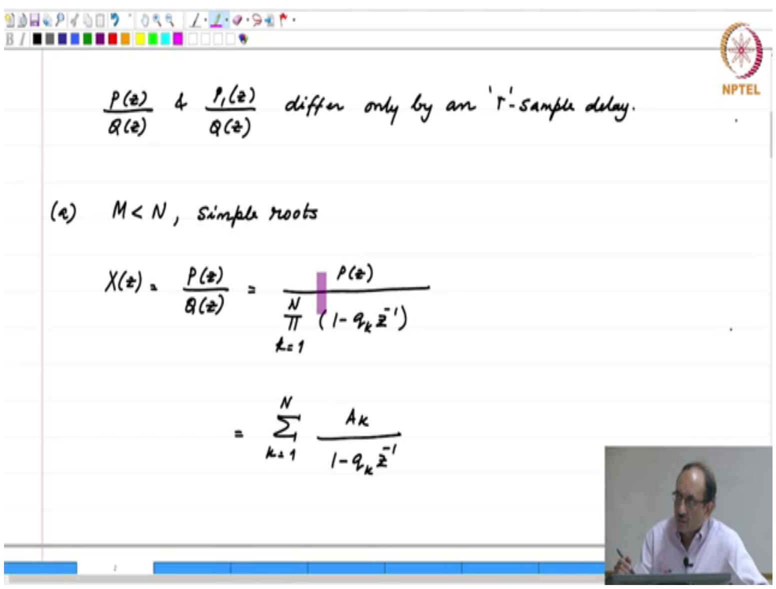

And now, what we will do is we will do, what we had done for inverse Laplace transform; there you had assumed $X(s)$ to be of the form $\frac{P(s)}{Q(s)}$ and one of the first techniques that was applied was partial fraction expansion. So, we will do the same thing here, it is very similar in principle, but we will point out an important difference.

So, the first case, we will consider is $M < N$, we will also assume simple roots. Therefore, $X(z)$ which is of the form $\frac{P(z)}{Q(z)}$, the very first thing you need to do when you do partial fraction expansion is factor the denominator. Yes, question.

Student: Simple root means.

Yeah, simple root means its each root has power 1. Therefore, for example, if you had $\frac{1}{1-az^{-1}}$, then $z = a$ is a pole, but it is a pole of order 1 whereas, $\frac{1}{(1-az^{-1})^2}$ is a pole at $z = a$ of order 2 and that is not designated as simple; simple means first order roots.

So, the very first thing that you do is factor the denominator and this of course is $P(z)$ and this is written as $\sum_{k=1}^{N} \frac{A_k}{1-q_k z^{-1}}$. Now, by the way, when you assume $M < N$, this ratio of polynomials is said to be in, you must have encountered this in Laplace. The term that is used when $M < N$ when you are applying partial fractions, what is the term that is used?

Student: Proper form.

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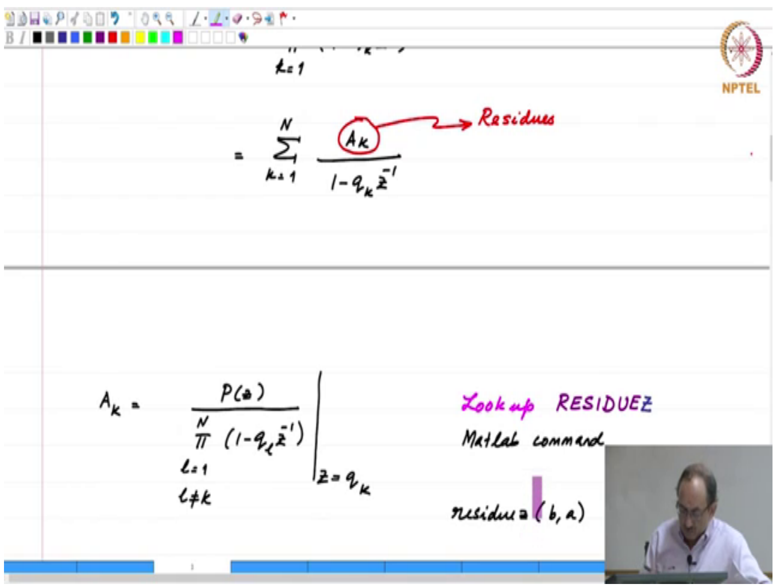

Proper form; so, this is in proper form and these are called residues; A_ks are called as the residues. And, A_k is given by $\frac{P(z)}{\Pi_{l=1,l\neq k}^{N}(1-q_l z^{-1})}$ and this whole expression is evaluated at $z = q_k$. And in this context, you can look up this one, *RESIDUEZ*, this is the MATLAB command and the counterpart of this for the Laplace is just *residue*.

And the difference is this so, if you do, after all you have a ratio of polynomials $\frac{b(z)}{a(z)}$. So, the b vector are the b coefficients that you have to feed to Matlab, the a vector are the denominator coefficients and then you have to give this command *residuez*. And this will assume rational transform of the form $\frac{b(z)}{a(z)}$, that is exactly, rather $\frac{P(z)}{Q(z)}$, just be consistent with what I had written earlier. The numerator coefficients as a vector fed to MATLAB, denominator coefficients as a vector fed to MATLAB and MATLAB will give you the residues and it will also factor the denominator.

And in this form, if you give it as a vector to MATLAB, it will assume the polynomial to be precisely of this form. The first coefficient will be p_0, the second coefficient will be p_1 which will correspond to z^{-1}

and so on. Similarly for the denominator. Whereas, if you use the command residue that is applied to the Laplace case and if you recall in the Laplace case, if the $X(s)$ is of the form $\frac{P(s)}{Q(s)}$, the polynomial is in powers of s rather than s^{-1}, ok. So, that is the difference between *residue* and *residuez*, the form of the assumed polynomial. In this case, its powers of z^{-1} that is why it makes sense to use this command in this context.

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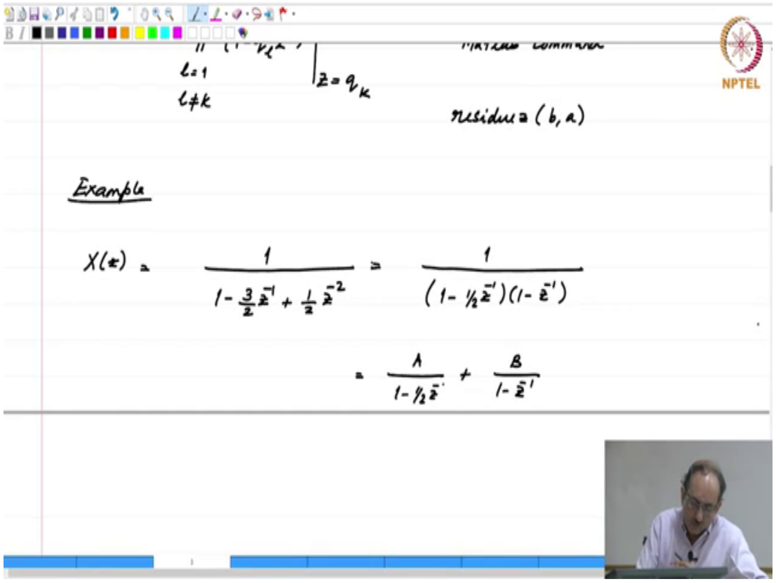

And as a simple example, if $X(z)$ is of this form $\frac{1}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}$, the very first thing that you need to do is to factor the denominator and this is easily seen to be $\frac{1}{(1-\frac{1}{2}z^{-1})(1-z^{-1})}$ and is this ok?

Student: (Refer Time: 12:52).

Yeah thank you, so, this is z^{-2}. So, this is $\frac{A}{(1-\frac{1}{2}z^{-1})} + \frac{B}{(1-z^{-1})}$ and all you need to do is multiply $X(z)$ by $1-\frac{1}{2}z^{-1}$ to cancel this and then evaluate it at $z = 1/2$, alright.

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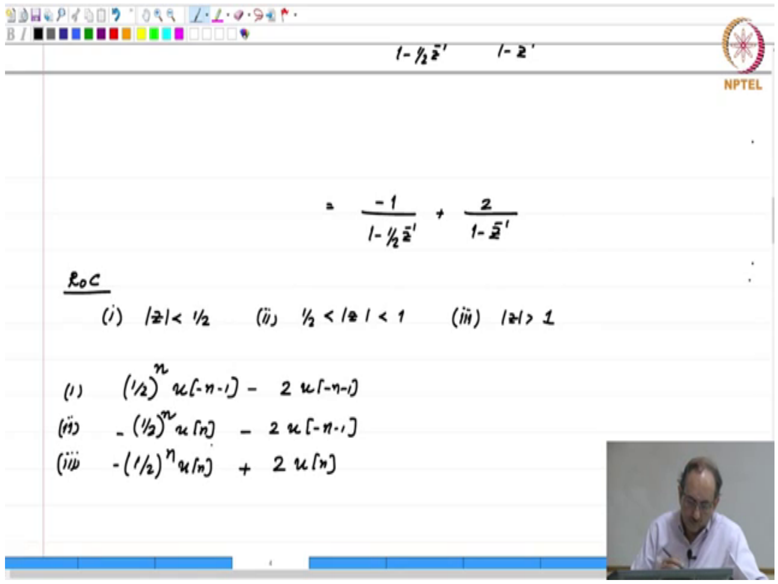

So, what is A, ok? So, this is $\dfrac{-1}{(1-\frac{1}{2}z^{-1})} + \Big(\,\Big)$.

Student: 2 by.

$\dfrac{-1}{(1-\frac{1}{2}z^{-1})} + \dfrac{2}{(1-z^{-1})}$. And, the next step is of course to find the time domain sequence, but to do that you need RoC information, From, so, it is very important to realize that at this point, you cannot find out what the time domain sequence is unless you are given the RoC and clearly there are three different possibilities here.

So, you can have $|z| < 1/2$ or you can have the RoC to be between $1/2 < |z| < 1$, and the final possibility is $|z| > 1$ because you have a pole at $1/2$, you have a pole at $z = 1$. Therefore, three different regions of convergence are possible. And, if this were $|z| < 1/2$, then this becomes $(1/2)^n u[-n-1] + 2u[n]$.

Student: Minus will cancels out Sir.

So, you remember both sequences are left sided, all right. So, this is already there is a minus sign here, right. Therefore, the corresponding time domain sequences is this, right, yeah because this is left sided, its inverse Z-transform is really $(1/2)^n u[-n-1]$.

Student: Second one.

The second one there is a minus sign, yeah right. $(1/2)^n u[-n-1] - 2u[n]$.

Student: (Refer Time: 16:11).

$u[-n-1]$ sure, thank you. $(1/2)^n u[-n-1] - 2u[-n-1]$. And, if it were between $1/2$ and 1 right, the this term must correspond to a right sided sequence. Therefore, this should be $-(1/2)^n u[n]$ and this stays as it is, i.e, $-(1/2)^n u[n] - 2u[-n-1]$. Final of course, is both of them are right sided therefore, this becomes and this is $-(1/2)^n u[n] + 2u[n]$.

So, this; so these are the three possibilities. Actually based on what we have seen so far this one, this should are you reminded of something that we have already seen in one of the properties. One of the properties that we had seen, there is a connection between that and what we have just now discussed, ok. I will tell you what the property is and then I want you to see whether you are able to catch the connection. The property that this is related to is final value theorem.

Student: Vector z has (Refer Time: 18:06).

Yes, remember the final value theorem if it had a nonzero value, we saw that that must contain what kind of sequence in the inverse Z-transform or rather in the time domain signal. It must contain a component that corresponds to $u[n]$, correct and that formula is no different from what is happening here. So, in this particular example, you can see it has this kind of term. Therefore, what will be the final value theorem as applied to this given transform assuming right sided sequence? It will be 2, right.

So, basically what you are doing, when you multiply by $1 - z^{-1}$ and compute the limit as z tends to 1, all you are doing is, you are evaluating the residue at that particular pole, that is all.

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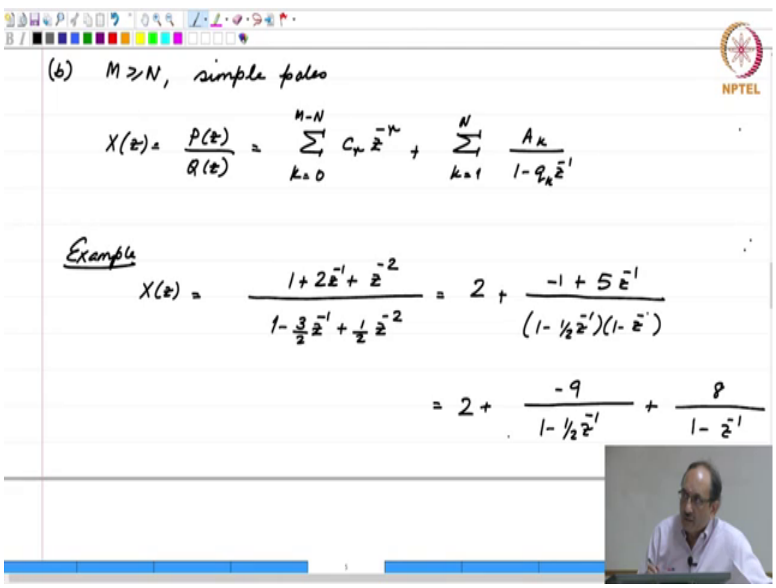

So, this is a simple pole. Now, again simple pole assumption, but it is no longer in proper form. So, $M \geq N$. Therefore, $X(z)$ which is of the form $\frac{P(z)}{Q(z)}$, before proceeding to partial fractions, you have to convert it to proper form and hence you divide and get quotient and remainder. Therefore, if you do that, this will be $k = 0$, the quotient will have degree, what will be the upper limit of this sum here?

Student: M.

$M - N$, $\sum_{k=0}^{M-N} c_r z^{-r}$; so, this will what the quotient will be. And, then of course, the remainder will be in proper form and this if you break it up into residues, you apply what we had seen earlier. So, this goes from going to $\sum_{k=1}^{N} \frac{A_k}{1 - q_k z^{-1}}$. And, as a simple example, if you had the same denominator as before and instead now if you had $\frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}}$, you need to first divide and then apply the partial fraction

expansion, you have to get the residues. And this if you divide so, this will be $2 + \dfrac{(\ )}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}}$.
This of course, can be written as $2 + \dfrac{-1 + 5z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - z^{-1})}$.

This you can check because you should take multiply this, 2×1, you will get $2 - 1$ that will give you 1 and $2 \times (-3/2)$ will give you $-3z^{-1}$, $(-3 + 5)z^{-1}$ will give you $2z^{-1}$. And this again has to be; so, this turns out to be $2 + \dfrac{-9}{(1 - \frac{1}{2}z^{-1})} + \dfrac{8}{(1 - z^{-1})}$ and again as before to proceed from this step to the next in finding the inverse Z-transform, you need RoC information.

Again, since the denominator is the same, you have three possibilities for the RoC; $|z| < 1/2$, $< 1/2|z| < 1$, and $|z| > 1$. So, this is pretty straight forward here. You had already seen partial fraction expansion in, for the Laplace case and this seems no different correct, but is there something that you can spot here that is slightly different or has an advantage over what we have been using before?

Student: (Refer Time: 23:38) is greater than the whatever the portion you get under will be that term (Refer Time: 23:47) it is shifted into.

That is ok. So, yeah actually I should have written that.

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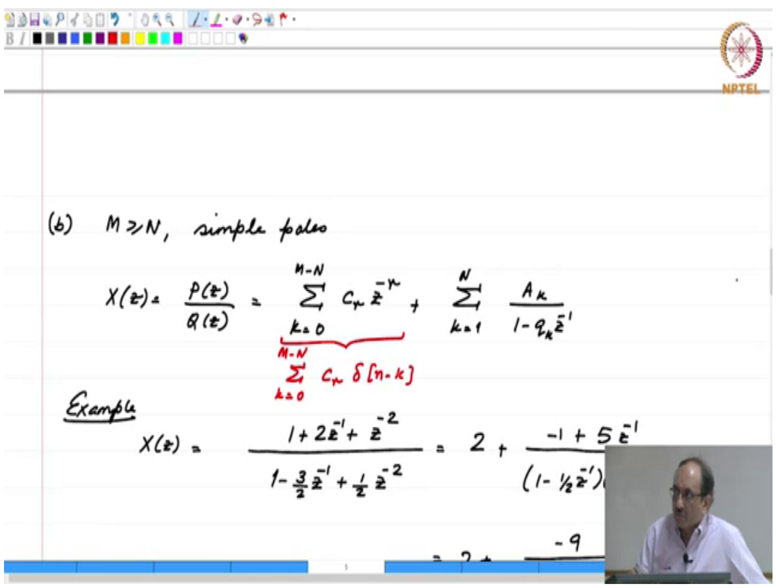

So, in this case, the inverse Z-transform is always $\sum_{k=0}^{M-N} c_r \delta[n-k]$, the inverse Z-transform independent of the RoC, this term will always be as it is, but that is not what I am after. For example, if you look at this particular example, does that trigger one difference in the way; we have been using partial fraction here versus what you mean used to doing partial fractions in the Laplace case.

Student: First of all.

Yes therefore, this a good point. Therefore, what is the consequence of that? Here, if you read if you express this in powers of z, the numerator will be z^2, right. So, it is not in proper form, right. Therefore, you have to do something else or the other shortcut that is normally done is what is what you have suggested, what you can do is you can take the divided by s, make it proper and then try to relate the inverse Laplace transform of $X(s)$ versus $sX(s)$. Whereas, here if this were in powers of z, this will not be in proper form, that is all; whereas, in powers of z^{-1}, this is in proper form and we are dealing with z^{-1} here which is the natural thing to use in this context.

And, just to illustrate this particular example in Matlab.

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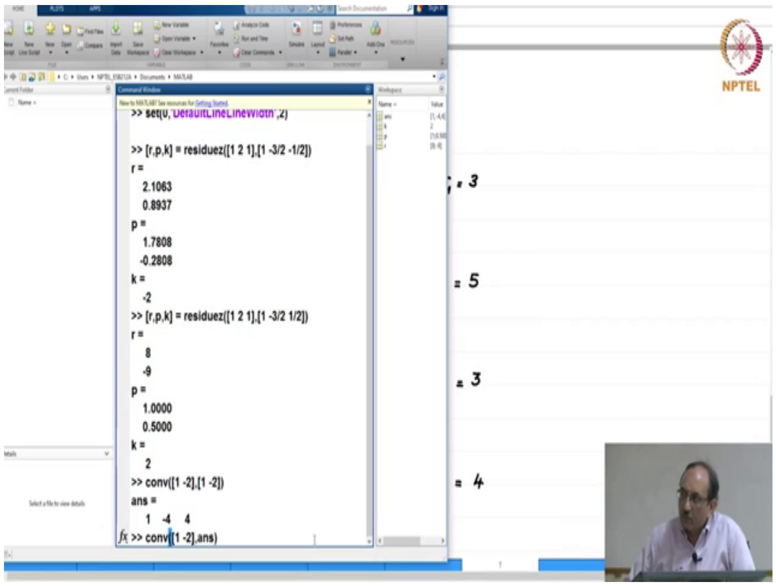

So, the general form is r which stands for the residue, p which stands for the pole and k stands for the quotient term and then this is *residuez*. The b vector in this particular case is [][1 2 1], the latest example that we have considered and the denominator is $[1\ -3/2\ 1/2]$, all right. So, what you will get here for the k will be the quotient polynomial, you will get all the c coefficients. Remember we had this right, the c coefficients is what you will get. So, in this particular case, what do you expect to get as for k, the quotient polynomial is.

Student: 2.

2 therefore, you can expect to get 2 here and then r are the residues and the residues that you will get are -9 and 8 and the p are the poles. So, you will be getting 1/2 and 1. So, let us see that here, did I do this right?

Student: Yes sir.

There you go. So, r are the residues 8 and -9 and look at the order of the poles is the first pole is 1 and the second pole is 1/2. So, corresponding residues are 8 corresponds to the pole at $z = 1$, -9 corresponds to the pole at $z = 1/2$ and the k term corresponds to the quotient. Therefore, MATLAB gives you this very easily. And, to go from this step to the time domain sequence, you need the RoC information.

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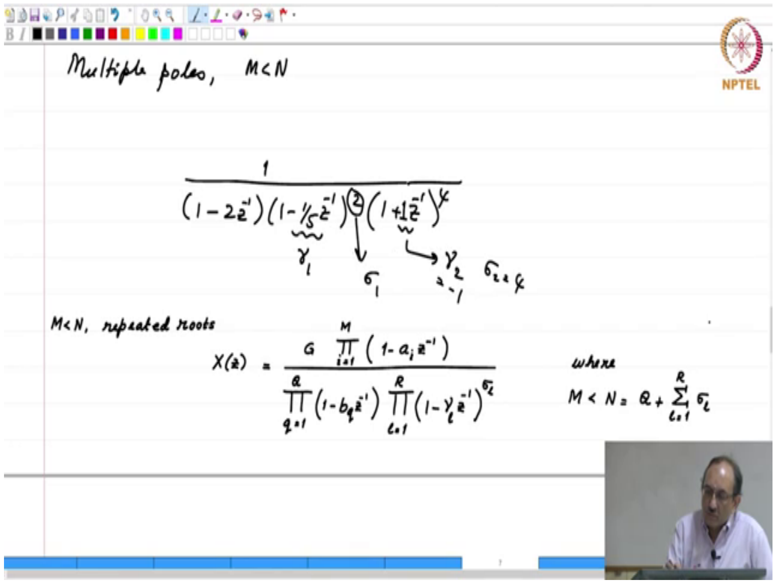

And, the last case is multiple poles that is non-simple, $M < N$. Again, if $M \geq N$, you can always divide and then quotient, you will get similar to what was happening in the previous case. And, once you express it in proper form, what the remainder term will have this. And, since this is messy, I have it already written down here. So, this is $M < N$, repeated roots so, you have the numerator here, all right and then the denominator is broken down into simple poles and multiple poles terms, all right. So, γ_l is a pole with multiplicity σ_l, that is all, all right.

So, as a simple example, if I had $\dfrac{1}{(1-2z^{-1}(1-\frac{1}{5}z^{-1})^2(1+z^{-1})^4}$. So, this is γ_1,; remember I am using gamma to denote the roots that have multiplicity. So, this will be γ_1 and this will be the corresponding σ_1. Similarly, this is actually $1 \times z^{-1}$ therefore, this is γ_2 which is 1, σ_2 is 4.

Student: γ_2 is -1.

γ_2 is -1, yeah right, all right. And, the overall degree of the denominator is, you sum up all these things.

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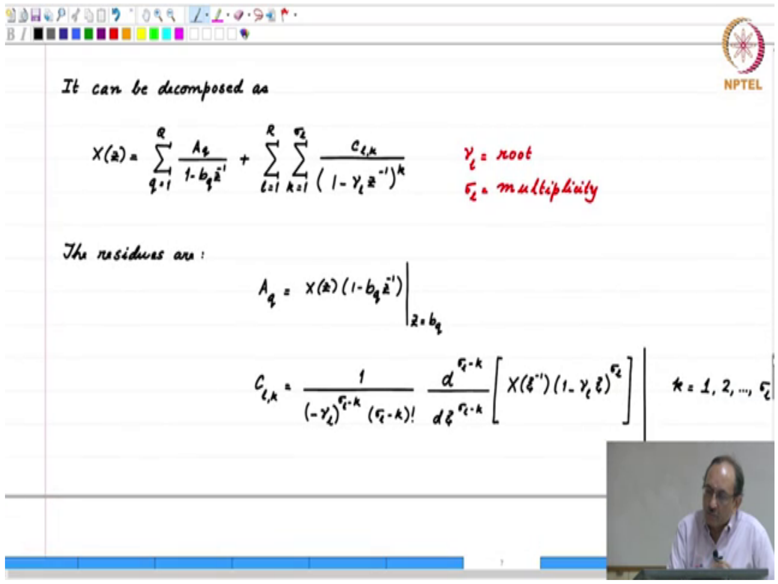

And, now this is how this can be decomposed as, the simple roots as before no change. For the multiple roots, you need this formula here all right. And this is how the residues are computed, for the simple roots no change. For the roots with multiplicity, you have to differentiate as many times as needed and you need to get the residues up to σ_l. So, what this tells you is for this particular example $(1+z^{-1})^4$, you will have 4 residues. For this term $(1-\frac{1}{5}z^{-1})^2$, you will have 2 residues and these are for the same pole locations, you will get multiple residues.

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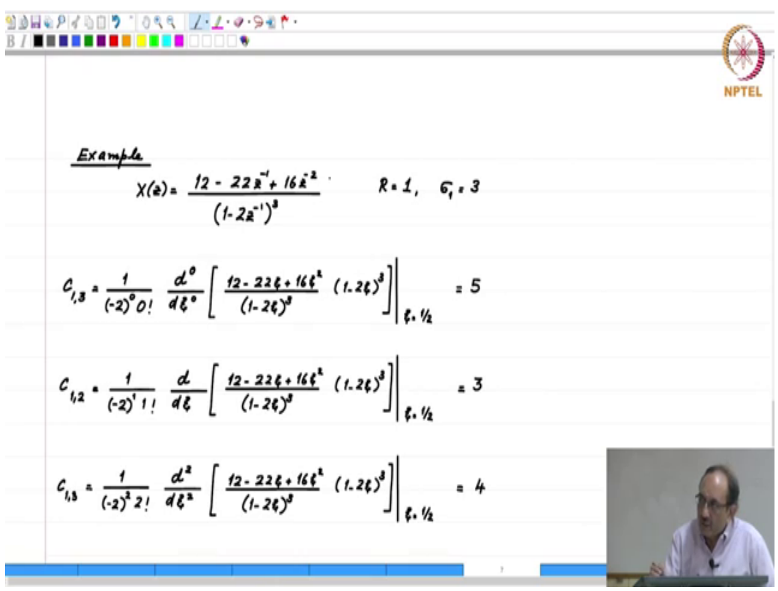

And, this is what the formula is. Again in MATLAB, you can implement this. So, this is $\dfrac{12-22z^{-1}+16z^{-2}}{(1-2z^{-1})^3}$, right.

So, if you had power 2 for this particular root at $z = 2$, you will have 2 residues all right; again this can be done in MATLAB very easily. So, all I need to do is, I need to feed in the numerator coefficient and the denominator coefficient and in this particular example, you see the residues are 5, 3 and 4 and this is already in proper form. Therefore, if I apply this to MATLAB, I will get pole which is at 2, I will get 2 with multiplicity 3. So, the vector p will have three elements, all 2 elements will be the same namely there will be 2, because that is where the pole is. And, then the residues will be 5, 3 and 4 and because this is already in proper form k will have a null vector, all right.

So, let me so, this is $[r, p, k] = residuez$ and then, the denominator is $(1 - 2z^{-1})^3$, right. Therefore, that remember you have to give this as a vector; you have to expand the denominator and give the coefficients as vectors, all right. So, this is $(1-2z^{-1})^3$ is $[1\ -2]$ convolved with itself 3 times. Therefore, you can get the denominator polynomial vector using convolution.

So, $[1, -2]$ convolved with $[1, -2]$. This of course, is if you remember the formula for $(a - b)^3$; you can get it right away. If you want to do it in MATLAB, the way to do that is like this, all right. And, then you need to convolve the answer one more time with $[1, -2]$.

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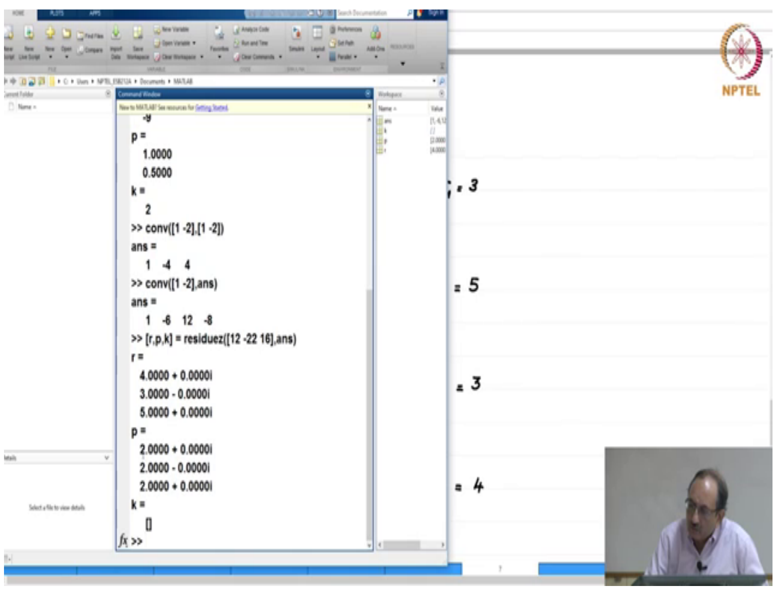

So, this is the denominator vector. Therefore, emphresiduez so, this needs to be of the form $[r, p.k]$ and the numerator is $12, -22, 16$; so, $[12, -22, 16]$ and the denominator vector is $[1, -6, 12, -8]$, since I already have that vector stored in the variable *ans*, I can use that. So, now, you see that the k coefficient is null because it is already in proper form. All the poles are at 2. So, you have p with 2 repeated thrice and these are the residues 4, 3 and 5 which is what we saw was happening; here 5, 3 and 4.

So, exactly MATLAB gives you this and again its good to work a couple of exercises using this formula, but again I do not expect you to remember this; it is not fair to ask multiple roots question on the exam right, because it is tedious, you can make mistakes to remember this is difficult. Therefore, I will not be asking this, may be multiplicity 2 is ok, but beyond that, it is not this one. And once you have found the residues, you still need to compute the inverse Z-transform.

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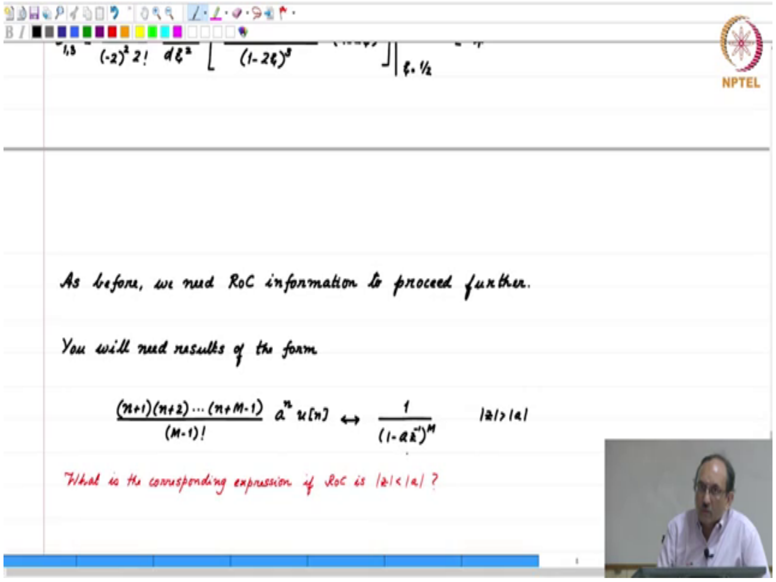

And, now we are talking about poles with multiplicities therefore, you need to know the formulas for $\frac{1}{(1-az^{-1})^M}$. Again, depending upon whether $|z| > |a|$ or $|z| < |a|$, the answer will change. So, the region of convergence will dictate what the left hand side will be.

If you look at this expansion here; if you look at this so, you have found out the residues and k itself will go from 1 to σ_l; therefore, you will have, for the first term you will have $k = 1$. So, it is on the form $\frac{1}{(1-az^{-1})^1}$, then $\frac{1}{(1-az^{-1})^2}$ and so on. So, you need to know the inverse Z-transform of $\frac{1}{(1-az^{-1})^M}$ in general.