Digital Signal Processing
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Lecture 36:
Properties of the Z-Transform (4)
- Final value theorem

So, we had looked at the Initial value theorem in the previous class, we will continue its counterpart which is the Final value theorem.

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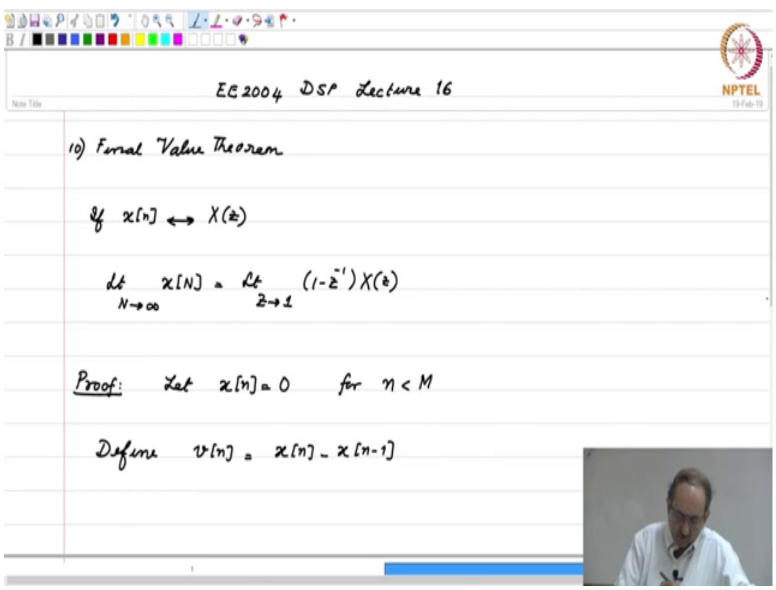

So, if $x[h]$ has Z-transform $X(z)$, then $Lt_{N\to\infty}\ x(N) = Lt_{z\to 1}(1 - z^{-1})X(z)$. So, this is the statement of the final value theorem and we will now prove this.

The assumption is, we will let $x[n]$ to be a right sided sequence. Therefore, $x[n] = 0$ for $n < M$. So, we will now define a sequence $v[n]$ which is nothing but $x[n] - x[n-1]$. And, once you have $v[n]$ defined like this, from the properties of the Z-transform, it is easy to see that $V(z)$ is nothing but $(1 - z^{-1})X(z)$ because this is nothing but $X(z) - z^{-1}X(z)$.

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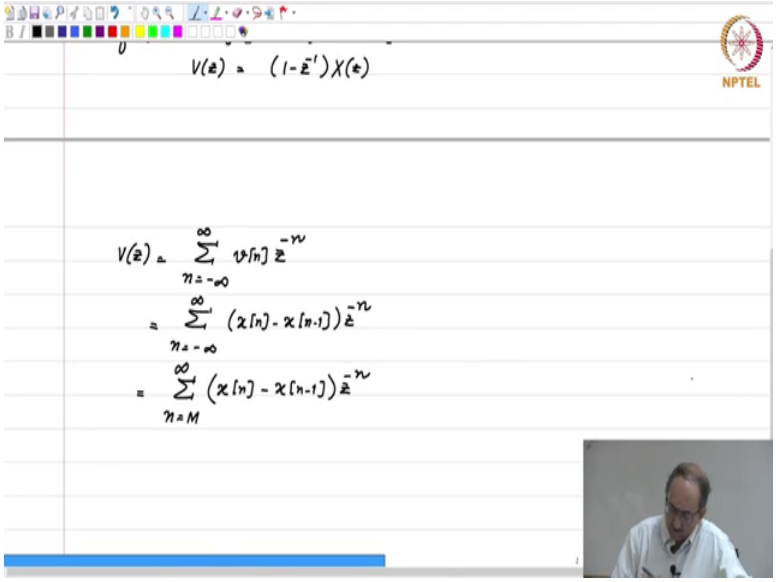

We can also get the Z-transform of $v[n]$ from first principles. Therefore, $V(z)$ is nothing, but $\sum_{n=-\infty}^{\infty} v[n]z^{-n}$. So, $v[n]$ after all is $x[n]-x[n-1]$, but $x[n]=0$ for $n<M$. Therefore, you can alter the range of summation to this.

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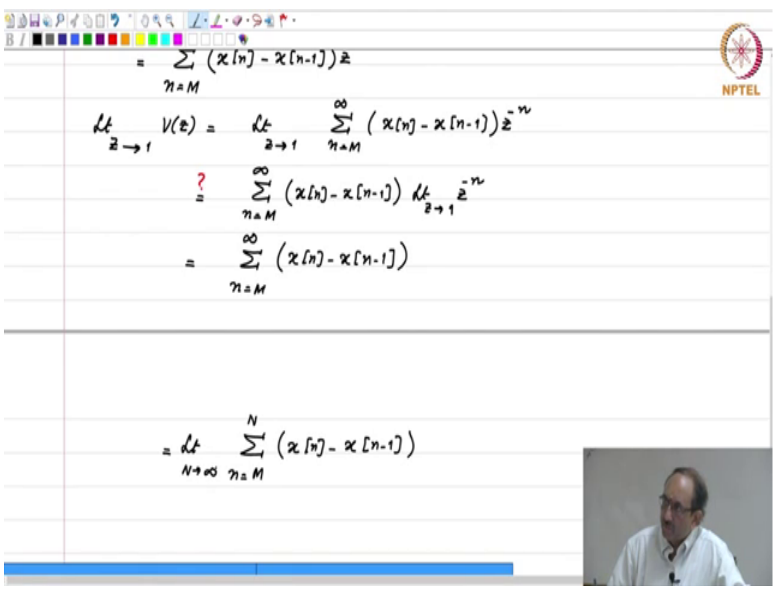

And, now let us take $Lt_{z\to 1}V(z)$. So, this in turn is $Lt_{z\to 1}(\ )$, $V(z)$ after all is $\sum_{n=M}^{\infty}(x[n]-x[n-1])z^{-n}$. Now, we will do what comes naturally to us. We will interchange two limiting processes, but then as always we will put a question mark here to remind us that this is not always true, but truly under certain conditions. Therefore, this is $\sum_{n=M}^{\infty}(x[n]-x[n-1])\ Lt_{z\to 1}\ z^{-n}$. And, this simplifies to $\sum_{n=M}^{\infty}(x[n]-x[n-1])$ because $Lt_{z\to 1}\ z^{-n}=1$ after all.

Now, we can rewrite this as $\sum_{n=M}^{N}(x[n]-x[n-1])$, but the upper limit really we want it to be ∞. We have replaced it with N; therefore, we will let $Lt_{N\to\infty}$.

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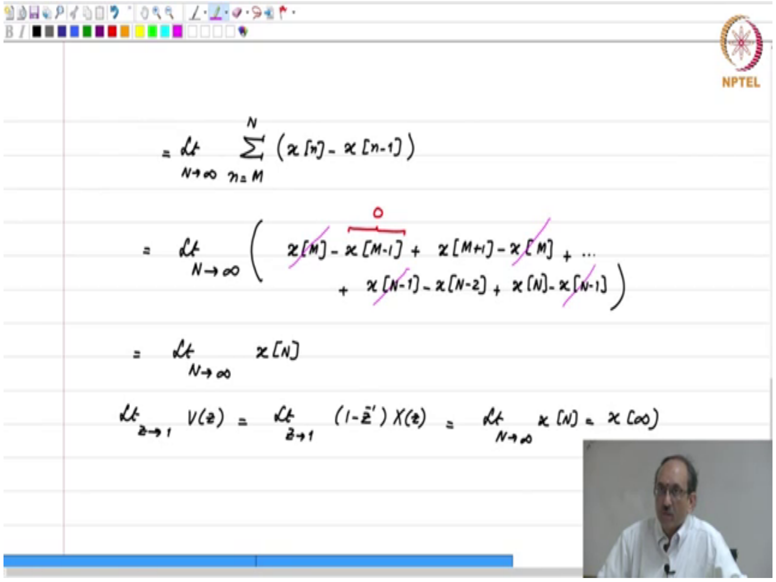

So, now, we have $Lt_{N\to\infty}$ and then we will write out terms of the summation. So, this after all is $Lt_{N\to\infty}\Big(x[M]-x[M-1]+x[M+1]-x[M]+\ldots+x[N-1]-x[N-2]+x[N]-x[N-1]\Big)$. When $n=M+1$, this becomes $x[M+1]-x[M]$. And this in turn when you put the upper limit to be $N-1$, this becomes $x[N-1]-x[N-2]$. And, for the last index namely N, this becomes $x[N]-x[N-1]$ because the sequence is 0 for $n<M$, $x[M-1]=0$ by assumption. And, this term $x[M]$ cancels with this.

Similarly, this $x[M+1]$ will get cancelled due to the next term and so on. And, this $x[N-1]$ gets cancelled with this. So, if you look at this based on the pattern that is out here, the only term that will survive within these parentheses is $x[N]$. Therefore, this is really $Lt_{N\to\infty}\ x[N]$. The left hand side is really $V(z)$, right because that is what we started off with.

We started off with the transform of $V(z)$; not only that the left hand side is $Lt_{z\to 1}\ V(z)$. Therefore, this is $Lt_{z\to 1}\ V(z)$ which in turn is $Lt_{z\to 1}\ (1-z^{-1})X(z)$.

On the other side, you have $Lt_{N\to\infty}\ x[N]$. So, this is N tending to ∞ of $x[N]$, this is really $x[\infty]$ which is really the statement of the final value theorem. So, just to make sure you do not get lost in the intermediate steps.

Some of the things we need to get a feel for; for example, going from this step to this what we did was, we replaced upper index of ∞ with N and then let N tends to ∞ and the reason you are going through this step, intermediate step of replacing the upper index with N and then taking $Lt_{N\to\infty}$ is needed because?

Student: (Refer Time: 09:55).

Yeah, say it again?

Yeah. So, for example, you might think if I expand this out of $x[n] - x[n-1]$, if I write this out, you might also think you can cancel adjacent terms, correct? If you write this out without going through the step of $Lt_{N \to \infty}$, you might also want to cancel terms pair wise and why is it that we are not doing that? You can naively write this term out and then do exactly what we did in this step, right.

Student: (Refer Time: 10:38).

So what about $x[\infty]$? Actually, there is this external real where you can have infinity to the set of real numbers and make it part of the set of numbers. Suppose, one argues like that, what would be your answer to that? Now, really if you this cancellation applies only to finite sums, you cannot cancel terms when you have infinite terms in the summation.

That is why we need to go through this intermediate steps of being converting this to finite sum, canceling and then taking the limit, that is all.

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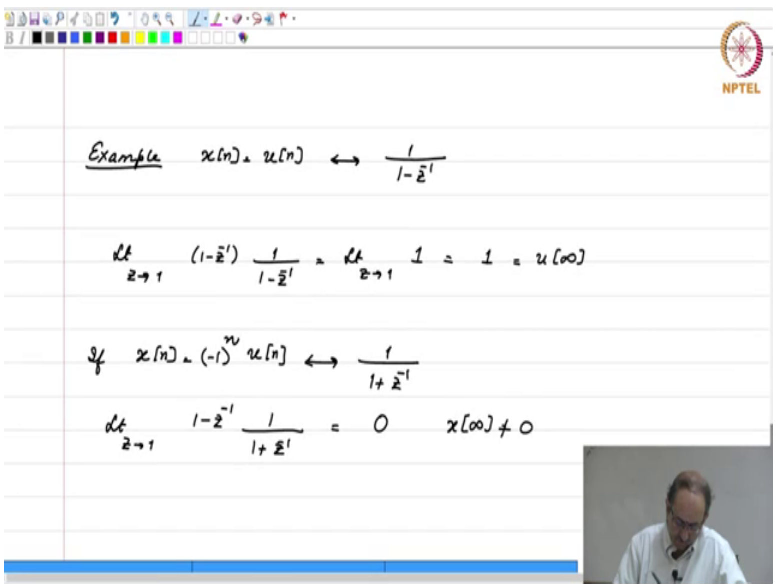

And, as a simple application of this, suppose you have $x[n] = u[n]$ so, this is really the transform of $u[n] \longleftrightarrow \dfrac{1}{1-z^{-1}}$ and therefore, if you now take $Lt_{z \to 1}\ (1-z^{-1})X(z)$; $X(z) = \dfrac{1}{1-z^{-1}}$. So, this is $Lt_{z \to 1}\ 1$ which is 1 and $u[\infty]$ is indeed 1.

And, similar to what was done in the Laplace transform case, when you were told about the initial and final value theorem and the final value theorem was illustrated with one example and it was immediately followed by another example. Do you remember what that was?

Student: (Refer Time: 12:42).

Very good. So, suppose, this is the counterpart to that; so, if $x[n] = (-1)^n u[n]$; so, this transform is $\dfrac{1}{1-z^{-1}}$ and then $Lt_{z \to 1}\ (1-z^{-1})\dfrac{1}{1-z^{-1}}$ which is $(1-z^{-1})X(z)$; $X(z) = \dfrac{1}{1-z^{-1}}$ and this turns out to be 0, but clearly $x[\infty]$ is not equal to 0.

Therefore, this the counterpart to that example. So, what this really means is if the limit exists in both the domains, this theorem will tell you that they are equal. Whereas for the second example, namely

$x[n] = (-1)^n u[n]$, the limit really does not exist as n tends to ∞.

Therefore, the sequence does not have a limit in the time domain therefore, you cannot hope to get that by using the final value theorem. And, the intuition behind that is very similar to what was happening in the Laplace case. There the final value theorem was, what was the statement of the final value theorem in the Laplace case? limit s tending to, of?

tudent: (Refer Time: 14:52).

$sX(s)$. So, here it is $(1 - z^{-1})X(z)$. So, really what is happening there is, suppose the sequence had a non-zero limit as n tended to ∞ similar to the function $x(t)$ having a non-zero limit as t tended to ∞ there in the continuous-time case, suppose if this had a nonzero limit, what is it in the signal that had to be present for this limit to exist?

Student: (Refer Time: 15:36).

Yeah, steady state and you have a non-zero limit. What is it that the signal contains?

Student: (Refer Time: 15:43).

We are talking about the time domain, why do you want to bring in pole? Let us go one step at a time.

Student: (Refer Time: 15:53) values are study state sinusoid.

If it were a steady state sinusoid, we do not have a final limit? It cannot have a limit if it were a sinusoid, correct. So, it had to have?

Student: (Refer Time: 16:12).

No, you are looking for a final non-zero value in the time domain as t tends to ∞ in the continuous-time case. So, what signal must be present, what as one of the components for this to be true?

Student: (Refer Time: 16:32).

Right, we are talking about the time domain.

Student: (Refer Time: 16:39).

It should contain $u(t)$, right. It should contain $u(t)$ and the $u(t)$ has transform what? $\frac{1}{s}$, right. So, now, you are canceling the pole at $s = 0$ when you multiply by s. So, that is why you have limit $Lt_{s \longrightarrow 0} sX(s)$, all right. Similarly, here in the discrete-time case, if the sequence has to have a non-zero final limit as n tends to ∞, what component must be present in the time domain sequence?

Student: $ku[n]$.

$ku[n]$, right and $u[n]$ has transform?

Student: (Refer Time: 17:28).

$\frac{1}{1 - z^{-1}}$. So, what is it that you are doing here? You are multiplying by $(1 - z^{-1})$ and then taking the $Lt_{z \longrightarrow 1}$, right because that is where the pole is. So, that is all. So, the intuition here is very similar to the intuition that that works or that is present for the final value theorem in the Laplace case, continuous-time case.

There you needed to have $u(t)$ present in the signal and you are extracting that by multiplying the transform by s and then taking the limit as s tending to 0, here the pole is at $z = 1$ and your canceling that by $1 - z^{-1}$ and then taking the limit as z tending to 1, that is all.

So, if you understand Laplace very well, you can see the very close similarities between Laplace and Z.