Digital Signal Processing

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Lecture 33:
Properties of the Z-transform (4)
- Convolution in the time-domain
- Frequency domain filtering

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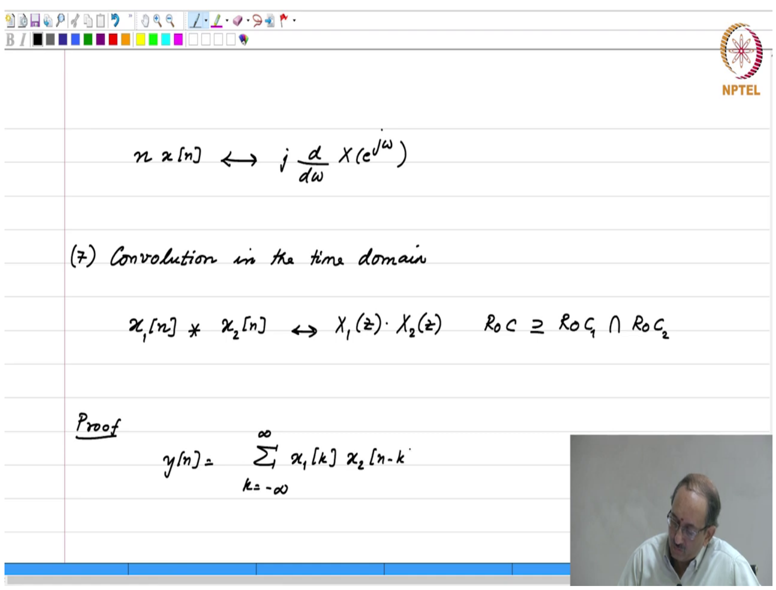

So, next property is similarities to what was happening in the continuous-time case. There, if you convolved two functions $x_1(t)$ and $x_2(t)$, the corresponding transforms got multiplied; exactly the same property holds here. Therefore, $x_1[n]$ convolved with $x_2[n]$ has Z-transform $X_1(z).X_2(z)$ with RoC at least as large as the intersection of RoC_1 and RoC_2. Again, as in the linearity case, the RoC can be larger if there are pole-zero cancellations. The proof is extremely simple. So, you have $y[n] = \sum_{n=-\infty}^{\infty} x_1[k]x_2[n-k]$. So, this is the definition of the convolution.

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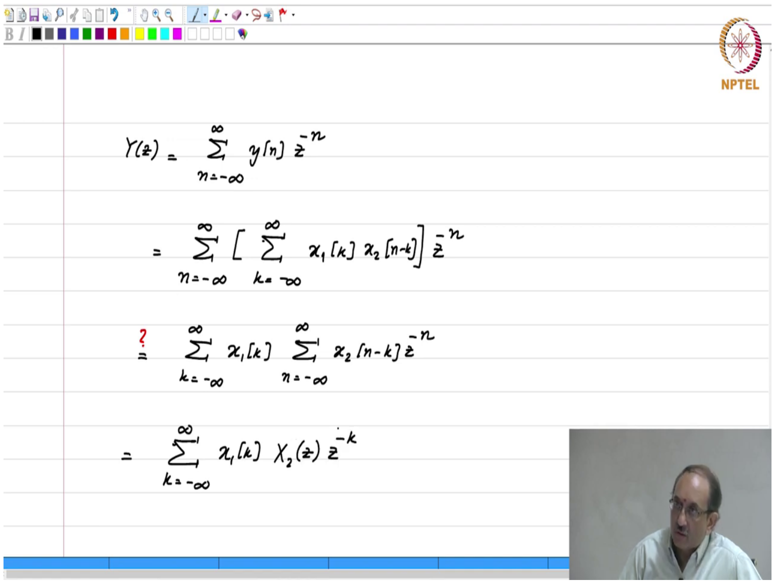

Therefore, $Y(z) = \sum_{n=-\infty}^{\infty} y[n]z^{-n}$ and this is wherever y is there, replace $y[n]$ by this expression. Therefore, this is $\sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k] \right] z^{-n}$. And, now again we will instinctively interchange these two summations and to satisfy the mathematicians in us, we will put a question mark because this is to only under certain conditions. Therefore, this becomes $\sum_{k=-\infty}^{\infty} x_1[k] \sum_{n=-\infty}^{\infty} x_2[n-k]z^{-n}$. It is actually $x_2[n-k]$.

Therefore, this is $\sum_{k=-\infty}^{\infty} x_1[k] \sum_{n=-\infty}^{\infty} x_2[n-k]z^{-n}$. This $\sum_{n=-\infty}^{\infty} x_2[n-k]z^{-n}$ is the Z-transform of $x_2[n-k]$. Therefore, we will use the delay property. Therefore, this becomes $\sum_{k=-\infty}^{\infty} x_1[k]X_2(z)z^{-k}$.

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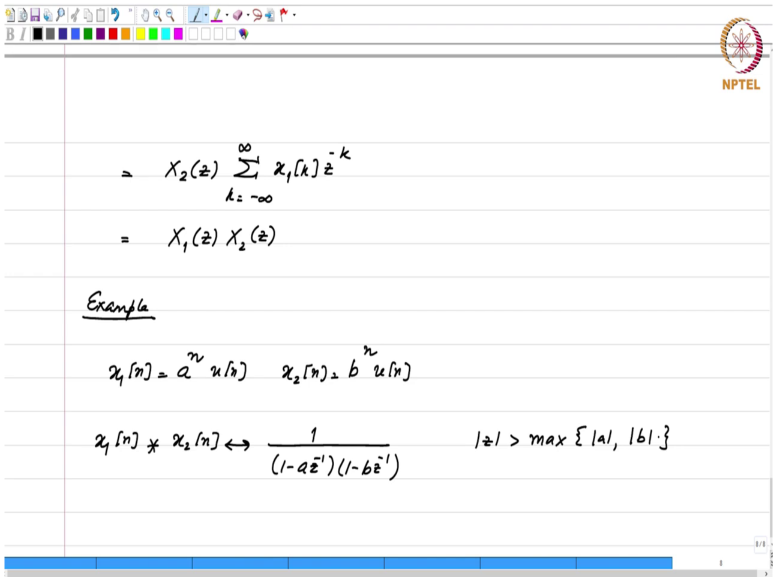

And, this is nothing, but $X_2(z) \sum_{k=-\infty}^{\infty} x_1[k]z^{-k}$, which my definition is $X_1(z)$. Therefore, this is indeed $X_1(z).X_2(z)$. And, as a simple illustration of this, if you had $z_1[n] = a^n u[n]$ and $x_2[n] = b^n u[n]$, then $x_1[n] * x_2[n]$ would have transformed $\frac{1}{(1-az^{-1})(1-bz^{-1})}$ and the RoC will be $|z|$?

Student : (Refer Time: 04:01).

Very good; so, $|z| > max\{|a|, |b|\}$.

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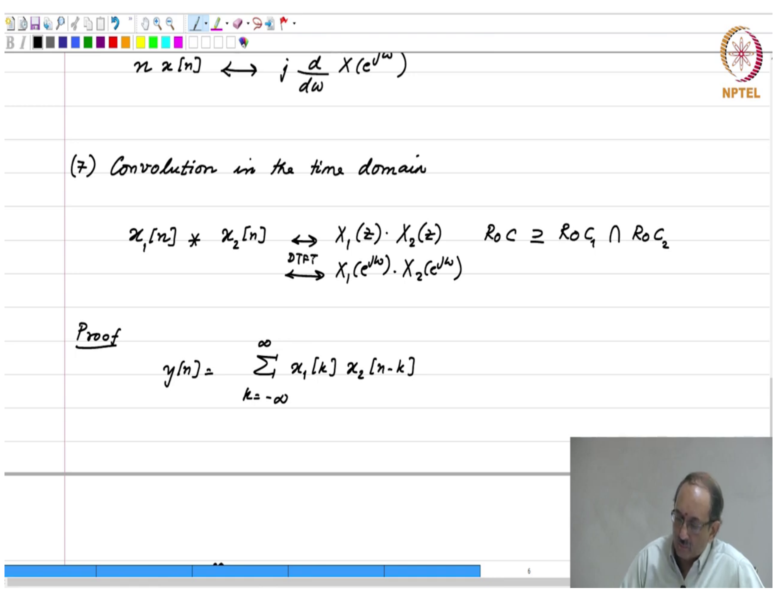

The corresponding DTFT is nothing, but $X_1(e^{j\omega}).X_2(e^{j\omega})$.

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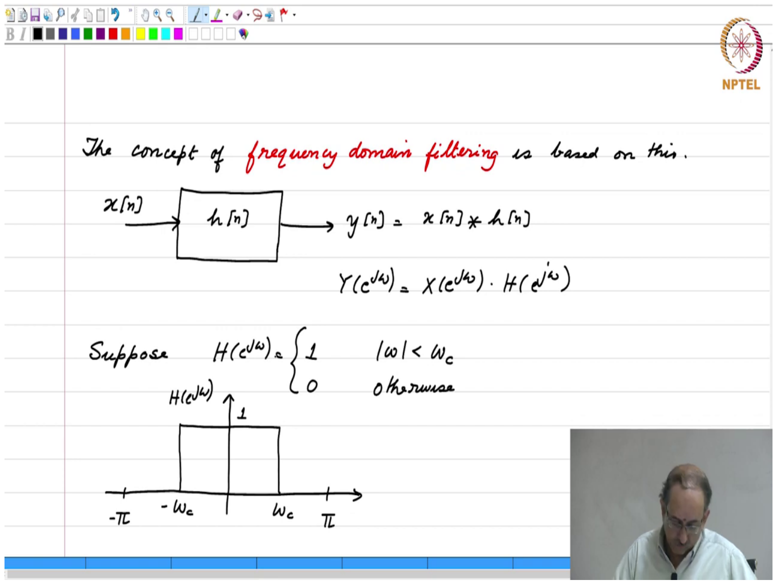

This is one of the most important properties because this is the property that gives rise to the concept of filtering. So, the concept of frequency domain filtering is based on this. So, suppose I have a system with input $x[n]$, its impulse response is $h[n]$ and then I have $y[n]$ which is nothing, but $x[n] * h[n]$ assuming the system is LTI.

Therefore, $Y(e^{j\omega}) = X(e^{j\omega}).H(e^{j\omega})$. Now, let us assume that $H(e^{j\omega})$ is 1 for $|\omega| < \omega_c$ and 0, otherwise. Therefore, frequency response $H(e^{j\omega})$ is like this. So, this after all is the DTFT. Therefore, this is between $-\pi$ to π.

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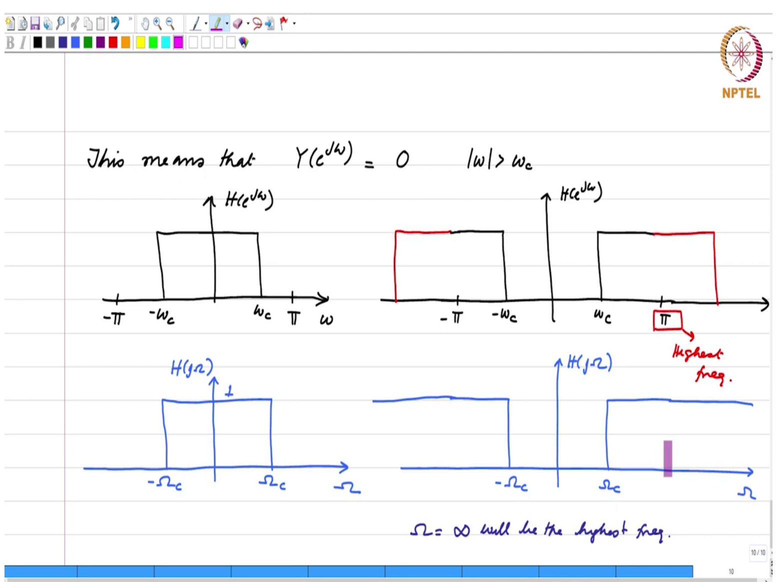

So, this means that $Y(e^{j\omega})$ is guaranteed to be 0 for $|\omega| > \omega_c$. So, any frequency component of the input that is outside $-\omega_c$ to $+\omega_c$ gets filtered out. So, this is no different from what was happening in the continuous-time case where you had looked at things like ideal low pass filter, high pass, band pass and band stop.

Now, let us revisit high pass and low pass just to reinforce some of the similarities and also see some of the differences. So, this is your low pass filter. Now in continuous-time case, this is your low pass filter. The crucial difference of course is this; if you draw this from $-\infty$ to $+\infty$, this figure will not change. From Ω_c to ∞, all frequencies will have zero weight. Similarly, from $-\Omega_c$ to $-\infty$, all frequencies will get completely cut out. Whereas, here remember the highest frequency in continuous-time is $+\infty$; whereas, the highest frequency in the discrete-time case is π.

Therefore, when you talk about filtering all frequencies above ω_c, you are really talking about frequencies in the range ω_c to pi. And, if you draw this picture beyond $-\pi$ to π, for example if you extended this, then you will see at $\omega = 2\pi$, exactly the same picture will get repeated. What happened at 0, will also happen at 2π, what happened at 0 will also happen at -2π and so on. So, if you extend this picture beyond the range that is plotted which is $-\pi$ to π, you will see periodic repetitions in the DTFT case whereas, in the continuous-time case, this is it.

So, this is your ideal low pass filter. If you now talk about the corresponding high pass case, if the cut off frequency is ω_c, it will pass all frequencies above ω_c. That is what the ideal high pass filter will do; from 0 to omega c, it will cut things out. So, this is what is going to happen. So, this is ω_c, $-\omega_c$, this is $H(e^{j\omega})$ and the corresponding high pass in continuous-time is this. The big difference of course is from Ω_c, it will pass all frequencies beyond $\Omega)_c$ therefore, this will go like this. On the other hand, in the discrete-time case, this will also go from ω_c to the highest frequency only, but now the highest frequency is π.

Now, if you complete the picture, this somehow the response will look, alright. So, this is the response for the ideal high pass filter in the discrete-time case because if you periodically repeat this what happened at 0 will also happen at 2π and also happen at -2π and so on. So, if you drew this with periodicity 2π, you will find that this is really part of the next repetition. Similarly this is part of the repetition, that is happening here because what happened at 0 will also happen at -2π. And, they are centred around -2π. If you drew this, you will see that these are the responses corresponding to the frequency response that occurs at -2π. Similarly, this is the arm of the response that happens at $\omega = 2\pi$.

Therefore, so this is the highest frequency in the discrete-time case whereas, in continuous-time case, $\Omega = \infty$ will be the highest frequency, alright. So, notice carefully the differences between these two cases. This is continuous-time; this is discrete-time. Continuous-time, things go from $-\infty$ to ∞; discrete-time case, you are always limited to the range between 0 to 2π or between $-\pi$ to π. And, in continuous-time case, the highest frequency is ∞; in discrete-time case, the highest frequency is π.

So, this is one of the most important properties because in practice, convolution and correlation happen all the time. A convolution is used for filtering, your cell phone does filtering all the time; your front end signal, it has to filter it to eliminate out of bad noise and to concentrate only on frequency of interest. In this context, we talk about Ω going from ∞ to ∞ here. And, then we had also talked about z actually, somebody made this remark in response to one of my questions. I did not respond to and elaborate on that answer, then somebody said $-\infty$ when it came to z, right. What about ∞ in the variable z, for example, because somebody did say z going from $-\infty$ to $+\infty$ and so on.

Student: It is complex.

Yeah. So, it is complex therefore?

Student: $|z|$ ∞.

So, $|z|$ being ∞, alright; so, if $|z|$, you can think of this as a circle $|z| = r$ and then let r going to ∞, right. So, does it mean that there are infinite number of points? When you are talking about real numbers, you have $x = +\infty$ and $x = -\infty$, right. What about $|z|$ then? Are there infinite number of infinities there?

Student: (Refer Time: 13:20).

Yeah. So, that is precisely the question. So, it doesn't mean there are infinite number of infinities, because we on a circle and if you let the angle vary continuously, there are uncountable infinite number of points, correct. So, have you thought about what it means infinity when the variable is complex. So, you need to look up this Riemann's sphere, alright; look up Riemann's sphere and what it does is, it is a mapping in which points in the 2D plane will be mapped on to the points on a sphere, and you will find that all points at infinity will get mapped to what is called the north pole of the sphere.

So, in this mapping, it is one to one an invertible. Therefore, when the variable is complex, ∞ is really only one point and that in the Riemann's sphere mapping, it maps to the north pole. So, you need to be clear that when we talk about complex variables, when we say $z = \infty$, it is only one point. Therefore, there it does not make sense to talk about $-\infty$ and ∞, alright.