Digital Signal Processing
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Lecture 30:
Properties of Z-transform (3)
-Complex conjugation

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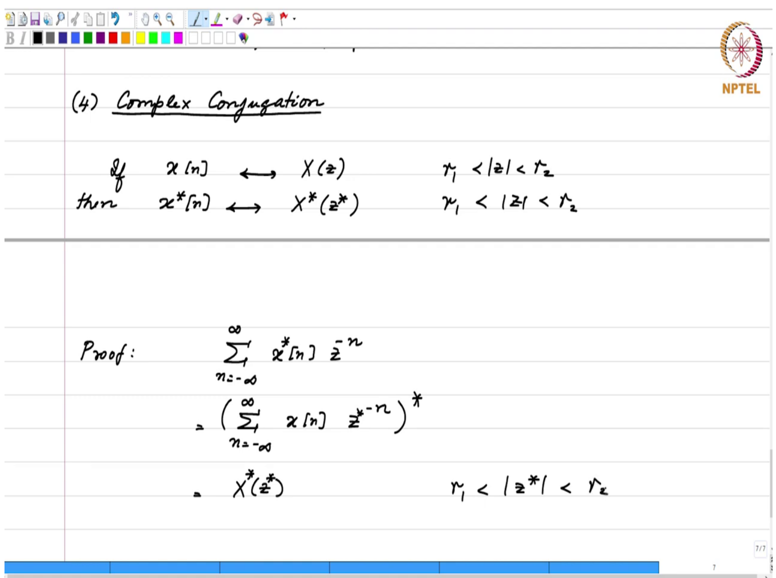

We will now look at Complex conjugation, so if $x[n] \longleftrightarrow X(z)$, then $x^*[n] \longleftrightarrow X^*(z^*)$. And RoC remains the same and the proof is really simple. Trying to find the Z-transform of $x^*[n]$, $\sum_{n=-\infty}^{\infty} x^*[n]z^{-n}$.

So, you can take the complex conjugate outside to get rid of this because you want to relate this to the transform of $x[n]$. But then if you do this, this is not what the previous line is. To make it equal to the previous line, you have to replace z by z^*. Therefore, this by definition is $\left(\sum_{n=-\infty}^{\infty} x[n]{z^*}^{-n}\right)^*$, what is inside the brackets is $X(z^*)$.

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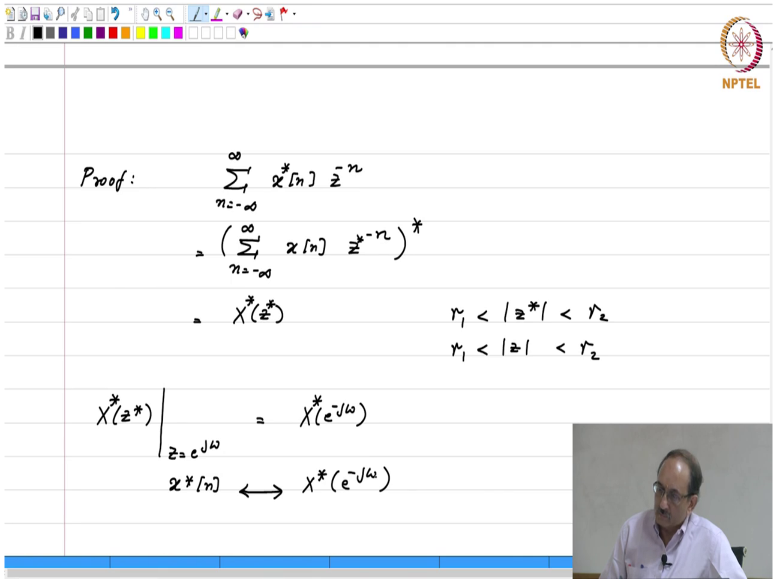

And to account for the final complex conjugate, you have $X^*(z^*)$. And therefore, now you have Z^* must now belong to the region of convergence which is nothing, but $r1 < |z| < r_2$ and the DTFT property is very simple. If you have $X(z^*)$ and if you replace z by $e^{j\omega}$, you get $X(e^{-j\omega})$. But, really you want $X^*(z^*)$ therefore, you have $X^*(e^{-j\omega})$. Therefore, this implies that $x*[n]$ has DTFT $X^*(e^{-j\omega})$.

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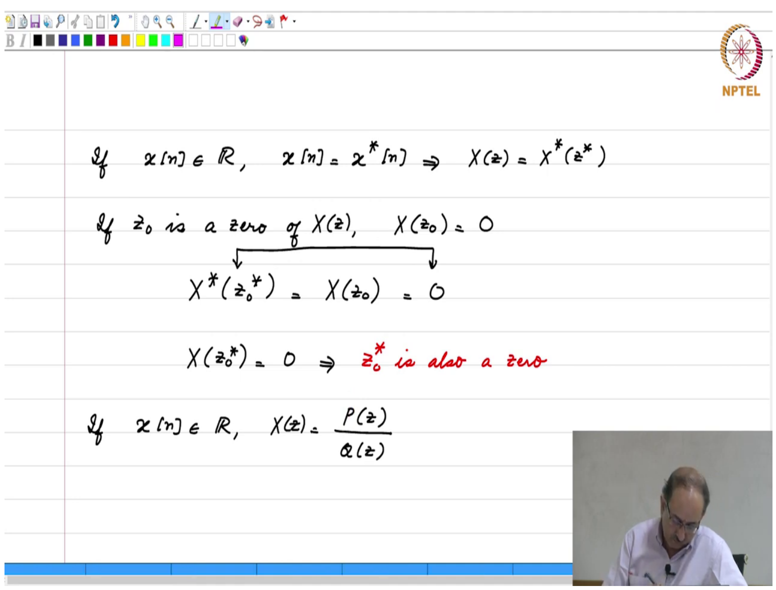

And, one consequence of this is if $x[n]$ were real valued, then $x[n] = x^*[n]$; therefore, this implies that $X(z)$ must be the same as $X^*(z^*)$. And a consequence of this is that if z_0 is a zero of $X(z)$, then $X(z_0) = 0$, but $X(z_0)$ is the same as $X^*(z_0^*)$, so this is the same as $X(z_0)$ and this is therefore, $X^*(z_0^*) = 0$.

Now, if you look at these two complex conjugate both sides; therefore, this becomes $X(z_0^*)$, 0's complex conjugate is 0; therefore, this implies that z_0^* is also a zero. Therefore, if the sequence is real valued, its Z-transform will have poles and zeros that are complex conjugates of each other. So, this argument applies also for poles as well, because you can write $X(z) = \dfrac{P(z)}{Q(z)}$ and this is the same as $\dfrac{P^*(z^*)}{Q^*(z^*)}$.

So, this is another way of saying that if you have a real valued sequence, if $x[n]$ were real valued, it is $X(z)$ if it is since you are restricting ourselves to the class of rational transfer functions. So, this of the form $\dfrac{P(z)}{Q(z)}$ and P and Q are after all polynomials in z and these polynomials will have real valued coefficients.

And when you study theory of equations, we have a polynomial that has real valued coefficients, you are guaranteed that the roots will be in complex conjugate pairs. So, this is another way of arriving at the same result, when you know the sequence is real valued based on the property that $X(z)$ is the same as $X^*(z_0^*)$.

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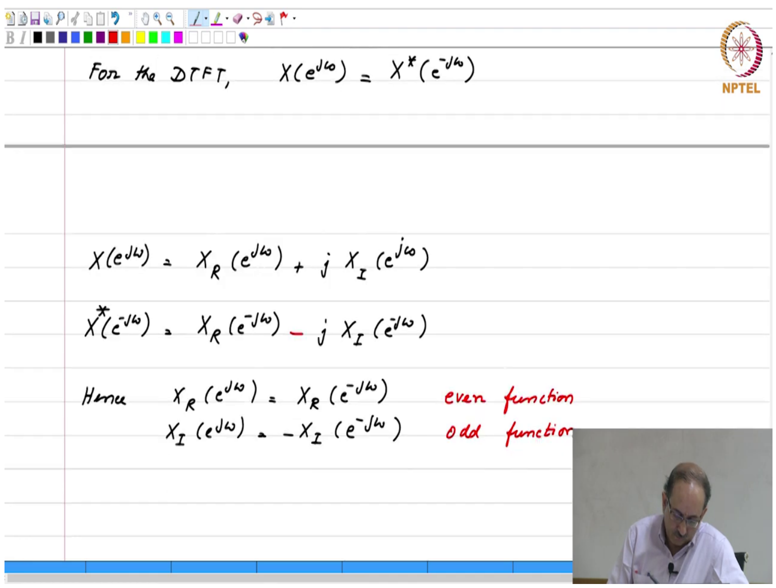

And for the DTFT, the corresponding property is $X(e^{j\omega}) = X^*(e^{-j\omega})$. And, $X(e^{j\omega})$ in general is a complex quantity as a function of ω; therefore, this can be written as $X_R(e^{j\omega}) + jX_I(e^{j\omega})$, where X_R and X_I are the real and imaginary parts of $X(e^{j\omega})$.

Therefore, if you replace now $X(e^{j\omega})$ by $X(e^{-j\omega})$, you have $X_R(e^{-j\omega}) + jX_I(e^{-j\omega})$. Here, wherever ω is there, I have replaced ω by $-\omega$, but what we really want is, we want $X^*(e^{-j\omega})$.

Therefore, $X^*(e^{-j\omega})$, now all you have to do is replace this by $X_R(e^{-j\omega}) - jX_I(e^{-j\omega})$. And these two have to be equal because the sequence is real. So, hence, you have $X_R(e^{j\omega}) = X_R(e^{-j\omega})$ and $X_I(e^{j\omega}) = -X_I(e^{-j\omega})$. Therefore, $X_R(e^{j\omega})$ is an even function and $X_I(e^{j\omega})$ is an odd function. And, hence if you look at magnitude squared.

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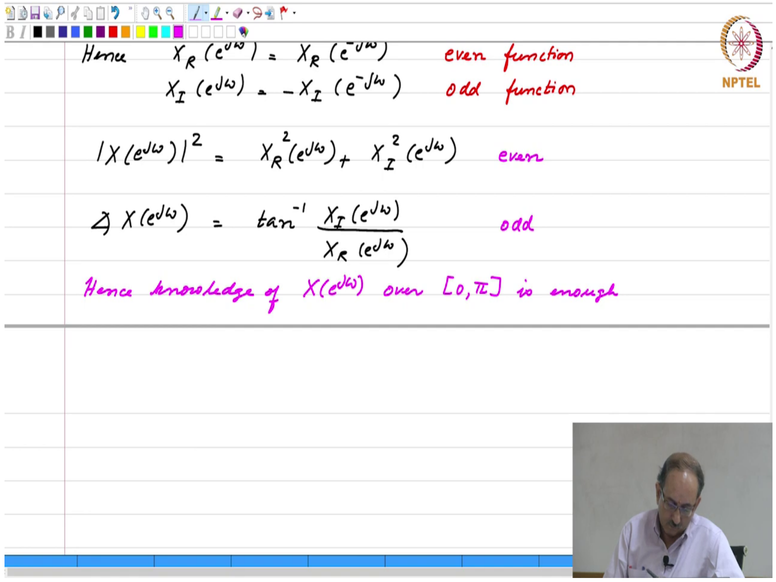

So, this is $|X(e^{j\omega})|^2 = X_R^2(e^{j\omega}) + X_I^2(e^{j\omega})$ and this again is even, because square of an even function is even, square of an odd function is again even. And, the angle is $\tan^{-1} \dfrac{X_I(e^{j\omega})}{X_R(e^{j\omega})}$. And, this is odd; the ratio of odd to even is odd, tan inverse is an odd function therefore, this is odd.

Therefore, exactly mirroring what happened in the continuous-time case for the Fourier transform. If we had a function that was real valued, your continuous-time Fourier transform, the magnitude was even and the phase was odd. And, there if you knew the transform from 0 to ∞, you knew the transform from $-\infty$ to $+\infty$.

Similarly, here because of this symmetry, if you know the transform from 0 to π, you can complete the transform from π to 2π. What is happening from π to 2π is the complex conjugate of what was happening in 0 to 2π. So, hence knowledge of $X(e^{j\omega})$ over 0 to π is enough to specify the transform over the entire interval between $-\pi$ and π or 0 to 2π. One thing that this exponentiation, the modulation example I had shown here.

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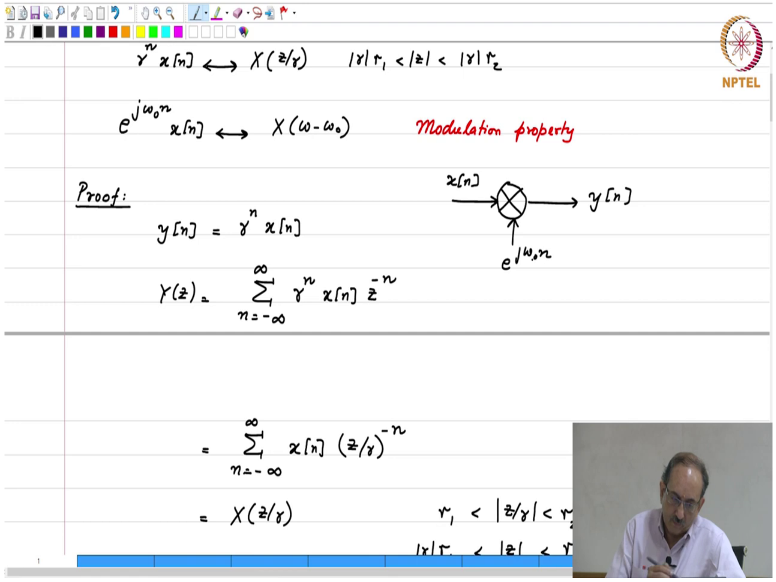

Typically, this is given by this block diagram, so you have $x[n]$ and you have $e^{j\omega n}$ and this is $y[n]$. And this modulator is a linear system. This is time variant because of the time varying gain.