Digital Signal Processing
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Lecture 29:
Properties of Z-transform (3)
- Exponential multiplication

Let us get started. We are looking at the Properties of Z-transform. We had seen linearity and time delay. And, time delay, we introduced the notion of fractional delay there. So, you can shift a sample by an amount that is not necessarily an integer number of samples and we illustrated that with a MATLAB examples, where we had taken $\cos\left(\frac{n\pi}{5}\right)$ and shifted it by 2.5 samples by applying $e^{-j2.5\omega}$ in the frequency domain.

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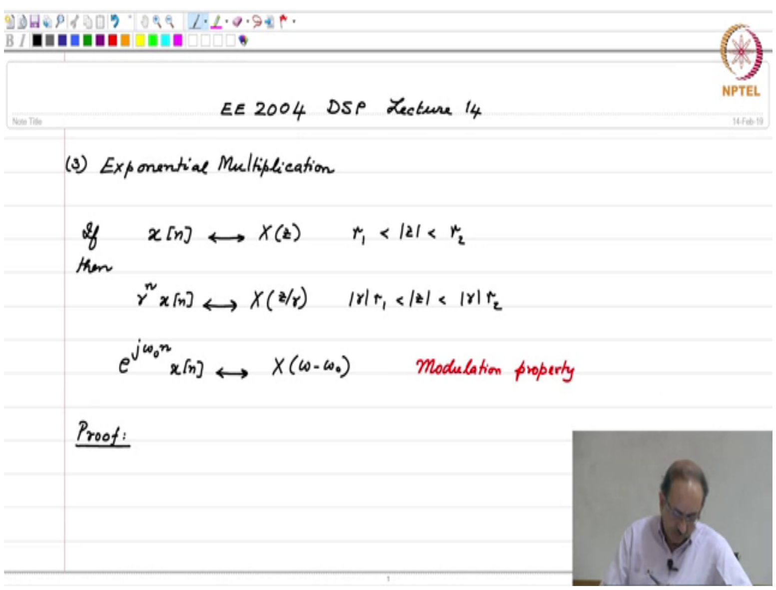

So, let us continue and look at the next property. So, this is exponential multiplication. Suppose, you had $x[n]$ with Z-transform $X(z)$ with this given region of convergence $r_1 < |z| < r_2$, then $\gamma^n x[n]$ has Z-transform $X(z/\gamma)$ and the ROC gets modified to $|\gamma|r_1 < |z| < |\gamma|r_2$. The corresponding property for the DTFT is typically this, $e^{j\omega_0 n}x[n]$, this has DTFT $X(\omega - \omega_0)$. And noticed here, I have used the $X(\omega)$ notation because it is more convenient and this is called as the modulation property. So, if you have to shift in the frequency domain, you have to modulate in the time domain. The proof is really very simple for this.

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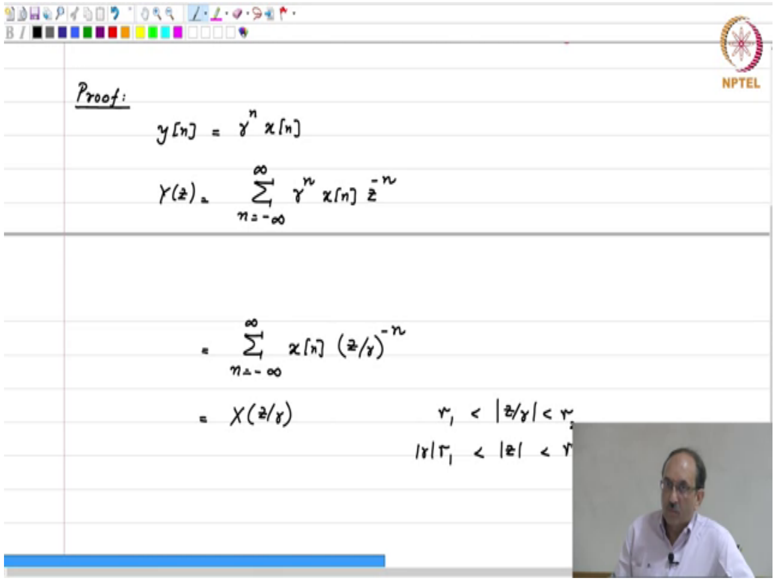

So, we are looking at the Z-transform of $y[n]$ which is nothing, but $\gamma^n x[n]$. Therefore, $Y(z) = \sum_{n=-\infty}^{\infty} \gamma^n x[n] z^{-n}$. And, this is written as $\sum_{n=-\infty}^{\infty} x[n](z/\gamma)^{-n}$. And, this is indeed $X(z/\gamma)$ and the RoC is, wherever z is there, you now have z/γ. Therefore, z/γ has to be part of the region of convergence and this is commonly written as $|\gamma|r_1 < |z| < |\gamma|r_2$. So, this proves the property there.

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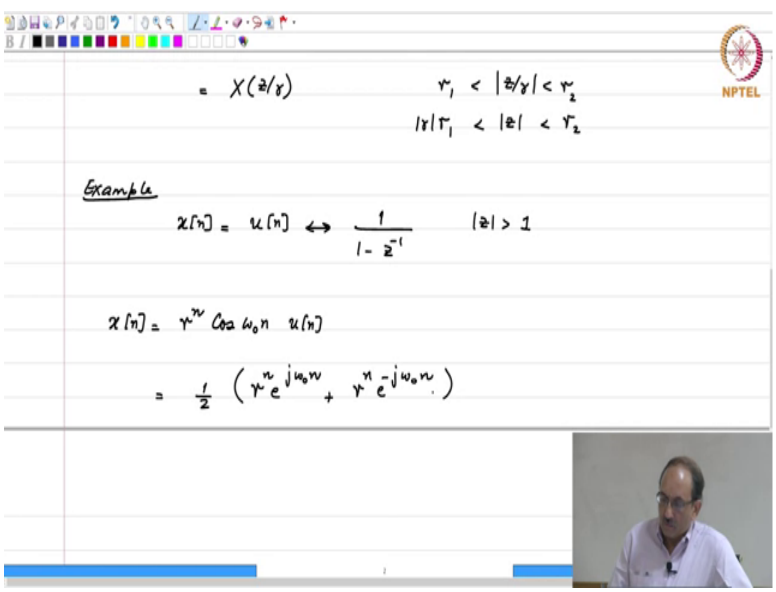

And as a simple example, suppose, we start off with this $x[n] = u[n]$. So, this is a special case of $a^n u[n]$, where $a = 1$. So, this Z-transform is clearly $\dfrac{1}{1 - z^{-1}}$ with $|z| > 1$. So, starting from this, we will get the Z-transform of $x[n] = r^n \cos(\omega_0 n) u[n]$. We will derive the transform of this. You can derive the transform of this from first principles, but we will use this property to derive the transform starting from the known Z-transform of $u[n]$. So, this is nothing, but $\dfrac{1}{2}\big(r^n e^{j\omega_0 n} + r^n e^{-j\omega_0 n}\big)$.

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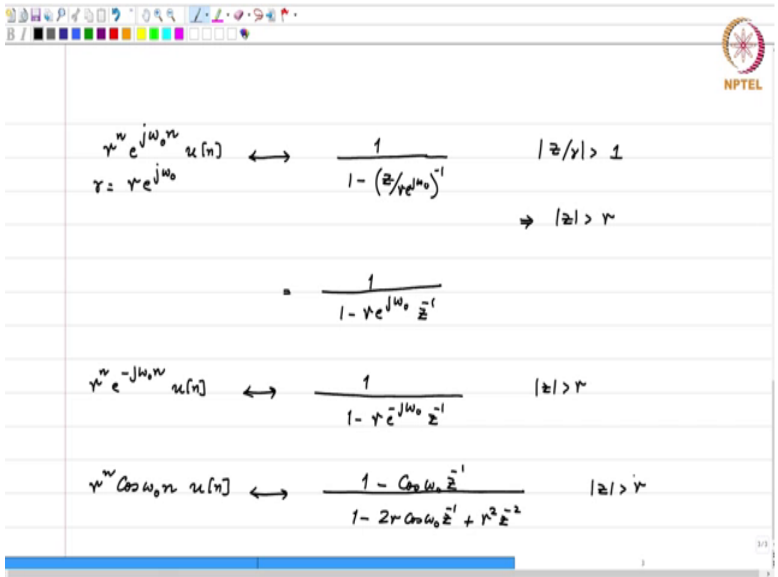

Therefore, if you consider $r^n e^{j\omega_0 n}u[n]$ so, we will immediately apply the modulation property. So, here, $\gamma = re^{j\omega_0}$. Therefore, this becomes $\dfrac{1}{1-()}$. Wherever z is there, you replace that by z/γ. Therefore, this is $\dfrac{1}{1-(z/re^{j\omega_0})^{-1}}$ and you also have to take care of the RoC.

So, this is now $|z/\gamma| > 1$ and this now becomes $|z| > r$ because γ after all is $re^{j\omega_0}$. You take modulus, this thing disappears and therefore, this becomes $\dfrac{1}{1-re^{j\omega_0}z^{-1}}$. Similarly, $r^n e^{-j\omega_0 n}u[n]$, this transform will be $\dfrac{1}{1-re^{-j\omega_0}z^{-1}}$. RoC continues to be $|z| > r$ and you add these two factors and then you also have a factor of $1/2$ because it is $\cos(\omega_0 n)$.

Therefore, $r^n\cos(\omega_0 n)u[n]$, the Z-transform, if you add these two factors up multiplied by $1/2$, you will get $\dfrac{1-\cos\omega_0 z^{-1}}{1-2r\cos\omega_0 z^{-1}+r^2z^{-2}}$. So, properties help you to get transforms quickly. You can plot the pole-zero plot for this $X(z)$, right.

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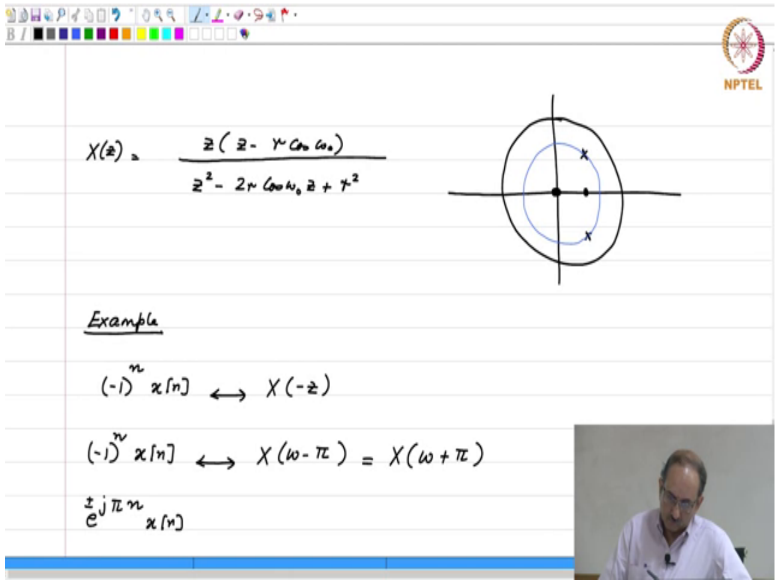

So, $X(z) = \dfrac{z(z - r\cos\omega_0)}{z^2 - 2r\cos\omega_0 z + r^2}$. So, there is a trivial zero and there are two poles because we have simplified this, you need to find the roots of this, but really the factors are $1 - re^{j\omega_0}z^{-1}$ and $1 - re^{-j\omega_0}z^{-1}$. These are the two factors. Therefore, the poles are at $re^{j\omega_0}$ and $re^{-j\omega_0}$. Therefore, you have two complex conjugate poles at radius r.

So, here I have chosen $r < 1$, just for illustration. So, this is the so, this is circle of radius r where the poles are. And the zero, one zero is at $z = 0$ which is the trivial zero, the other zero is at $r\cos\omega_0$. Therefore, it is right here. It is on the line vertical line joining the two poles. If $r = 1$, these poles will lie on the unit circle; if $r > 1$, these poles will lie outside the unit circle. Here I have shown r to be less than 1.

Similarly, you can work out the transform of $r^n \sin(\omega_0 n)u[n]$ along similar lines. Suppose, you had $\gamma = -1$, this will be $X(-z)$ because it is $X(z/\gamma)$; $\gamma = -1$. The corresponding DTFT is, this the same as $e^{j\pi n}$. If it is $e^{j\omega_0 n}$, it is $X(\omega - \omega_0)$; ω_0 happens to be π. Therefore, this is $X(\omega - \pi)$ and this is also same as $X(\omega - \pi)$ because $(-1)^n$ is $e^{\pm j\pi n}$. Let us look at what this exponential multiplication does in the z-domain.

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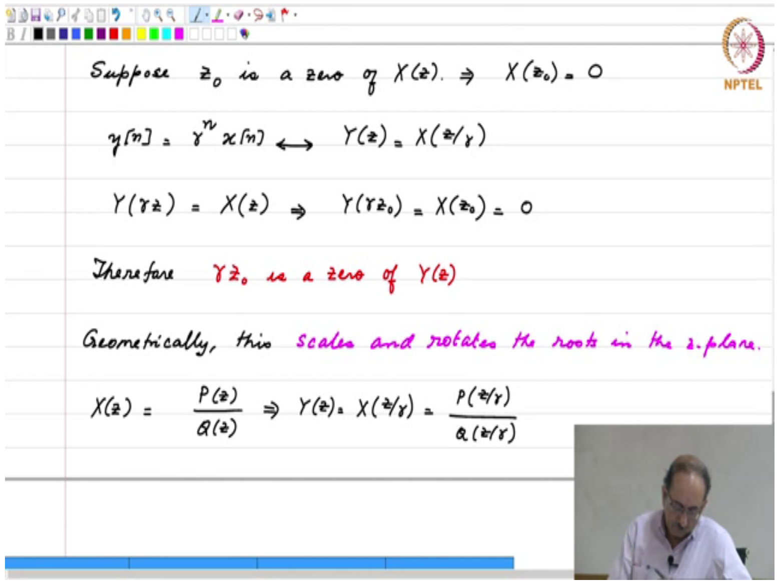

Suppose, z_0 is a zero of $X(z)$. So, this implies that $X(z_0) = 0$. Now, if you consider $y[n] = \gamma^n x[n]$. This Z-transform is $Y(z)$. So, this is $X(z/\gamma)$, just repeating the property here. Now, what is $Y(\gamma z)$? $Y(\gamma z)$ is nothing, but $X(z)$; wherever z is there, replace it by γz. So, this implies $Y(\gamma z_0) = X(z_0)$, but $X(z_0) = 0$. Therefore, γz_0 is a zero of $Y(z)$.

The implication of this is that, remember γ is in general a complex constant; z_0 is a zero in the complex plane. Therefore, γz_0 means you are taking the original zero, z_0 and scaling it by magnitude of γ and rotating it by the angle of γ. Therefore, exponential multiplication causes scaling and rotation of the roots in the z-plane. Therefore, so, geometrically, this scales and rotates the roots in the z-plane.

Note that, I had started off by illustrating this by showing that if z_0 where a zero, the new zero for the exponentially multiplied sequence is γz_0. So, this scaling and rotation is not just limited to the zeros, it is limited to both poles, as well as zeros. Because, this is after all $X(z)$ is of the form $\dfrac{P(z)}{Q(z)}$. So, this implies the transform of $\gamma^n x[n]$ is $Y(z)$ which is nothing, but $X(z/\gamma)$. So, this in turn is $\dfrac{P(z/\gamma)}{Q(z/\gamma)}$. So, this means that both poles and zeros get scaled and rotated.

Again, this is not just an idle property, in the sense that it is not just mathematical. It has an important application in say for example, like a speech coder. In speech coding, what they do is they do perceptual weighting.

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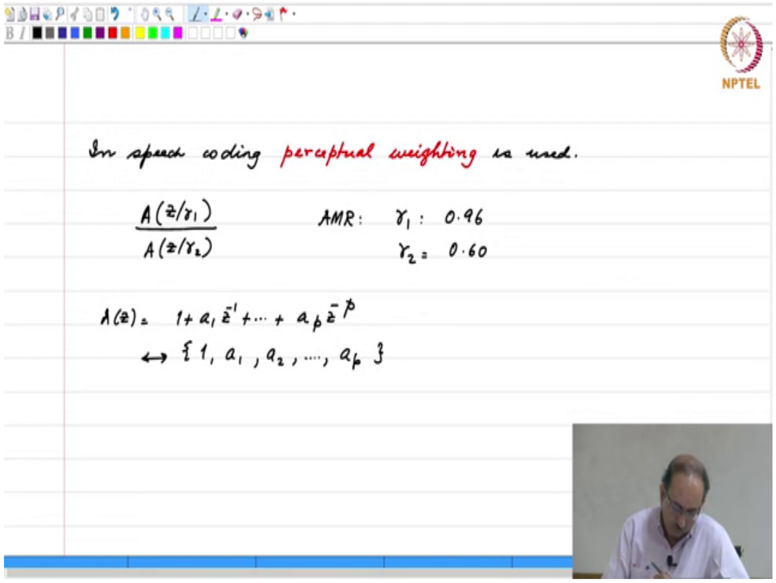

So, perceptual weighting is used. And the motivation for this is after all for coded speech, the final judge of what has been coded is your ear and ear has some special properties. It has what is called masking. So, if you have one tone that is played, you will hear it; if there is another tone, by tone we mean a pure sinusoid.

If that tone, the next tone that is played along with the first tone say it is weaker, that is it is smaller in amplitude. But, if you adjust the frequency such that the weaker tone's frequency is close to the stronger tone, if it comes close enough, you will not hear the weaker tone. The weaker tone will be perceived only if it is a reasonably away in frequency from the stronger tone. And, this is what is called as the masking property of the ear.

So, what they do is when they quantize and when there is this error spectrum that is there, if you shape the error spectrum such that you put error energy more in frequencies where there is more energy in the signal itself. In those cases, the error will be masked. So, these perceptual weighting, these are very heavily used in your actual speech coders. So, when you are able to compress your music to much smaller values compared to your raw sampling that is used, if you say you sample it at 16 kilo Hertz and even if you use 16 bits or if you sample it at 44.1 and use 16 bits, that sell a lot of samples and bits. And the file size will be very large.

On the other hand, if you compress it using say mp3, you get significant reduction in file size that is the storage. But, you do not perceive too much of a difference and this is because all these psycho acoustic properties are exploited and the error spectrum is shaped, so that the error is distributed in regions of the spectrum where you do not hear it, all right. So, in that context, this perceptual weighting is used. And in speech, the vocal tract which is the technical term for the mouth and that is modelled as an all pole filter. And this all-pole filter notationally is typically called as $A(z)$. It is modelled as $1/A(z)$ and $A(z)$ is estimated. And the perceptual weighting that is used is $\dfrac{A(z/\gamma_1)}{A(z/\gamma_2)}$.

So, this is the perceptual weighting filter that is used and one of the common commonly used speech coders is AMR adaptive multi rate and in AMR $\gamma_1 = 0.96$, $\gamma_2 = 0.6$. And, typically $A(z) = 1 + a_1 z^{-1} + \ldots + a_p z^{-p}$ and these coefficients are estimated, $a_1, \ldots, a_p$. And for the perceptual filter, you will multiply this sequence.

So, in the time domain, this is nothing, but $\{1, a_1, a_2, \ldots, a_p\}$ and this time domain sequence is multiplied by γ^n, where $\gamma = 0.96$ and 0.6 and so on. And using this, perceptual weighting filter is created and this is used to improve the quality of the speech coders. So, this property has very significant practical use in speech coding.