Digital Signal Processing
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Lecture 23:
Z - transform (3)
Recursive implementation of FIR filters

Let us continue from where we left last class. We got introduced to poles and zeros, and the poles and zeros in the z plane is not different from the concept of poles and zeros that you had encountered in the Laplace transform case. And, one difference is we are going to call poles or zeros at $z = 0$ as trivial poles or zeros and we will see why they are called trivial poles or zeros when we later look at frequency response.

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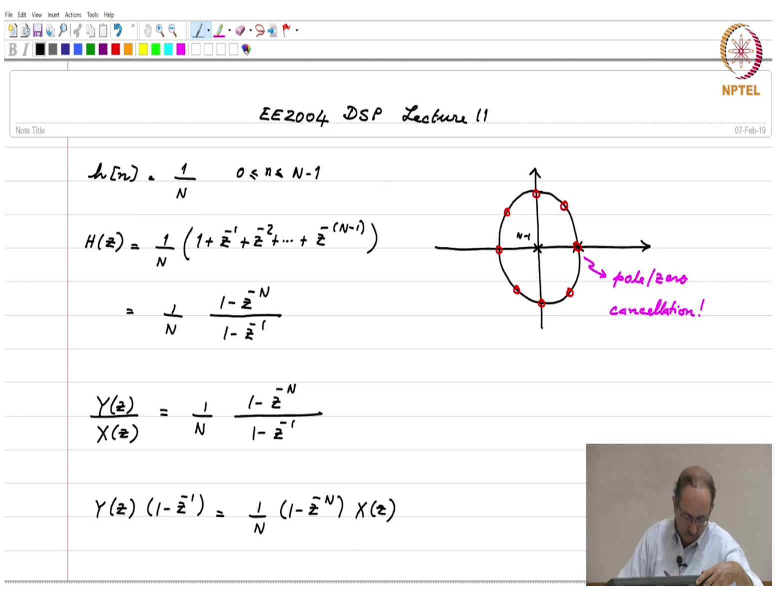

And, we saw some examples of systems having poles and zeros and we also introduced the concept of poles or zeros at ∞. And, the example that we had looked at towards the end of last class was this, we had $h[n] = \dfrac{1}{N}$, $0 \le n \le N-1$. And, then the corresponding $H(z)$, we apply this straight forward definition, it is $H(z) = \dfrac{1}{N}\Big(1 + z^{-1} + z^{-2} + \ldots + z^{-(N-1)}\Big)$. And, this can be written as $\dfrac{1}{N}\cdot\left(\dfrac{1 - z^{-N}}{1 - z^{-1}}\right)$. And in terms of pole-zero plots, there is a pole at, we also saw that there is an $(N-1)$th order trivial pole. And the remaining roots or the zeros of the transfer function and they are the Nth roots of unity and if you take as an example $N = 8$, you will have 8 zeros distributed on the unit circle uniformly.

Therefore, here you will have a zero at $z = 1$ and these are the remaining zeros. And, we pointed out that at $z = 1$, there is a pole-zero cancellation. And, we made the remark that if there were an uncancelled non-trivial pole in the transfer function, the corresponding inverse Z-transform will have a factor of the form $\frac{K}{1-az^{-1}}$. And, if you have a factor of the form $\frac{K}{1-az^{-1}}$, the inverse transform depending upon the RoC will either be of the form $a^n u[n]$ or $-a^n u[-n-1]$.

In either case, it is an exponential that last forever, it is a one sided exponential that is of infinite duration. Therefore, if you had an un cancelled non trivial pole, it will give rise to an exponential that is of infinite duration, a semi infinite duration; either to be right sided or left sided. So, moment you have such a component in the impulse response, such a system cannot be FIR. Whereas, the system given here is indeed FIR and in this form, a pole appears at $z = 1$, it has to necessarily get eventually cancelled. And to see the connection between this and the equation that we had written earlier where we had taken this exact same system and then we wrote it in two different form; one was the non-recursive form, the other was the recursive form.

So, to see the connection in that and this, $H(z)$ is after all $\frac{Y(z)}{X(z)}$, we have not formally introduce the notion of transfer function yet. But this is similar to what was happening in the Laplace transform case, where you have $H(s)$ is nothing but $\frac{Y(s)}{X(s)}$. So, drawing upon that analogy, $H(z)$ can be thought of as the system transfer function which in turn is $\frac{Y(z)}{X(z)} = \frac{1}{N}\left(\frac{1-z^{-N}}{1-z^{-1}}\right)$ and if you cross multiply, you have $Y(z)(1-z^{-1}) = \frac{1}{N}.\left(1-z^{-N}\right)X(Z)$.

After we are done with this, we are going to look at properties of Z-transform and after linearity, the second property we are going to look at is the delay property. So, I will make use of that property now and write this in the time domain and very soon you will be able to see why this is indeed true. So, if you take the inverse Z-transform and write the above in the time domain.

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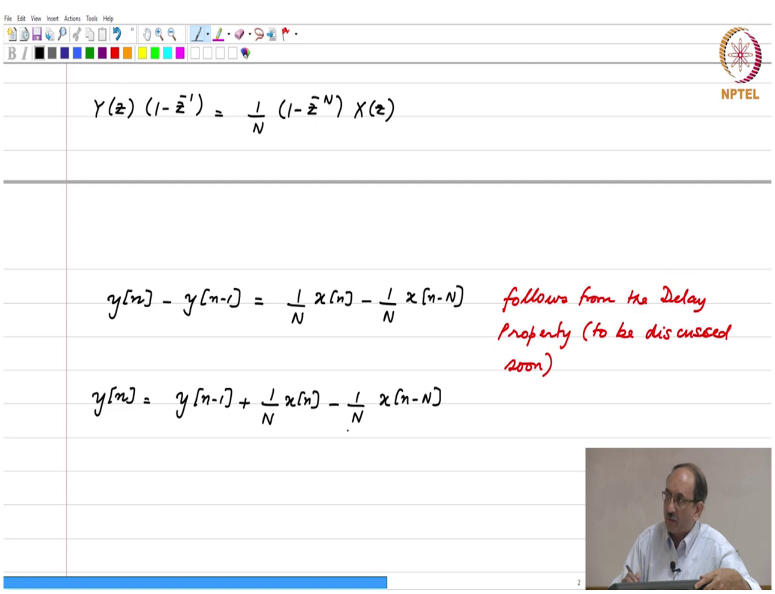

This can be written as $y[n] - y[n-1] = \frac{1}{N}x[n] - \frac{1}{N}x[n-N]$ and this follows from the delay property.

So, this will be seen very soon. So, assuming for the moment that this is indeed true, you get this and rearranging this we can see that this is exactly the same equation that we had seen earlier. So, this is $y[n] = y[n-1] + \frac{1}{N}x[n] - \frac{1}{N}x[n-N]$, so we were expressing the n point average as an update over the previous average, take the previous average add in the newest sample and then subtract out the oldest sample.

So, this is indeed the recursive implementation of the non-recursive system and this follows as the inverse Z-transform of the expression given here. And, now you are able to see that, if you have an FIR system and then if you implement it in a recursive manner, moment you have recursion, you will introduce poles, but the system is indeed FIR.

Therefore, the introduced pole cannot be uncancelled. Here, in this case, the introduced pole is at $z = 1$, it is going to get cancelled by the zero at $z = 0$. And, the one advantage of this is that it has fewer additions, the recursive implementation has fewer additions. You might wonder why you need to implement it like this rather than taking N samples and averaging all the time, you have fewer additions in this recursive implementation.