Digital Signal Processing

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Lecture 22:
Z - transform (2)
-Poles and Zeros at infinity
-All-pole, all-zero, and pole-zero filters
-Trivial poles and zeros

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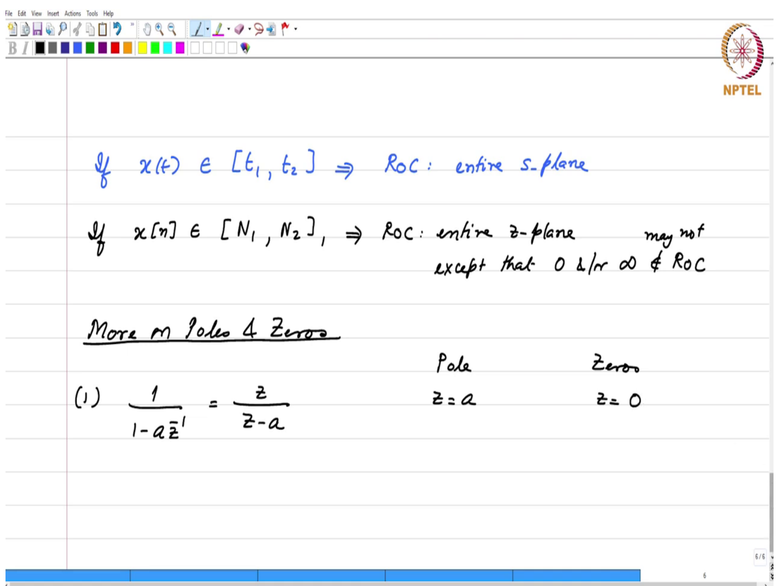

Now, let us get a feel for poles and zeros a little further. Suppose, you had $\dfrac{1}{1-az^{-1}}$. So, this of course, is $\dfrac{z}{z-a}$. We look at poles and zeros, this is something we already seen, $z = a$ is a pole, $Z + 0$ is a zero.

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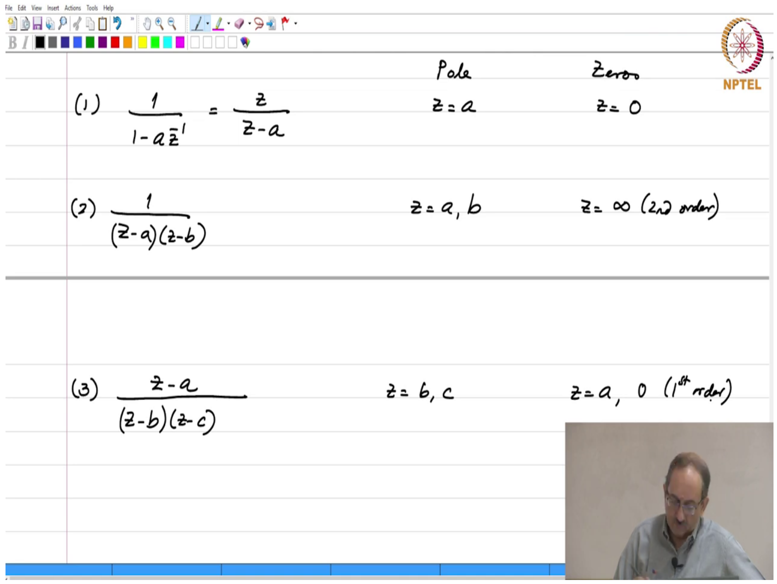

Suppose, you have $\dfrac{1}{(z-a)(z-b)}$. So, the poles are $z = a, b$, alright. What about zeros? So, what is the definition of a zero? Zero is when the function goes to 0, alright. So, where will this go to 0?

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At z equal to

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∞, alright. So, for very large values of z, this decays as, falls of as $\dfrac{1}{z^2}$. Therefore, $z = \infty$, there is a zero and this is of second order. On the other hand, suppose you had $\dfrac{z-a}{(z-b)(z-c)}$, then clearly $z = b, c$ are poles. You have a zero at $z = a$ but then for extremely large values of z, numerator behaves as z, denominator behaves as z^2 and the ratio behaves as $\dfrac{1}{z}$. Therefore, you can think of this as having a zero at $z = \infty$ and in this case the zero is of first order.

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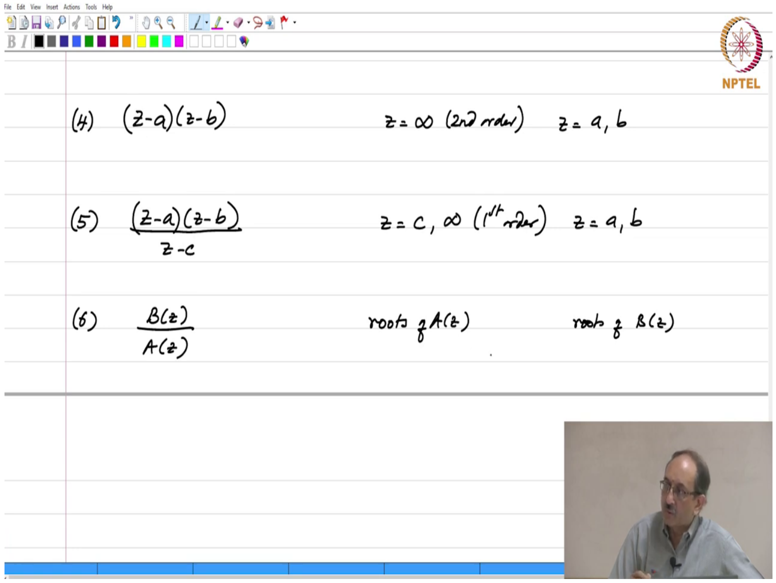

And, once you see these examples, you can very quickly, look at examples like this; $(z-a)(z-b)$ right. Clearly $z = a, b$ are the zeros, whereas, you have also pole here and the poles are actually at ∞, and because this behaves like z^2 for large z, you can think of this as a second order pole at ∞. And the other variants are just counterparts of this.

$\dfrac{(z-a)(z-b)}{(z-c)}$. So, here you have $z = c$ is a pole in the finite plane. So, these are called finite plane poles or zeros and you have zeros at $z = a, b$ and you also have a pole at ∞ and this is of first order. And in the general case, when you have $\dfrac{B(z)}{A(z)}$, then roots of $A(z)$ are the poles and then roots of $B(z)$ are the zeros and then you need to worry about whether there are poles and zeros at ∞.

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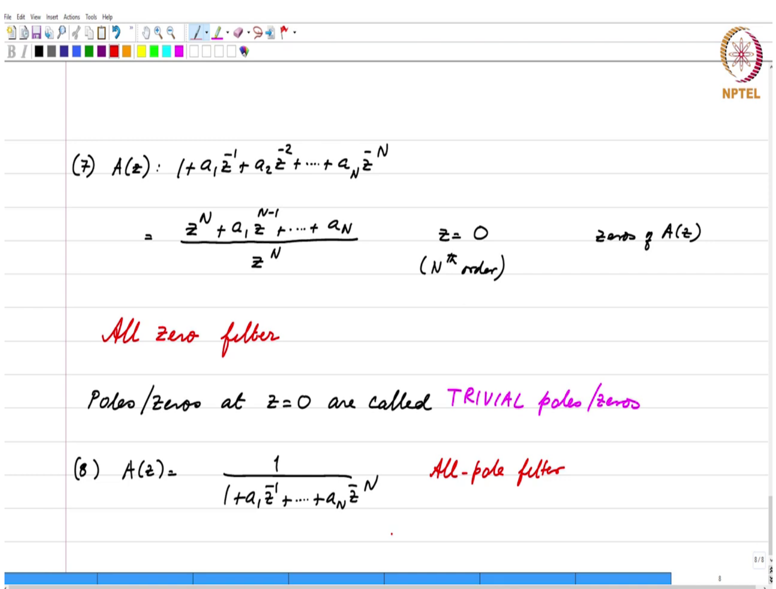

So, these are the finite plane poles and zeros. And we look at two special cases. Suppose, you had $A(z) = 1 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-N}$. If you want to talk about its poles and zeros, where are the poles where are the zeros?

So, this can be written as $A(z) = \dfrac{z^N + a_1 z^{N-1} + \ldots + a_N}{z^N}$. So, you have $z = 0$, you have a pole and this is of N^{th} order and you have zeros of this one; the numerator polynomial. Of course, here I am assuming there is no pole-zero cancellation. If there were some zeros at $z = 0$, some of those zeros will cancel with the poles at $z = 0$. This is called an all-zero filter and this is called an all-zero filter even though there are poles at $z = 0$. This is because poles and zeros at $z = 0$ are called trivial poles or trivial zeros. So, this is an important concept.

So, poles or zeros at $z = 0$ are called trivial poles or zeros and later we will understand why this is called trivial; that is zeros or poles at the origin are called trivial poles or zeros, because later when we look at magnitude response, we will see what role they play in contributing towards the magnitude response. Therefore, even though $A(z)$ in this form has both poles and zeros, this is still called a all-zero filter, because the only poles are trivial poles. And the counterpart to that of course, is you have $A(z) = \dfrac{1}{1 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-N}}$ and this is called as an all-pole filter. This is after all is the inverse of that the zeros of the previous example become poles and vice versa.

Even though there are zeros for this all pole filter, all those zeros are at $z = 0$. Therefore, since all the zeros are trivial zeros, this is called an all-pole filter. If you had both non trivial zeros and non trivial poles, that will be called as pole-zero filter.

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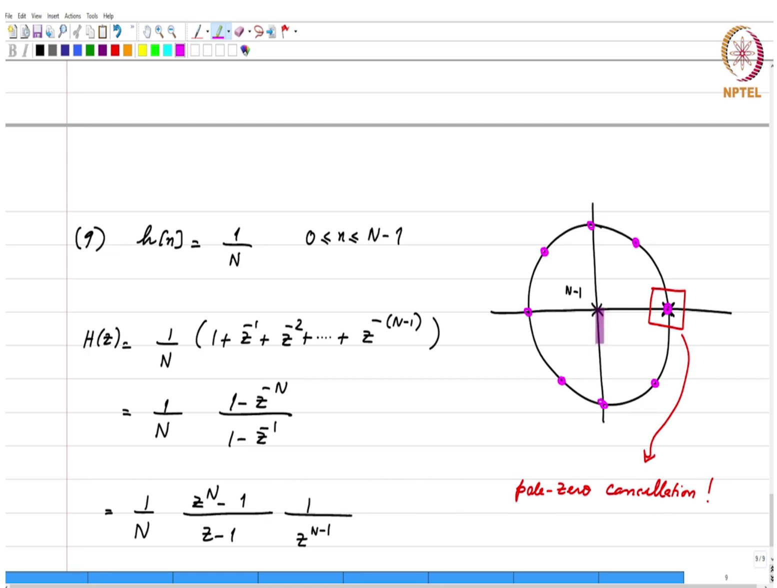

Let us look at this particular example. So, this is $h[n] = \dfrac{1}{N}$, $0 \le n \le N-1$. Therefore, $H(z) = \dfrac{1}{N}\Big(1 + z^{-1} + z^{-2} + \ldots + z^{-(N-1)}\Big)$. This of course, can be written as, you can sum up this geometric series. So, this is $\dfrac{1}{N}\dfrac{(1 - z^{-N})}{(1 - z^{-1})}$.

Now, if you look at the poles and zeros of this transform, you can rewrite this as $(z^N - 1)/z$, denominator also can be written as $(z-1)/z$. Therefore, one of the powers of z will get cancelled, so you will be left with z^{N-1} and then of course, you have this scale factor of $1/N$. So, $H(z) = \frac{1}{N}.\frac{(z^N-1)}{(z-1)}.\left(\frac{1}{z^{N-1}}\right)$. Now if you start to look at the poles and zeros of this, there is a pole at $z=1$; there is also a trivial pole of order $N-1$, so you have this. What about zeros; zeros are the roots of the numerator.

So, numerator polynomial is $z^N - 1$; therefore, the roots are?

Student: z to the power (Refer Time: 12:17) Nth roots of unity.

N^{th} roots of unity, ok. Therefore, if you now plot this, say for example, if I assume $N=8$, I will have 8 roots. This is assuming $N=8$ and if you look at this point, you have a pole at $z=1$, you also have a zero at $z=1$. Therefore, what is happening here is, pole-zero cancellation.

Therefore, in this form, you see that this is an all-zero filter, because the only poles are the trivial poles, so this is indeed a all-zero filter. In this form, it seems you have both poles and zeros, but it so happens that the non trivial pole at $z=1$ gets canceled by the non trivial zero at $z=1$. Therefore, the only non trivial pole that is there in this representation actually gets cancelled out. Therefore, there is no uncancelled non trivial pole. And, immediately you can also see why such a function that does not have uncancelled non trivial poles has to be FIR and the reason it is very easy to see why this is the case.

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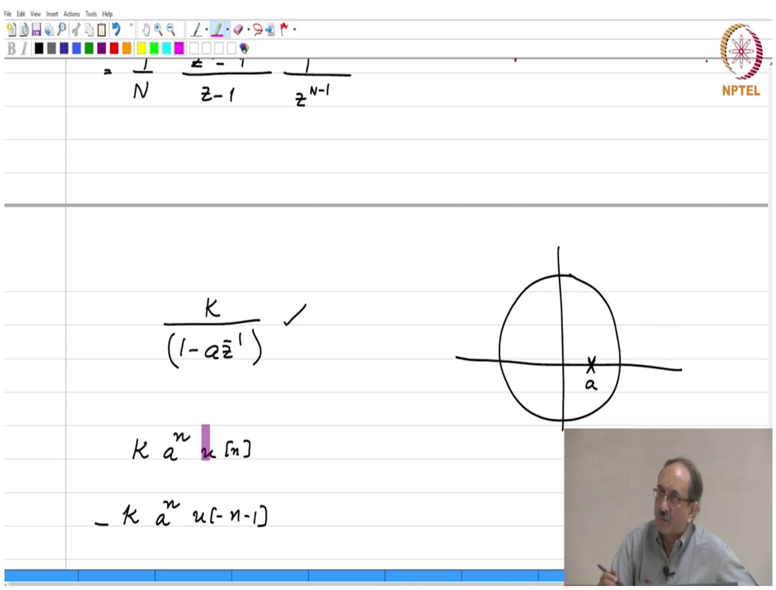

Suppose, you had a pole here that is uncancelled. Clearly this is non trivial, because this is not at $z=0$ and for simplicity I have shown the pole to be on the real axis. And, suppose if the pole is at $z=a$, then if you look at the transform. In general we are considering ourselves only in the class of rational transfer functions and you will see that $\frac{1}{1-az^{-1}}$ will be a factor in the transform. And, if you do the inverse transform, then if you do partial fraction expansion, you will get $\frac{K}{1-az^{-1}}$ as one of the terms in the partial fraction expansion. And, depending upon the region of convergence, whether the region of convergence is $|z|<|a|$ or $|z|>|a|$, the inverse transform corresponding to this will either be of the form $Ka^n u[n]$, if the ROC is $|z|>|a|$ or it will have the form $Ka^n u[-n-1]$, if the ROC is $|z|<|a|$.

So, given this factor, depending upon the region of convergence the corresponding time domain will either be this or this and these are infinite duration sequences. This is right sided infinite duration, this is a left sided infinite duration. Therefore, if you have a pole that is non trivial and that has not been cancelled out, when you do the inverse transform, it will give rise to sequence that is an exponential; that is either left sided or right sided. Therefore, that is why, if you had an uncancelled non trivial pole, this system will always be IIR. So this kind of domains of that example.

So, we will continue this and see the connection between the recursive implementation that we looked at last class and tie these two things together.