Digital Signal Processing
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Lecture 21:
Z - transform (2)
-Z-transform of left-sided, two-sided, and right-sided sequence
-Comparison with Laplace Transform

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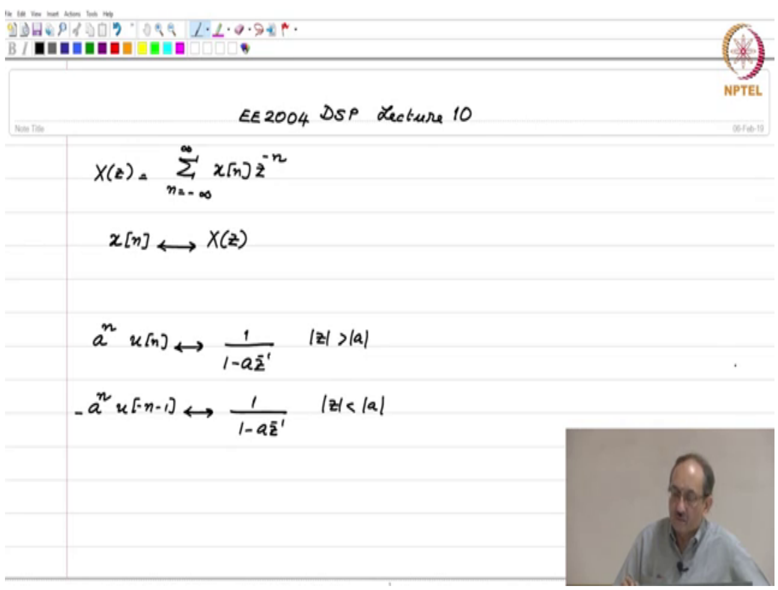

Let us continue from where we left off last time. We got introduced to the Z-transform, and the Z-transform of sequence $x[n]$ is defined as $x[n] = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$ and the notation that is commonly used to denote as Z-transform pair is similar to what was used for the Laplace transform and for the continuous-time Fourier transform. So, you denote this as $x[n] \longleftrightarrow X(z)$, and the algebraic expression and the region of convergence is a shorthand notation for the power series; Z-transform after all is a power series expansion. And the region in the z-plane over which this transform is valid is called as the region of convergence (RoC).

So, this parallels closely what was happening in the Laplace transform case. And then we saw the transform of $a^n u[n]$. So, this transform, the algebraic expression is $\dfrac{1}{1-az^{-1}}$, but this is valid for $|z| > |a|$. And the related sequence that is left sided is $-a^n u[-n-1]$ and this has the identical algebraic expression, the only difference is of course, the region of convergence. And then we also saw examples of finite duration sequences, where the region of convergence is the entire z-plane except possibly for 0 and or ∞.

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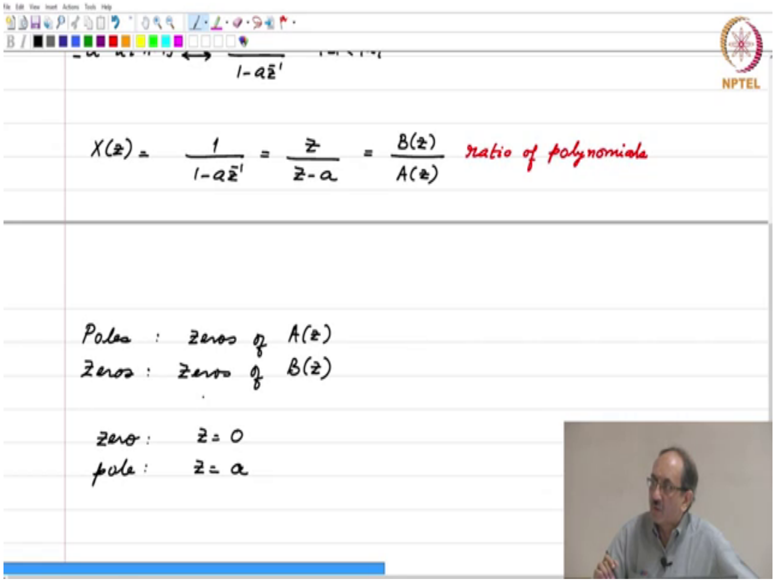

Let us look at this $X(z) = \dfrac{1}{1 - az^{-1}}$, and this is nothing, but $\dfrac{z}{z-a}$ and the class of Z-transforms that we will be looking at in this course will be of the form $\dfrac{B(z)}{A(z)}$ and this is ratio of polynomials. So, this is very similar to the class of Laplace transforms that you had considered in signals and systems, there it was a ratio of polynomials in s, it was of the form $\dfrac{B(s)}{A(s)}$.

So, this is parallel to that. And this class of transforms namely ratio of polynomials, be it either $\dfrac{B(z)}{A(z)}$ for Z-transform or $\dfrac{B(s)}{A(s)}$ for the Laplace, this class is called? Very good. So, these are the class of rational functions and they are the most well behaved class of functions you can think of, as well behaved as you guys are. These are not crazy functions, they will not cause trouble to us and to the mathematicians, and this is the only class of functions we will be considering in this course.

This was the only class of functions that you were considering in signals and systems as far as Laplace transform went. And again I had asked you the question, why you were considering only LTI systems whose input output relationship was governed by linear constant coefficient differential equations. Counterpart here would be linear constant coefficient difference equations.

So, again if you had an input output relationship that was governed by linear constant coefficient differential equation continuous-time, the corresponding transfer function had the form that was rational; $\dfrac{B(s)}{A(s)}$. We would also see that input output relationships that are described by linear constant coefficient difference equation in the discrete-time case we will give rise to transfer functions in the z-domain which are again rational. At that point, I had asked why are you enamored so much by this class. So, why I said that this class is so important.

So, coming back to $X(z)$, moment you realize that this of the form $\frac{B(z)}{A(z)}$, we will again invoke ideas that were used in continuous-time. When you had a system transfer function, you talked about poles and zeros, so the same notion of poles and zeros applies here as well. Therefore, poles are zeros of $A(z)$ and zeros are zeros of $B(z)$.

So, in this particular example, $\frac{1}{1-az^{-1}}$, one zero is $z = 0$ and you have a pole at $z = a$, and again you may have learnt as to why this is called a pole. If you have $X(z)$, $X(z)$ is the complex function of a complex variable, so it cannot be plotted, because it requires 2 dimensions for the independent variable; real and imaginary parts and then 2 dimensions for the dependent variable. So, again since we cannot visualize things in 4D you cannot plot $X(z)$ versus$z$. How are these functions then studied?

Student: (Refer Time: 07:40) plot the actual (Refer Time: 07:42)

Ok, but that will not be a complete study of the function. In terms of studying these, how these studied? So, these are studied using conformal mapping. You have the z plane where you trace curves in the z plane and then you study the corresponding behavior in the $X(z)$ plane and that is studied using conformal mapping. But as was pointed out one way of getting a feel for this is to plot magnitude of $X(z)$ versus z. Some magnitude is a real quantity, z you take 2 dimensions and then magnitude of $X(z)$ is the third dimension. And whenever you hit a pole, the function will blow up. And the neighborhood of the function around the pole if you plot $X(z)$ versus z has the appearance of a physical pole.

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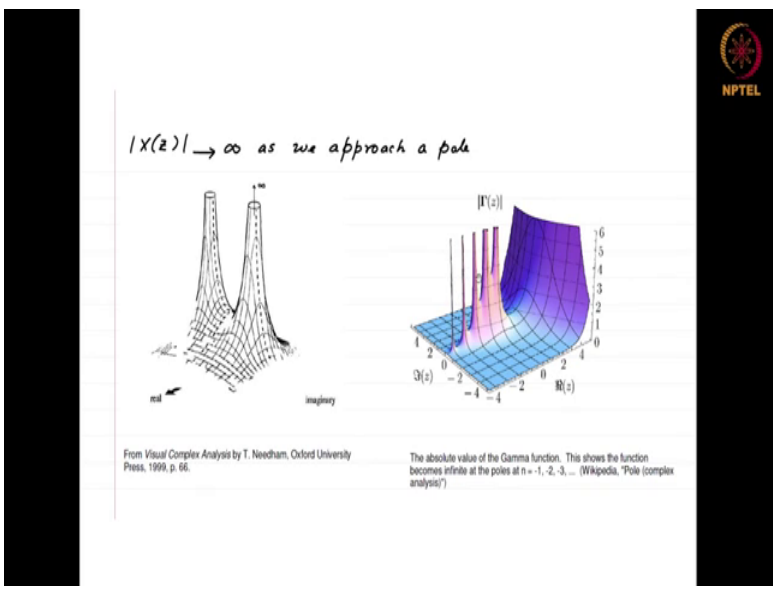

So, so here are two examples. So, this is the real part and this imaginary part and this is the magnitude as z varies. And in these cases where z approaches these two points; the function becomes, magnitude becomes larger and larger and at the pole location the function will actually go to ∞. And this is the magnitude of $\Gamma(z)$, where $\Gamma(z)$ is the Gamma function. And if n is an integer, $\Gamma(n)$ is $(n-1)!$. And, whenever you evaluate the gamma function at -1, -2, -3, and so on, it will blow up.

So, you see that, at -1 this blows up, again at -2, -3, -4, -5 and so on, it has poles. And this is real z versus imaginary z versus magnitude, because it physically has the appearance of a pole, these are called poles.

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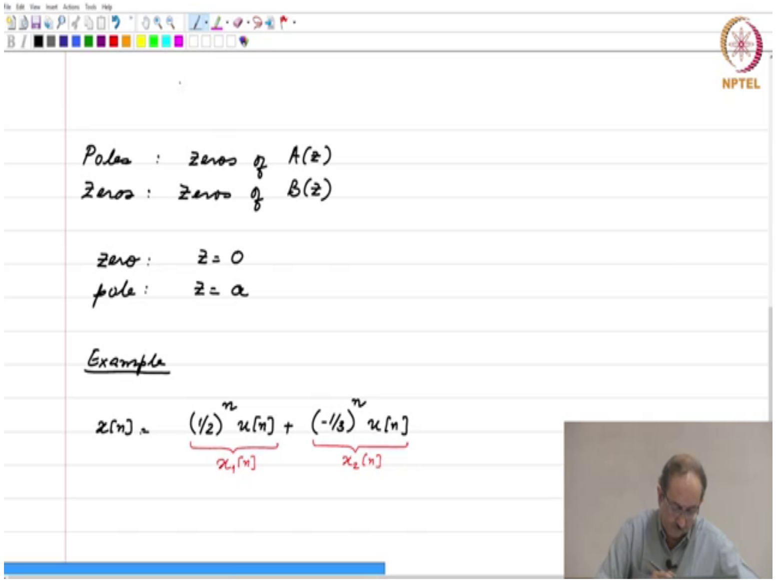

Now, let us, gamma function is defined by an integral equation, I do not quite recall this, but this is from Wikipedia that plot I have taken. So, look up $\Gamma(z)$ definition and. I mean, it is defined for all values of z, not just necessarily at the integer points, ok.

So, now let us consider simple extension of this. Suppose, I have $x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(\frac{-1}{3}\right)^n u[n]$. If this is my given sequence, I can think of $\left(\frac{1}{2}\right)^n u[n]$ as $x_1[n]$ and $\left(\frac{-1}{3}\right)^n u[n]$ as $x_2[n]$ and each of these sequences has the same form as what we had considered earlier, $a^n u[n]$.

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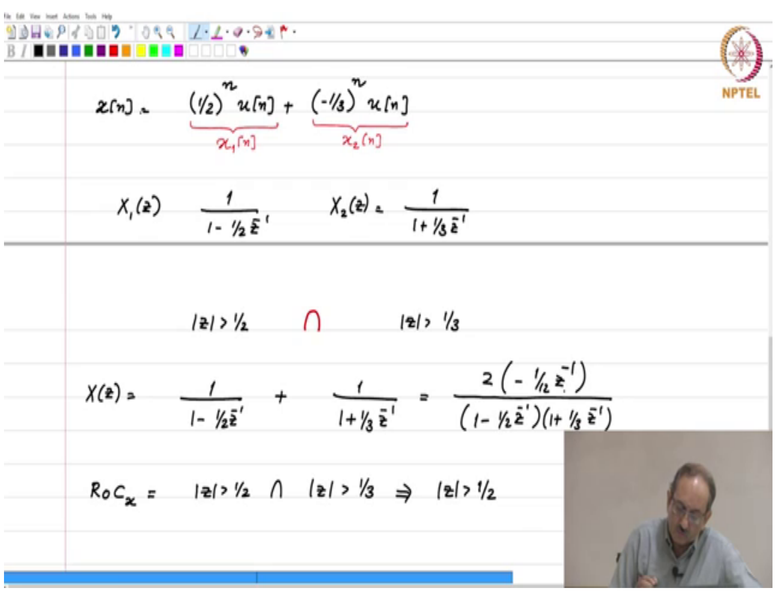

So, if I consider $X_1(z)$, so this is $X_1(z) = \dfrac{1}{1 - \frac{1}{2}z^{-1}}$, but this transform is valid only when $|z| > \frac{1}{2}$. Similarly, $X_2(z) = \dfrac{1}{1 - \frac{1}{3}z^{-1}}$ and this is valid when $|z| > \frac{1}{3}$, but really you are interested in the whole transform $X(z)$. So, this is $X(z) = \dfrac{1}{1-\frac{1}{2}z^{-1}} + \dfrac{1}{1-\frac{1}{3}z^{-1}}$. If you want both of these expressions to be valid, then the region of convergence is the intersection of these two regions.

Therefore, you need $|z| > \frac{1}{2}$ and $|z| > \frac{1}{3}$. So, this implies the final region of convergence is $|z| > \frac{1}{2}$, and if you simplify this algebraic expression for this turns out to be $X(z) = \dfrac{2\left(1-\frac{1}{12}z^{-1}\right)}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{3}z^{-1}\right)}$.

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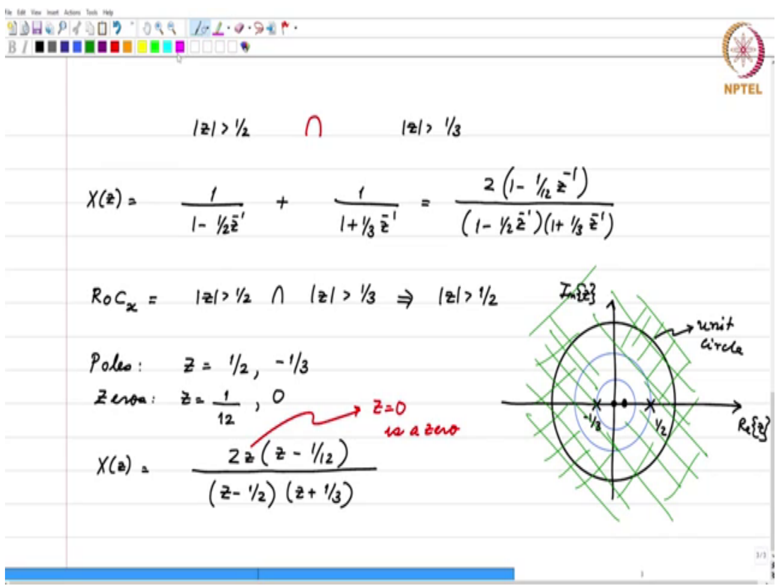

And this, if you simplify, this turns out to be this and this can always write it as $2\left(1-\frac{1}{12}z^{-1}\right)$. And, then if you look at the poles, the poles are at $z = \frac{1}{2}$ and $z = \frac{-1}{3}$. What about the zeros? Clearly, there is a zero at $z = \frac{1}{12}$, alright. Anything else?

Student: (Refer Time: 14:34).

Yes very good. This is zero at also $z = 0$ and the way to see this is the given X of z is nothing, but $\dfrac{2z\left(z-\frac{1}{2}\right)}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)}$. And in this form, you are able to see $z = 0$ is a zero. A same thing here, if you look at $\dfrac{1}{1-az^{-1}}$, if you put $z = a$, the denominator will go to 0.

So, superficially if you look at this, you would think that there is a pole at $z = a$ and you might think there is no other route, there is no other zero for example. Whereas, if you write it as $\dfrac{z}{z-a}$, you clearly able to see the zero at $z = 0$. Therefore, the important point to note is when you are plotting poles and zeroes, you are likely to miss the pole or zero at $z = 0$, if you look at the function in terms of z^{-1}. To make sure you do not miss any of these poles or zeros at $z = 0$, you have to write it in powers of z, rather than powers of z^{-1}. And, if you plot the pole zero diagram, so this is real z, this is imaginary z.

This is what is called the unit circle. So, this is a circle with radius one and this is an important boundary point in the z plane. An important boundary point in the s plane is the Laplace transform was the $j\Omega$ axis. So, we will see that the unit circle plays the role of the $j\Omega$ axis as far as the Z-transform is concerned. So, this is the unit circle.

And here you have a pole at $z = \frac{1}{2}$, you have a pole at $z = \frac{-1}{3}$, you have a zero at $z = \frac{1}{12}$, you also have a zero at $z = 0$. And, the first sequence, the transform was $\dfrac{1}{1-\frac{1}{2}z^{-1}}$ and the region of convergence was $|z| > \frac{1}{2}$ and this is the circle $|z| = \frac{1}{2}$ and this rough sketch. So, let me mark this.

So, for the first sequence, the region of convergence was outside the circle $|z| = \frac{1}{2}$, $|z| > \frac{1}{2}$ was the RoC for the first sequence $X_1(z)$, so you had this. And for the second sequence the region of convergence was $|z| > \frac{1}{3}$; therefore, for the second sequence this was the RoC and hence the overall RoC is the intersection of the individual RoCs.

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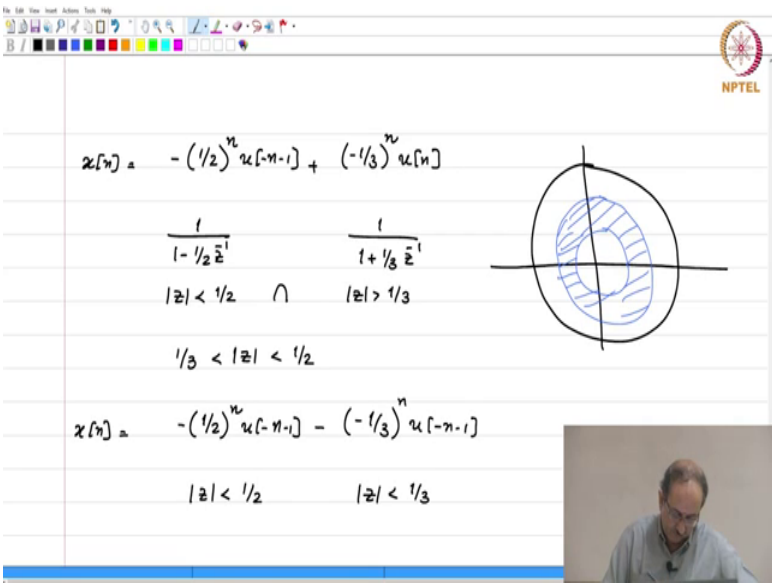

So, this is as far as this example is concerned, now we will look at variant of this. Suppose you had $x[n] = -\left(\frac{1}{2}\right)^n u[-n-1] + \left(\frac{-1}{3}\right)^n u[n]$. So, the algebraic expressions are not going to change. This of course, is $\dfrac{1}{1-\frac{1}{2}z^{-1}}$. This continues to be $\dfrac{1}{1+\frac{1}{3}z^{-1}}$. The main difference of course, is now the region of convergence is $|z| < \frac{1}{2}$ whereas, this region of convergence is $|z| > \frac{1}{3}$.

Therefore, the overall region of convergence is the intersection of these two regions and hence you have final RoC to be $\frac{1}{3} < |z| < \frac{1}{2}$. So, exactly the same pole zero plot holds. So, this of course, is the unit circle and then you have $|z| = \frac{1}{2}$ and $|z| = \frac{1}{3}$. So, the region of convergence is the intersection of $|z| > \frac{1}{2}$ and $|z| < \frac{1}{2}$.

Therefore, for this particular case, this is the final RoC, its the annular region between these two circles. And, and you can almost guess what the other variants are going to be. If you had now $x[n] = -\left(\frac{1}{2}\right)^n u[-n-1] - \left(\frac{-1}{3}\right)^n u[n]$, algebraic expressions are identical, this continues as in the previous case, its RoC being $|z| < \frac{1}{2}$ and in this case you have $|z| < \frac{1}{3}$, right.

Student: $u[-n-1]$.

So, this should be $x[n] = -\left(\frac{1}{2}\right)^n u[-n-1] - \left(\frac{-1}{3}\right)^n u[-n-1]$. Thanks.

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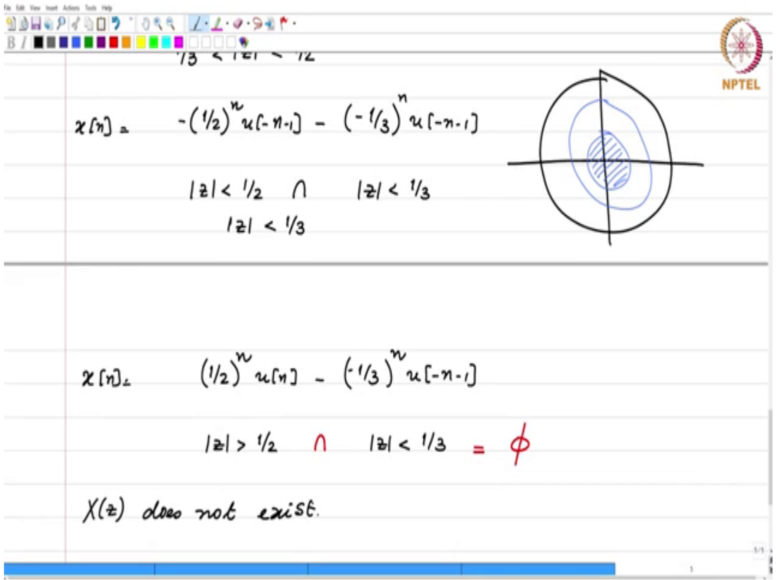

Therefore, the final RoC is $|z| < \frac{1}{3}$. And hence this is the unit circle, these are $|z| = \frac{1}{2}$, $|z| = \frac{1}{3}$ and the final RoC is this. If you notice in the very first example, where you had the sequence $\left(\frac{1}{2}\right)^n u[n]$ and $\left(\frac{-1}{3}\right)^n u[n]$, both of them are right sided. And later we will see that, we had a right sided sequence, the region of convergence will always be outside of a certain circle, what that circle is will depend upon the sequence. But in general, we had a right sided sequence, the RoC will always be outside of a certain circle.

In the first example, both sequences were right sided therefore, RoCs, both RoCs were outside of their respective circles and the region of convergence was dictated by the pole that had the largest radius. In the second example, one sequence was left sided the other sequences right sided. This RoC is inside of a certain circle, this RoC is outside of a certain circle and the final RoC is the intersection of these two.

The third example is both sequences being left sided; therefore, the RoC is going to be inside of a certain circle in both cases and the final RoC is the intersection of these two regions and it is now dictated by the smallest pole. And the last variant is $x[n] = \left(\frac{1}{2}\right)^n u[n] - \left(\frac{-1}{3}\right)^n u[-n-1]$. So, this region of convergence is $|z| > \frac{1}{2}$. This region of convergence is $|z| < \frac{1}{3}$ and the intersection is null. There is no common region of convergence and hence the Z-transform of this given sequence $x[n]$ does not exist, right. So, this is an important point that is $X(z)$ does not exist.

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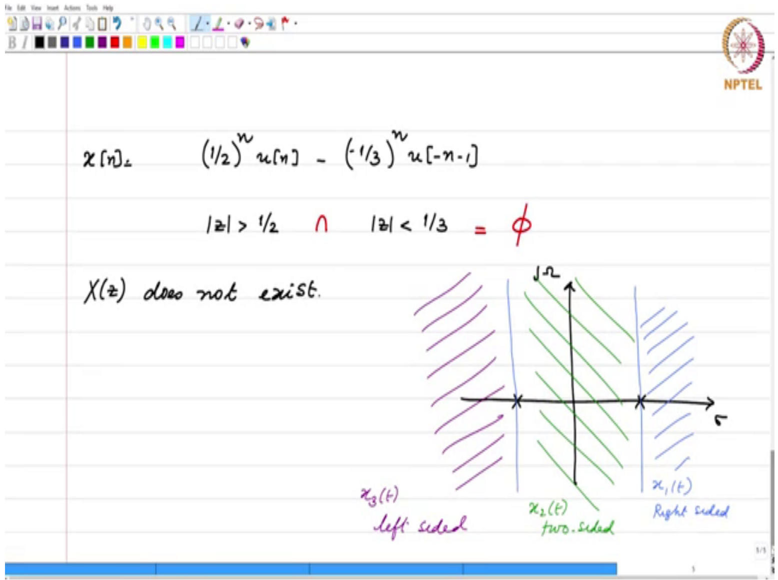

So, this is completely analogous to what you had learned in Laplace. In the Laplace transform case, you had real s. So, this was really the σ axis, this was $j\Omega$. And then if we had two poles like this, what was happening there was? You had vertical lines like this and then if the region of convergence was this, this gave rise to $x_1(t)$, that was right sided. If the region of convergence was between these two vertical lines, you had $x_2(t)$ that was two sided.

On the other hand, we had the RoC to be given like this; you had an $x_3(t)$ that was left sided. So, when we look at the convergence criterion for Z-transform, we will see that the regions of convergence will be for rational transfer functions be bounded by circles. Whereas, in the Laplace transform case, the region of convergence is bounded by vertical lines.

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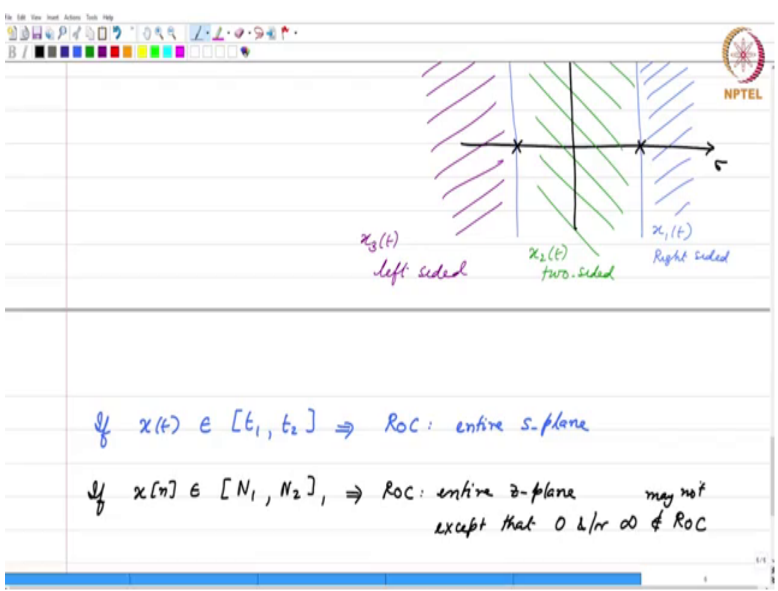

One difference between Laplace and Z is that if you had, if $x(t)$ is between t_1 and t_2. So, this is a finite duration function and then the function is absolutely integrable, then RoC is, what is the RoC in the Laplace transform case?

Student: (Refer Time: 28:22).

So, it is the entire s plane. Now, since we have already seen what the RoC is for the finite duration sequence in the Z-transform case, what is one difference that is, that should strike you when you look at this RoC versus RoC in the Z-transform case of finite duration sequences?

Student: (Refer Time: 28:52).

Very good. So, if $x[n]$ is between these two intervals; RoC is the entire z plane except that 0 may not belong. So therefore, if you have a finite duration function, $s = 0$ is always part of the region of convergence. Whereas, $z = 0$ need not be part of the RoC for a finite duration sequence.