Digital Signal Processing

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Lecture 20:
Karplus-Strong Algorithm, Z-transform (1)
-Z-transform definition
-Region of Convergence (RoC)
-Finite-duration sequences example

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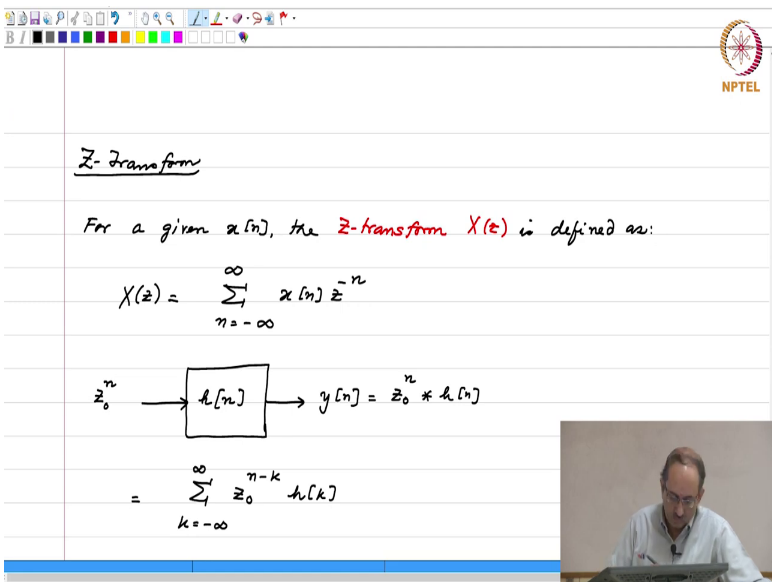

For a given $x[n]$, the Z-transform $X(Z)$ is defined like this $X(Z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$. And the maths people call this as you knows what the, what this is called by them? Z-transform, no not quite. Generating functions. And Z-transform is the other commonly used term. So, this is almost as if God given, here is the Z-transform, take it. Whereas, we can motivate this in a slightly different manner from what we have learned so far.

So, the way we would like to think of this is like this. Here, we have a system, its impulse response is $h[n]$. To this, we feed this everlasting exponential z_0^n and $y[n]$ of course, is nothing, but $z_0^n * h[n]$. So, this is nothing, but $x[n-k]$, which is z_0^{n-k}, $y[n] = \sum_{k=-\infty}^{\infty} z_0^{n-k}h[k]$. And, because the summation involves only k, we can take z_0^n outside. So, this is $z_0^n \sum_{k=-\infty}^{\infty} h[k]z_0^{-k}$.

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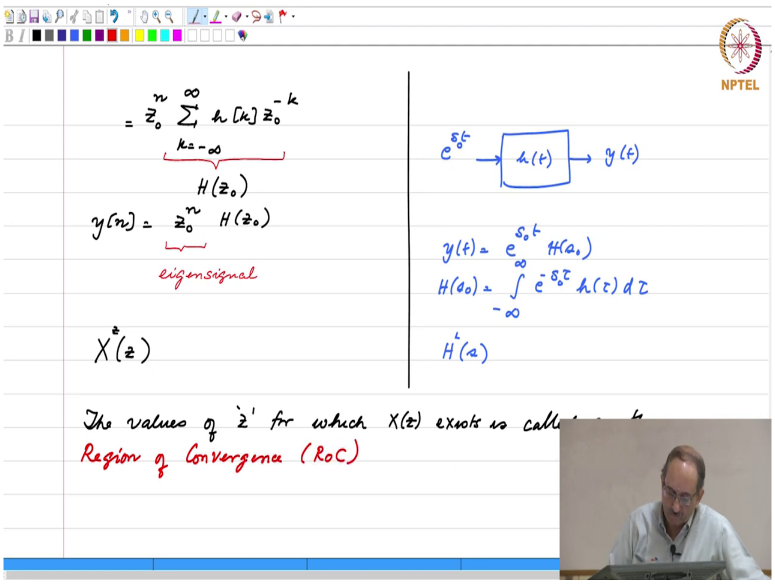

Now, this, when you sum up over all k, this is not going to depend on k. It surely will depend on h. To show its dependence on h, we will denote it as H. It will also be a function of z_0, so, we call this as $H(z_0)$. And hence, $y[n] = z_0^n H(z_0)$ of course, we are assuming that this infinite summation exists and we call this as $H(z_0)$.

So, if you look at this particular output, for the given input, the input was z_0^n, output also is z_0^n, except that it is modified by a constant, which is complex in general. And, if you apply an input to an LTI system and you get exactly the same signal as the output except for a scale factor, signal is called an eigen signal.

So, this parallels very closely another way of looking at Laplace transform in continuous-time. There, the eigen signal takes the form e^{st} and if you apply to continuous-time system with impulse response $h(t)$, you will get e^{st} back, except it is multiplied by a constant namely $H(s)$ and that $H(s)$ is called as the Bilateral Laplace transform.

So, this is exactly along those lines. So, just to show the parallel here, if you have $e^{s_0 t}$ and if you apply this and you will see that, $y(t) = e^{s_0 t} H(s_0)$ and $H(s_0)$ is nothing, but $H(s_0) = \int_{-\infty}^{\infty} e^{-s_0 \tau} h(\tau) d\tau$ and this is the Bilateral Laplace transform and this should also immediately tell you something. Do you see a problem here, between Laplace and Z-transform, the way we have been using the notation? Of course, the mathematicians will be appalled by this, but we take it in our side, we are so much used to it any other notation will seem not quite what it should be.

So, what I want you to realize is, we are using the independent variable to tell us what the transform is. When we say $H(z)$, we say Z-transform. When we say $H(s)$, we say it is the Laplace transform. s is after all the independent variable and s is a complex number, here z also is a complex number. And, so both Laplace transform and Z-transform are complex functions of complex variables. And, merely calling the independent variable as s does not make $H(s)$ the Laplace transform whereas, calling it as $H(z)$ does not make it as the Z-transform.

So, if you want to use precise, more precise notation, there are some books that use this. They denote $X(z)$ as $X^z(z)$, some books that are more careful use this and they will call this as $H^L(s)$. So, one more change in notation is used, not merely relying on the independent variable to tell us what the transform is. But, as far as this course is concerned, we will continue to use $X(z)$ or $H(z)$ to denote the Z-transform. And, when we write $H(s)$, we will mean the Laplace transform, Bilateral Laplace transform, but you need to be aware that this is not quite good notation, it is sloppy, but that is well entrenched in the literature. And, moment you have an infinite summation, you need to worry about the existence of this alright.

So, the values of z for which the Z-transform $X(z)$ exists is called as the, what you are already familiar with in the Laplace transform case, it is called as the Region Of Convergence and abbreviated as ROC. In the Bilateral Laplace also, you had to worry about region of convergence.

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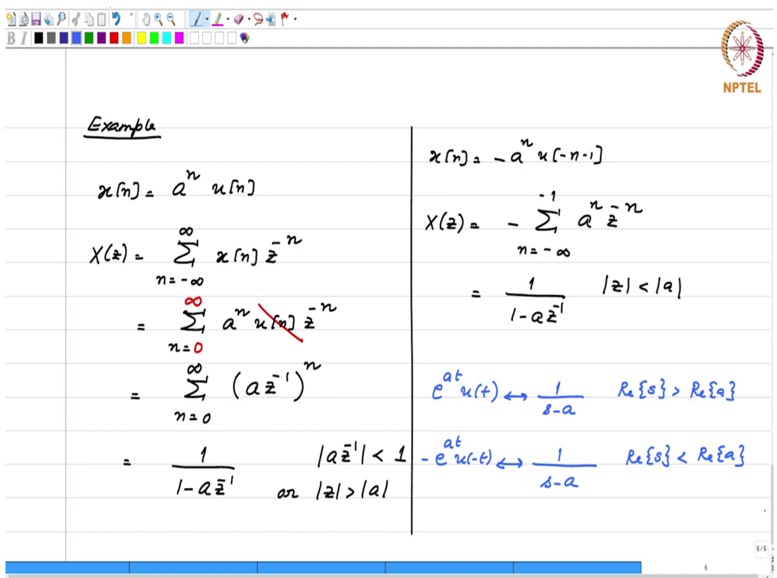

Moment you have that definition, a simple minded application of the formula is $x[n] = a^n u[n]$, $X(z)$ of course is $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$. In this case, $x[n]$ happens to be $a^n u[n]$ and, because $u[n]$ is there, we can get rid of $u[n]$ and then limit our sum from 0 to ∞. And, if you simplify this, this is nothing, but $\sum_{n=0}^{\infty}(az^{-1})^n$. And this is a simple geometric series, $a(r)^n$ and the answer is $a/(1-r)$, geometric series.

So, this is nothing, but $X(z) = \dfrac{1}{1-az^{-1}}$. This of course, assumes that the geometric ratio is strictly less than 1, you need $|az^{-1}| < 1$. So, this is the same as $|a| < |z|$ or it is usually written like this, $|z| > |a|$. So, here is an example of a simple geometric sequence $a^n u[n]$ having this Z-transform. And the only reason why this is the first example is, because we really want the second example to be this, alright.

So, now we have $x[n] = -a^n u[-n-1]$. So, what is its Z-transform? So, this is, because you have $u[-n-1]$, the summation goes from $-\infty$ to -1, $X(z) = \sum_{-\infty}^{-1} a^n z^{-n}$. And if you simplify it, what will be the answer? So, you can now make a change of variable l to be $-n$. The only difference of course, is the condition under which this is valid turns out to be now is $|z| < |a|$.

Therefore, this tells you that the algebraic expression of the Z-transform alone is not enough to pin down the time domain sequence for which this is the Z-transform of, you also need the associated region of convergence. So, if I tell you the Z-transform is $\dfrac{1}{1-az^{-1}}$, tell me what the corresponding sequence is, you will not be able to tell me the answer, because unless I tell you whether it is $|z| > |a|$ or $|z| < |a|$; you will not know whether the sequence is $a^n u[n]$ or $-a^n u[-n-1]$.

So, this is absolutely no different from what was happening in the continuous-time case with respect to Laplace transform. There you had $e^{at}u(t)$, it is Laplace transform was $\frac{1}{s-a}$ and the region of convergence was $Re\{s\} > Re\{a\}$. And what was the corresponding counterpart? It is e power?

Student: (Refer Time: 15:43).

$e^{at}u(-t)$. Anything else missing here?

Student: Minus sign.

You have minus sign and then this is $Re\{s\} < Re\{a\}$. So, now, you see the parallel between Laplace and Z. There you had two functions which had identical algebraic expression for the Laplace, they differed only in the region of convergence. So, this counterpart in the discrete-time case is $a^n u[n]$ and $-a^n u[-n-1]$.

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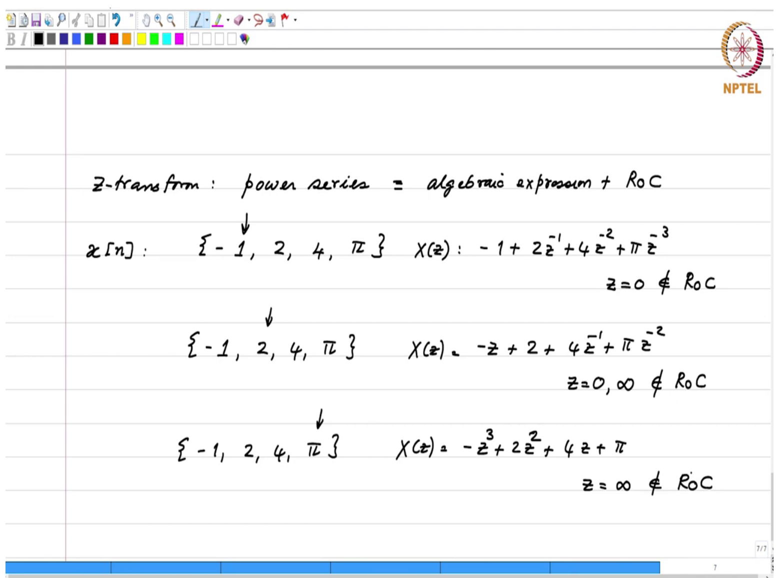

So, Z-transform is a power series and in at least textbook examples, you will find that this power series has a closed form expression and associated region of convergence. So, typically our series will have algebraic expression plus ROC. Suppose, you had $x[n]$ to be $\{-\overset{\downarrow}{1}, 2, 4, \pi\}$ and we will use this arrow to denote that this is the $n = 0$ term.

The corresponding $X(z)$, all you need to do is merely apply the definition. So, this is $-1 + 2z^{-1} + 4z^{-2} + \pi z^{-3}$ and for what values of z is this expression valid. So, in the previous example, when we saw $a^n u[n]$, we found that that was valid only for $|z| > |a|$. So, here it is a simpler example and this is valid for what values of z?

Student: z (Refer Time: 18:27).

So, $z = 0$ does not belong to the region of convergence and immediately you can see variance of this.

Suppose, you had this as the sequence except now, this is $\{-1, \overset{\downarrow}{2}, 4, \pi\}$, $X(z)$ will now be $-z + 2 + 4z^{-1} + \pi z^{-2}$. And for what values of z is this valid or what values of z this is not valid? Again, $z = 0$, it will this will blow up. And, then for which are the value of z will this not be finite?

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So, both 0 and ∞ are not part of the region of convergence. So, all other values are. And the final variant is this. So, this is the $\{-1, 2, 4, \overset{\downarrow}{\pi}\}$. So, clearly $X(z)$ is $-z^3 + 2z^2 + 4z + \pi$ and the ROC will exclude ∞ right. Therefore, $z = \infty$ will not belong to the region of convergence.

So, later we will see that this is a general property of finite duration sequences. When you have a finite duration sequence, then the region of convergence will be the entire z-plane except possibly for 0 and or ∞. So, later what we will do is, we will start to look at ideas like poles and zeros and these are no different from the concept of poles and zeros that you had encountered in the Laplace transform case.