Digital Signal Processing
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Lecture 14:
Systems and their Properties (2), LTI Systems (1)
-LTI systems
-impulse response
-convolution

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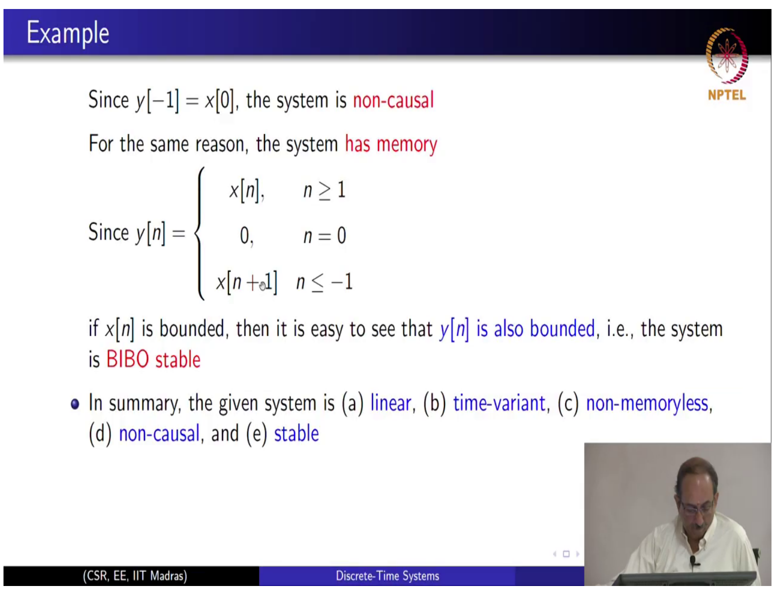

Ok. So, now, let us focus on that subclass of systems that are both linear as well as time-invariant.

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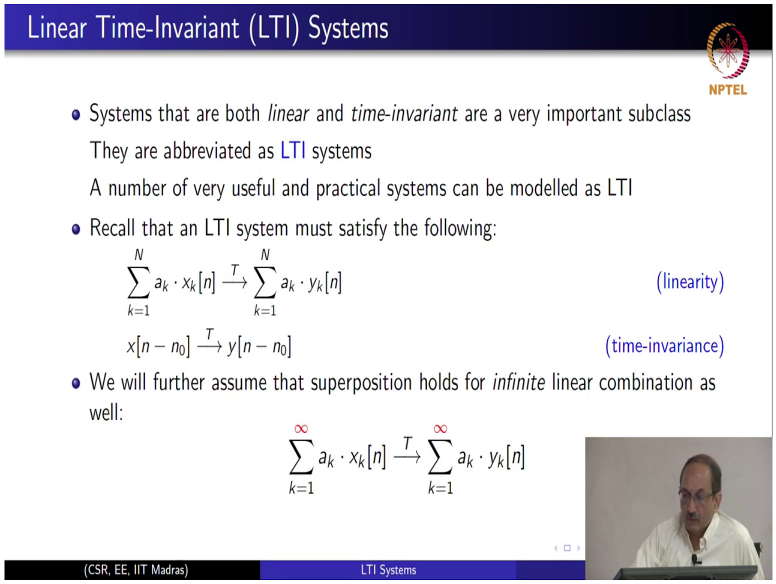

So, this is a very very important subclass and they are abbreviated as LTI or LSI (Linear Time Invariant or Linear Shift Invariant). And a number of very useful and particle systems can indeed be modeled as LTI and you will see that there is a very powerful theory associated with LTI systems, mathematically its very much tractable. But, if that mathematical tractability alone were the criterion, this is not be as useful. Its mathematically tractable, so what?

The fact that, many practical systems can be modeled as a LTI systems is the key point. So, you have the advantage of mathematical tractability plus it being practical for a wide number of cases. So, LTI system must satisfy superposition and time-invariance and we will further assume that this is true for infinite linear combination as well.

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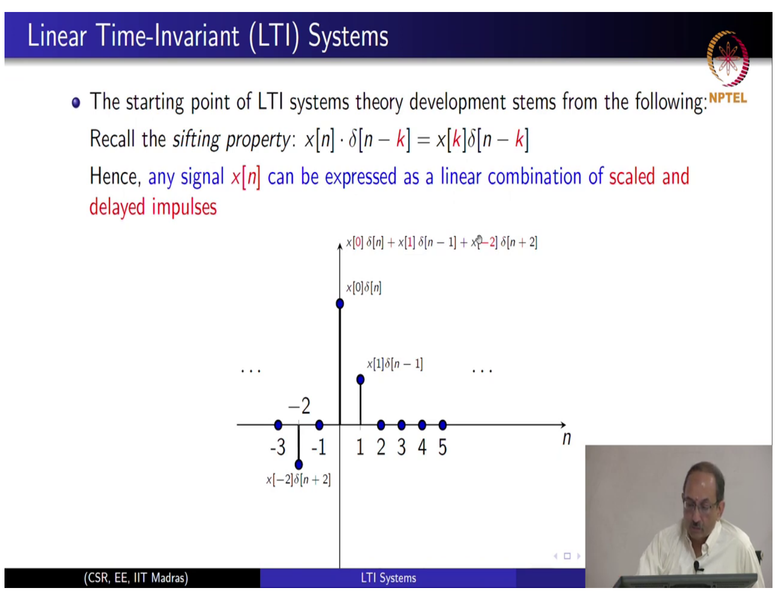

So, the very starting point of LTI theory is as follows. So, recall the shifting property of the unit impulse, $x[n].\delta[n-k]$ is, it picks out the k^{th} sample. $\delta[n-k]$ is located at the k^{th} index, it picks out that particular sample. And any signal can be expressed as a linear combination of scaled and delayed impulses.

So, this is $x[n].\delta[n]$, it picks out $x[0]$, this is $x[n].\delta[n-1]$, it picks out $x[1]$, all the other samples are 0. Similarly, this is $x[n].\delta[n+1]$ picks out the sample at -2. So, if you consider $x[n].\delta[n] + x[n].\delta[n-1] + x[n].\delta[n+2]$, then these products pick out the individual samples at those locations.

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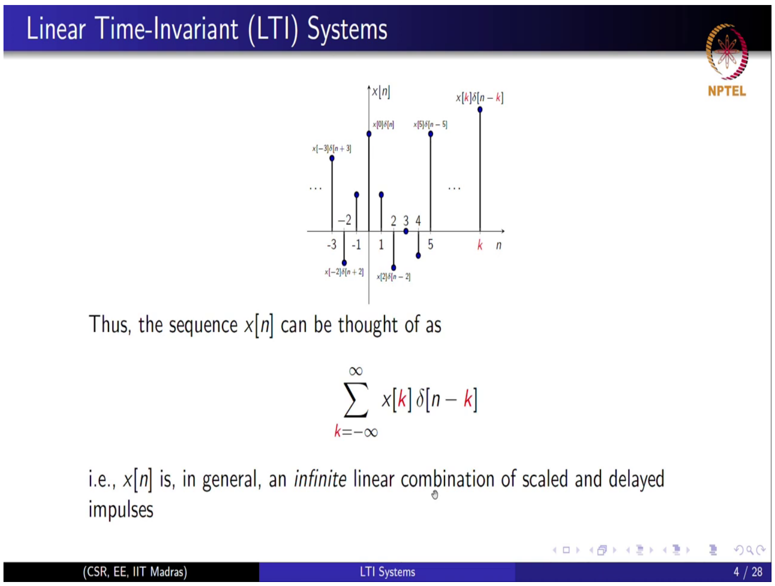

And in general, if you have an arbitrary sequence, if you focus on $x[n].\delta[n-k]$, so here you have an impulse at the k^{th} location. Even easy to see that an arbitrary sequence can be thought of a sum of scaled and delayed impulses.

Therefore, if you let k go from $-\infty$ to ∞, that is another way of thinking of the arbitrary input $x[n]$. So, in general this is an infinite linear combination of scaled and delayed impulses.

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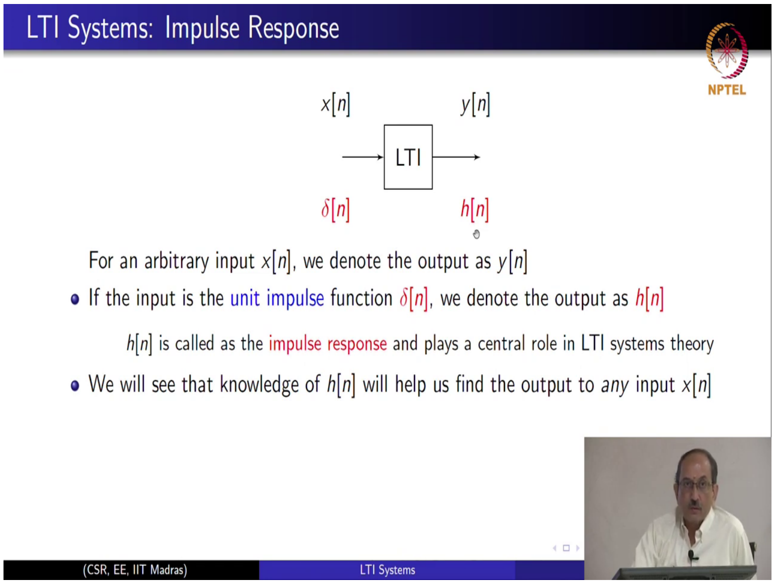

So, this brings us to the impulse response. Arbitrary input if the input is $x[n]$. For the LTI system, the output is $y[n]$. If the input happens to be $\delta[n]$, it has a special name and its usually denoted by $h[n]$. In continuous-time system, this is denoted by $h(t)$ and this is called as the impulse response.. So, this impulse response plays a central role in LTI systems theory. So, if you have knowledge of $h[n]$, you will find that you can get the output to any input $x[n]$ and the fact that you are able to find output to any input $x[n]$ follows from both linearity being true as well as time-invariance.

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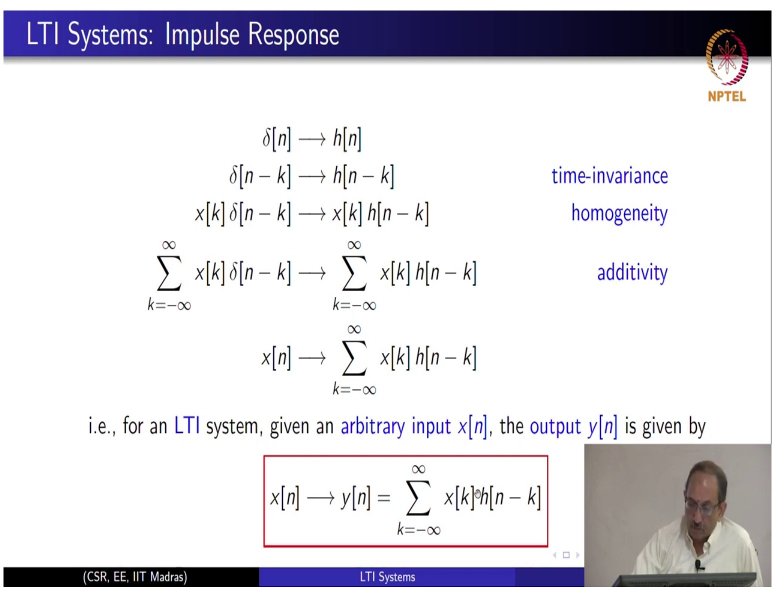

So, $\delta[n]$ produces $h[n]$, $\delta[n-k]$ produces $h[n-k]$ this is of course, time-invariance. $x[k]\delta[n-k]$ producing $x[k]h[n-k]$ is homogeneity. And $\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$, then you get $\sum_{k=-\infty}^{\infty} x[k]h[n-k]$.

So, this is additivity. But here, we are using additivity for infinite number of terms which we have assumed it is true. But, $x[k]\delta[n-k]$ summed up over all k is nothing, but $x[n]$. Therefore, for an arbitrary input $x[n]$ and applied to an LTI system, you will get $\sum_{k=-\infty}^{\infty} x[k]h[n-k]$ as the output. So, $x[n]$ producing $y[n]$ and $y[n]$ is indeed this is well known famous convolution result.

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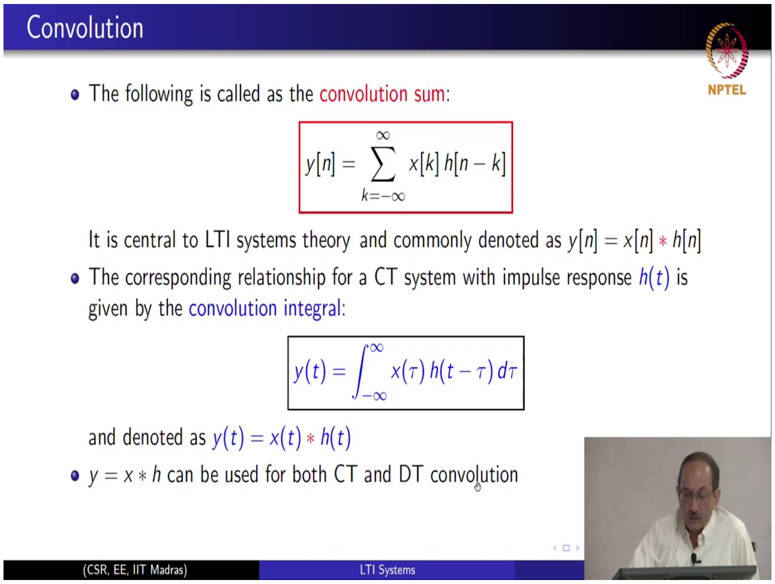

So, this is called as the convolution sum. The corresponding counterpart for CT systems is the convolution integral and this is very central LTI systems theory right and this is denoted as $x[n] * h[n]$. So, this stands for the convolution and this is the counter part for the continuous-time case and this is denoted as $x(t) * h(t)$. If you write it as $x * h$, this notation is used for both CT as well as DT convolution.

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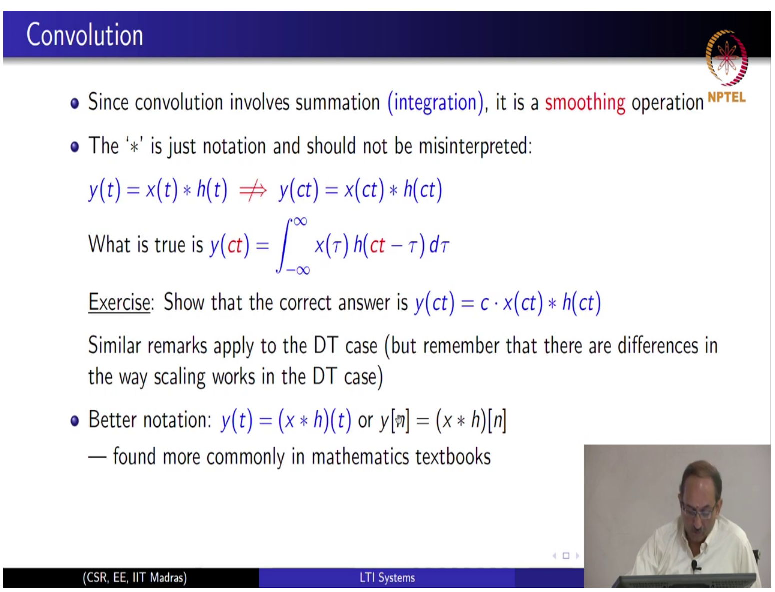

Since convolution involves summation or integration, its a smoothing operation. So later, I will show java applet that brings this point clearly. So, this $*$ is just notation, it should not be misinterpreted. So, if $x(t)$ convolved with $h(t)$ is $y(t)$, right.

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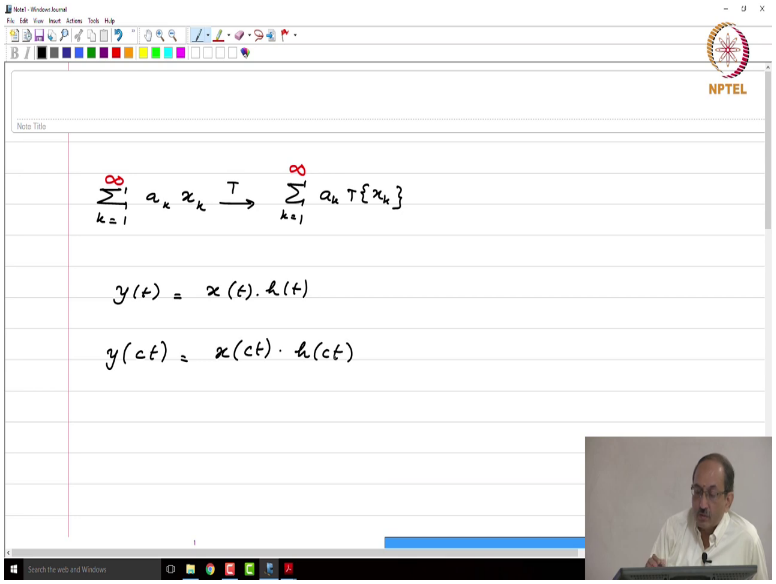

Let us go back to this. So, $y(t) = x(t).h(t)$, what is $y(ct)$? This is $x(ct).h(ct)$. So, this is very clear. Now let us look at this; $y(t) = x(t) * h(t)$, this does not mean $y(ct)$ is $x(ct) * h(ct)$, alright. But what is true is, if you write down the actual convolution integral definition.

So, $y(t)$ was $\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$. There if I replace t by ct, on the right hand side also I can replace t by $c \times t$ whereas, this is not true because $*$ is just notation. And the correct answer is $y(ct)$ equals, actually this has slight error, here I have to fix this error, it is not actually c. I want you to find out what the error is, what see very good yeah, you have already seen this before. Then, c is actually mod c.

So, verify that $y(t)$ is indeed (mod c). $x(ct) * h(ct)$. So, similar remarks apply to the DT case except that there are slight differences in the way scaling works for CT versus DT. And the maths textbooks usually use this notation for convolution.

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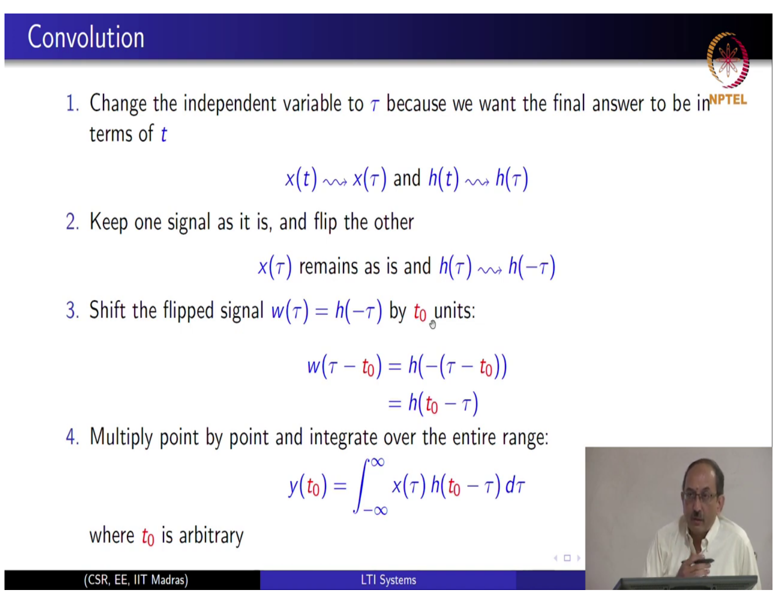

Again this is all review, let me quickly go through this. This is continuous-time convolution, replace the independent variable t by τ, because you want the final answer to be in terms of t. So, you change the dummy $x(t)$ from t to τ. This remains fixed, this gets flipped $h(\tau)$ becomes $h(-\tau)$, and then you shift the flipped signal by t_0 units. So, if $w(\tau) = h(-\tau)$, then you need to look at $w(\tau - t_0)$ because you are going to shift the flipped signal.

So, wherever τ is there, replace τ by $\tau - t_0$. So, since $w(\tau) = h(-\tau)$, replace τ alone by $\tau - t_0$ and you get $h(t_0 - \tau)$. Then you multiply point by point and then integrate over the entire range, where t_0 is arbitrary, number t_0 is a shift and since this is an arbitrary shift you need to test for all possible shifts and all possible shifts means shifting the entire signal from $-\infty$ to ∞. So, replace t_0 by the general t, now t has to take the value between $-\infty$ to ∞.

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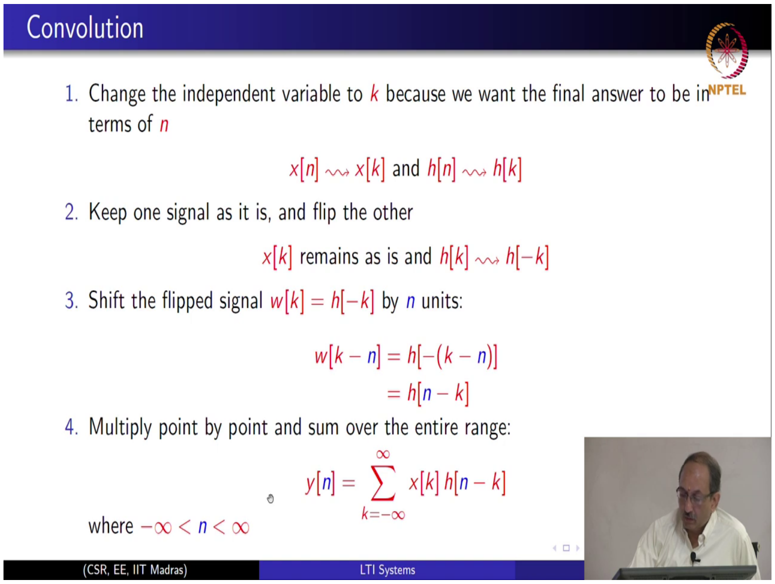

So, the corresponding discrete-time is identical. Replace n by k because we want the final answer to be in terms of n, keep one signal fixed, flip the other, shift the flipped signal, multiply point by point and sum up and you get the usual convolution sum.

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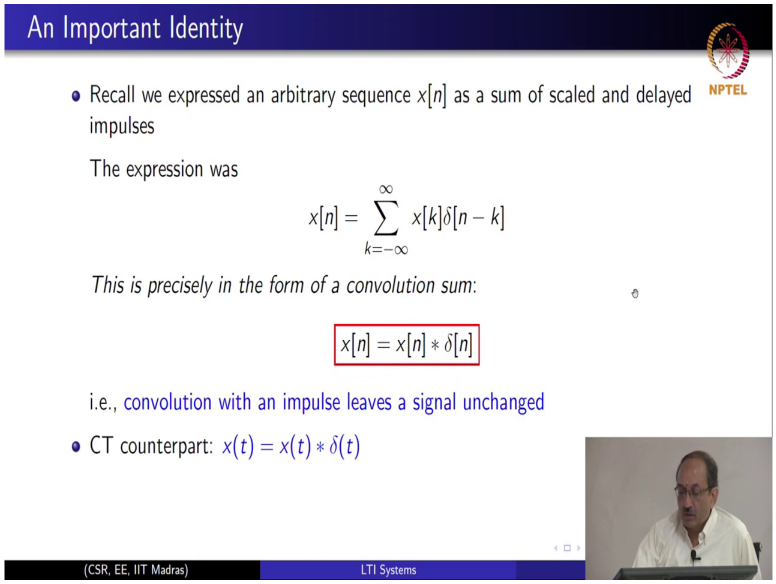

Now, if you understand the convolution in the continuous-time domain, then convolution with this k time is no different. And some of the points related to convolution I want to show with respect to the java applet which I will do next class.

And an arbitrary sequence can be expressed as the sum of scaled and delayed impulses is what we saw and this is exactly in the form of a convolution sum. So, this is exactly in the form of $x[n]$ convolved with $\delta[n]$. Therefore, this is an important identity, convolution with impulse leaves the signal unchanged, similarly in the continuous time case $x(t)$ convolved with $\delta(t)$ you get back the original signal and here is an important identity.

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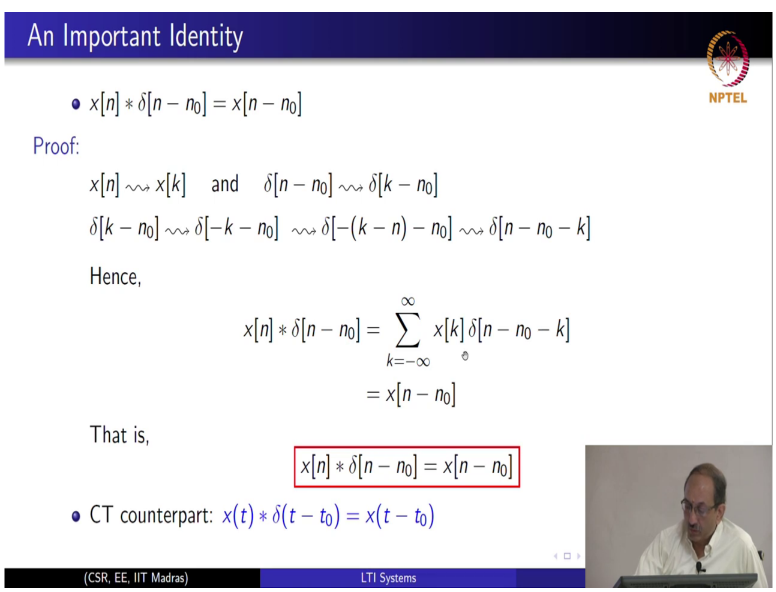

If you take $x[n]$ and convolved with $\delta[n-n_0]$, you shift the input by the by the shift of the impulse. So, let us go through this step by step. So, replace n by k. So, $n-n_0$ becomes $k-n_0$, wherever n was, replace n by k. Then, flip, wherever k is there, replace k by $-k$ and then replace k by $k-n$ for the shift by n unit. So, wherever k alone is there, replace k by $k-n$.

So, you get $\delta[n-n_0-k]$ and then you multiply point by point and then sum up over all possible indices and here the only term that will survive is when $k = n - n_0$. For everywhere else, this is going to be 0 therefore, you get this. And hence $x[n]$ convolved in $\delta[n-n_0]$ is indeed $x[n-n_0]$. If you are not careful, you will make mistakes in how you reflect and how you shift, right. So, this is where some of the errors can occur if you are not careful. Therefore, $x[n]$ convolved with $\delta[n-n_0]$ is indeed $x[n-n_0]$. The corresponding continuous-time counterpart of course, is a very similar.

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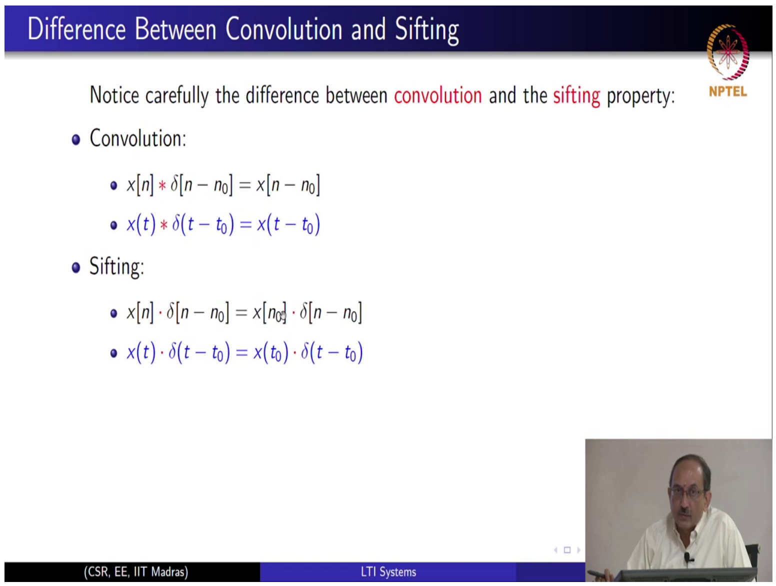

And you should have already by now seen the difference between convolution and sifting. Convolution is, $x[n] * \delta[n-n_0] = x[n-n_0]$, this is the continuous-time counterpart whereas, sifting you take $x[n]$ multiply by $\delta[n-n_0]$. It picks out the sample at n_0 and the $\delta[n-n_0]$ still remains because the product has to be 0 outside whenever the index is not n_0, ok.

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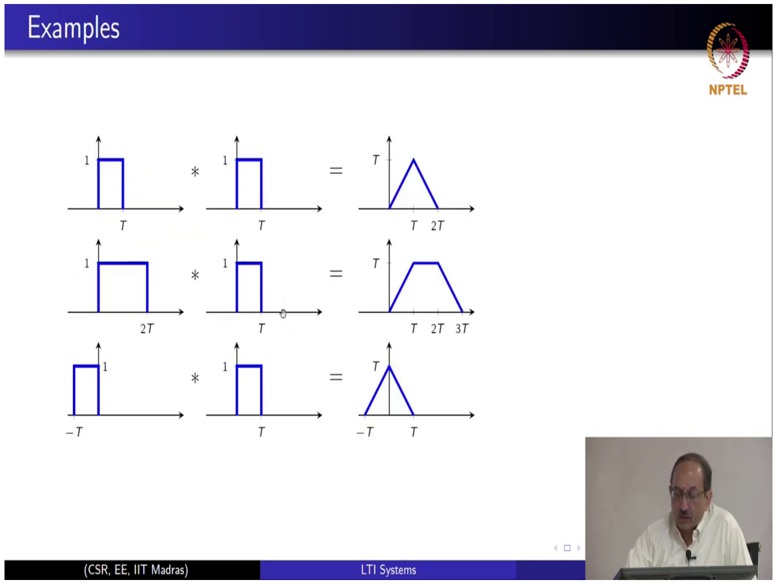

So, notice carefully the difference between these two cases and these are examples, again I will explain up on these examples later, you must have seen this in your earlier course.

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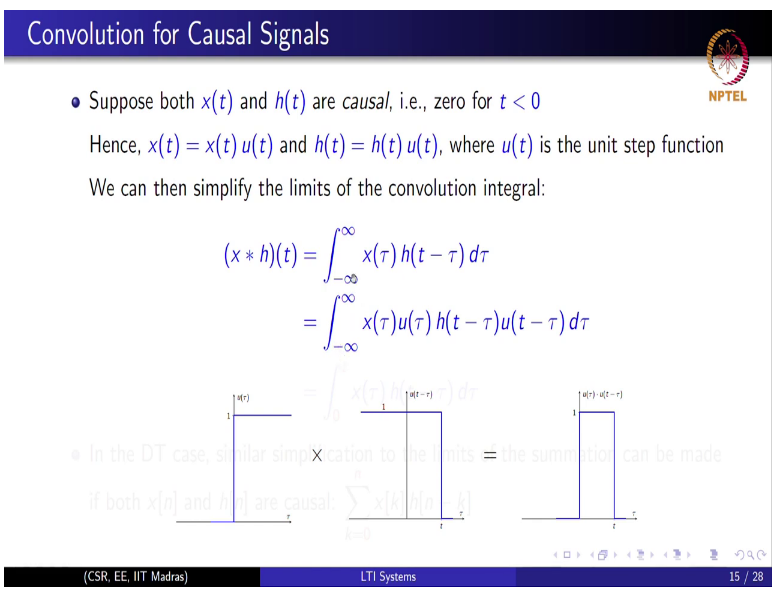

And again if both signals are causal, then you can think of causal signals as, the term we are using causality earlier we had used the term casual for systems right, here we are borrowing that term and we use the term 'the signal is causal' if it is 0 for $t < 0$.

So, if you have both x and h, if they are 0 for $t < 0$, you can replace $x(t)$ with $x(t)u(t)$, $h(t)$ by $h(t)u(t)$, where $u(t)$ is of course, the unit step function and then you can simplify the limits of the convolution integral like this. So, replace $x(\tau)$ by $x(\tau)u(\tau)$, similarly for $h(t-\tau)$. And then if you look at the product $u(\tau)u(t-\tau)$, this is $u(\tau)$ and this is $u(t-\tau)$. So, the product exists only from 0 to t and everywhere else the product is 0, which means you can now change the limits from $-\infty$ to ∞ to 0 to t.

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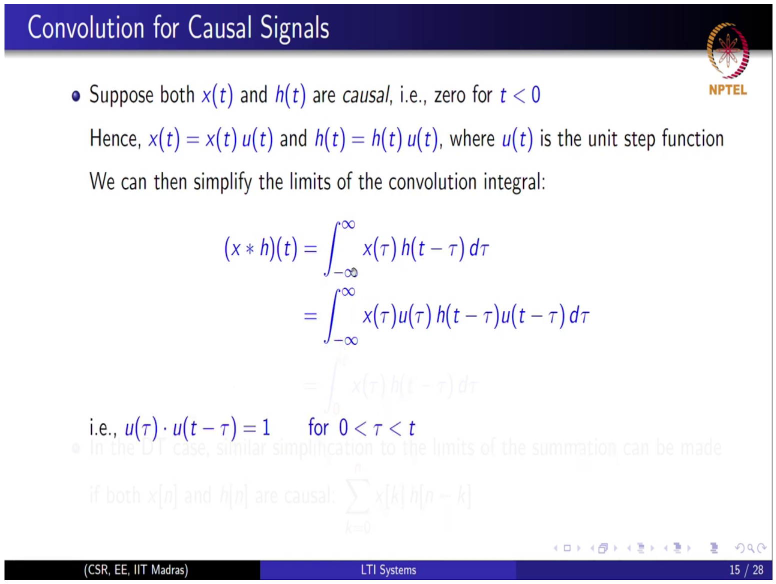

So, when the signal is 0 for $t < 0$, you can make this change in the limit.

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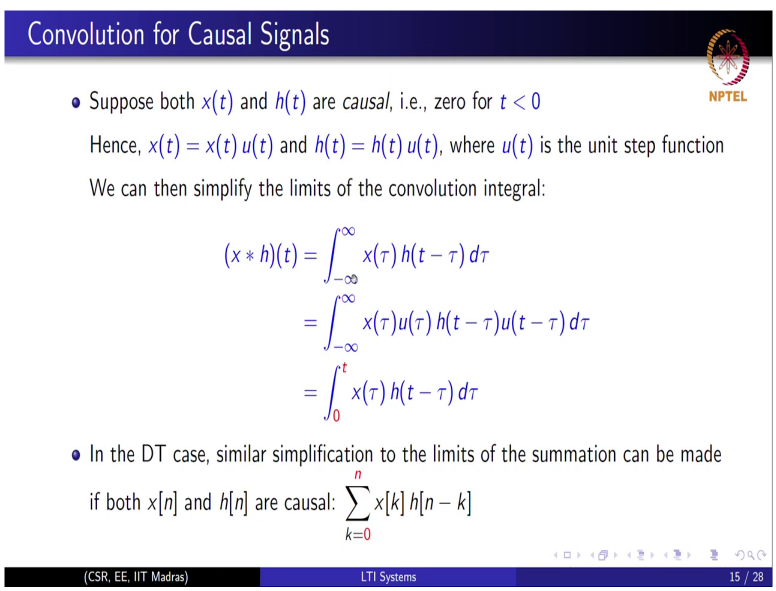

Similarly, the discrete-time summation also can be simplified from $-\infty$ to ∞. The summation will collapse to 0 to n when both $x[n]$ and $h[n]$ are causal, may be they are 0 for $n < 0$.

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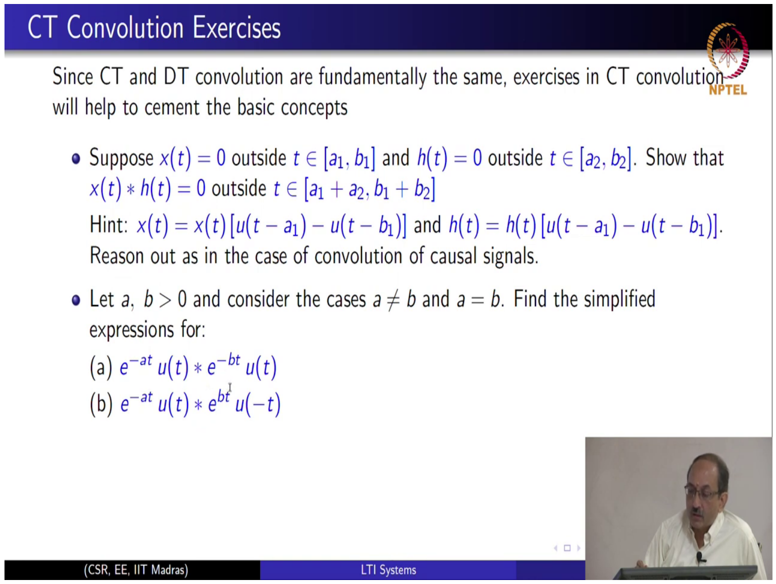

And these are some simple convolution exercises that you can try, and this is an important property. If $x(t) = 0$ outside a_1 and b_1 and if $h(t) = 0$ outside a_2 to b_2, then you can show that the convolution result is 0 outside $a_1 + a_2$ to $b_1 + b_2$. You can reason this out very similar to what does for the case of causal signals.

So, these are important exercises, they have a discrete and counter parts as well where you take one signal that is causal and the other signal is non causal. So, $e^{-at}u(t)$ is 0 for $t < 0$ whereas, $e^{bt}u(-t)$ is 0 for $t > 0$. How you are able to simplify this step by step is important, only then you are fully understood. You may understand graphically convolution very well, but you should also be able to write down the equations, fix the limits, simplify and get the final expression that is correct. So, only if you are able to do that, you completely understood all the individual parts.