Digital Signal Processing
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Lecture 13:
Systems and their Properties (2), LTI Systems (1)
– System properties: Time-invariance, static and dynamic systems, causality, stability – Worked-out example

Let us get started for the day. Let us continue from where we left last time. We were looking at the property of linearity and that in turn consisted of two sub properties; namely additivity and homogeneity.

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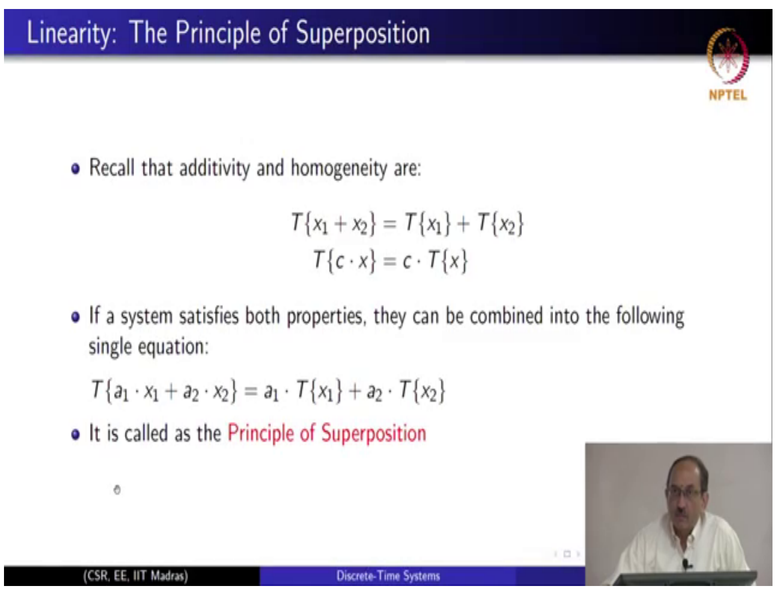

And, if the system is more additive and homogenous, it is linear. And, these two properties can be combined like this and this gives rise to the well known principle of superposition.

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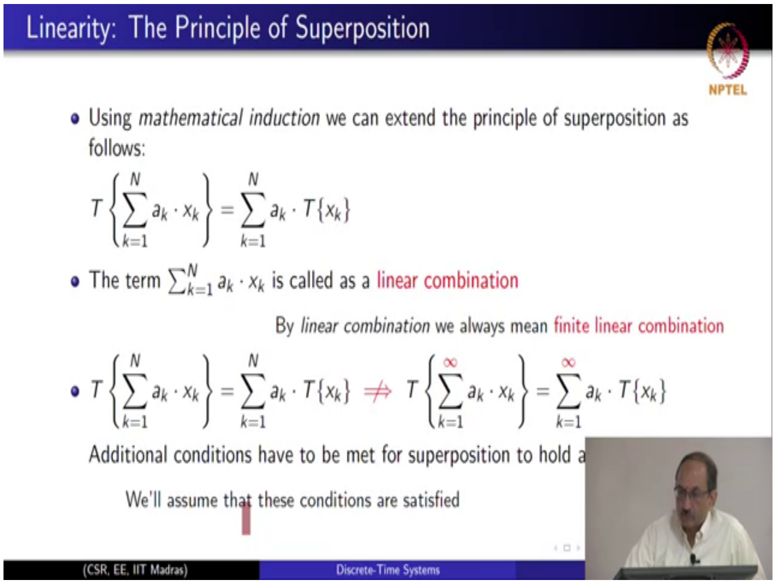

And, using mathematical induction, we can extend this principle of superposition as follows. Remember, superposition applies only to just two inputs using induction, this can be extended to N inputs. And, this term $\sum_{k=1}^{N} a_k.x_k$ is called as a linear combination.

Now, let us try to extend this. Let us look at this first. So, we know this is true; additivity for two terms can be extended up to N terms.

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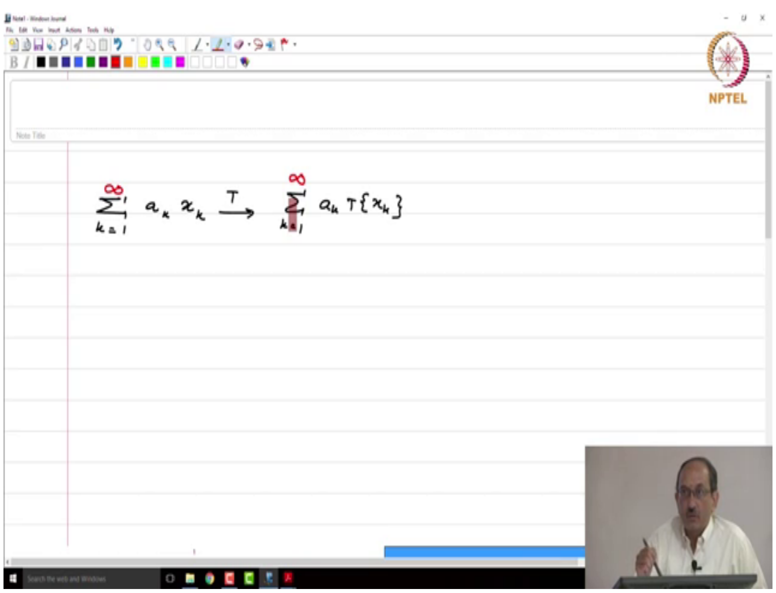

So, the question is, $\sum_{k=1}^{N} a_k.x_k$, this is the input. So, this is passed through the system.

Student: Sigma.

The system output of course, is $\sum_{k=1}^{N} a_k.T\{x_k\}$. The question that we want to ask is, is this true? Additivity, when it is extended using induction is true for N terms, is it true for infinite number of terms is the question. If it is true for N terms, is it true for infinite number of terms? Actually, you have assumed that it is indeed true for infinite number of terms in signals and systems.

Student: (Refer Time: 02:19).

Very good, Fourier series. So, what happen in Fourier series? Yeah. So, you have to input that has periodic that was expanded in Fourier series and then. So, where did you use this, yeah very good. So, indeed in applying periodic signals to an LTI system, you expanded the input in terms of its Fourier series components, in general the Fourier series as infinite number of terms.

And, when it is applied to an the LTI system, you indeed assume that the output was of this form. Without even questioning whether finite additivity is true for infinite additivity. We will assume that infinite additivity is indeed true, all I want to point out to you is it does not automatically fall.

This is a linear combination and by linear combination we always mean finite linear combination. And, if it is true for finite then it does not automatically true it is, it holds for infinite number of terms, additional conditions have to be met for superposition to hold as $N \to \infty$. We will assume that these conditions are satisfied and we will leave the worrying to be mathematicians, we will happily use this.

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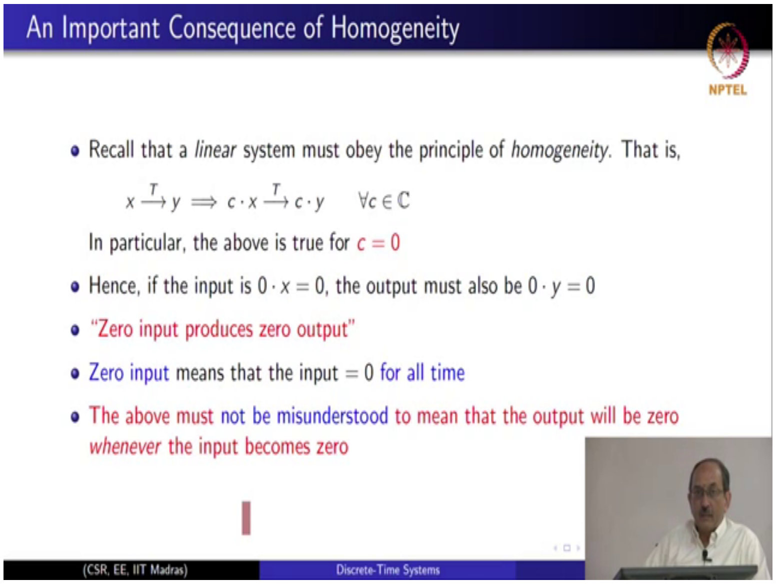

Now, let us try to deal one important consequence of homogeneity. Though if $x \xrightarrow{T} y$, $c.x \xrightarrow{T} c.y$ for all c. In particular, the above is true for $c = 0$. And, hence if the input is $0.x$ which is 0, the output also must be $0.y = 0$.

So, "Zero input produces zero output" is what, this is normally termed as, when we say the input is zero we mean that the input is 0 for all time. And, it is important that this should not be misunderstood to mean that, whenever the input is zero, the output will become zero, that is not what is meant by this.

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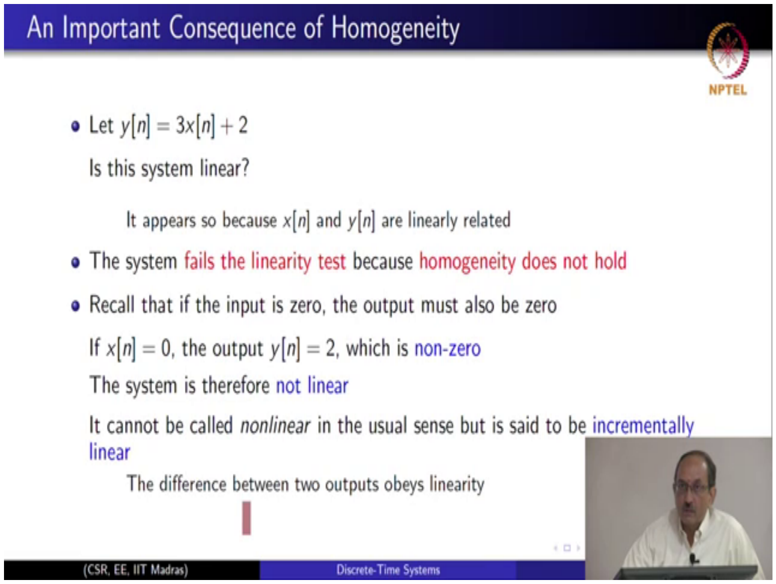

So, let us look at a system: $y[n] = 3x[n] + 2$. So, if this system linear, y and x appear to be linearly related, but the system fails the linearity test because homogeneity does not hold. If the input is zero, the output must also be zero, whereas if $x[n] = 0$, the output is $y[n] = 2$ which is non-zero. Therefore, the system is not linear, but it is also the case that the system is not non-linear, all right. This is what is called incrementally linear, which means that the difference between two outputs must obey linearity.

So, we will not get into incremental linearity beyond this point we will just mention what this term means and then we will leave at that.

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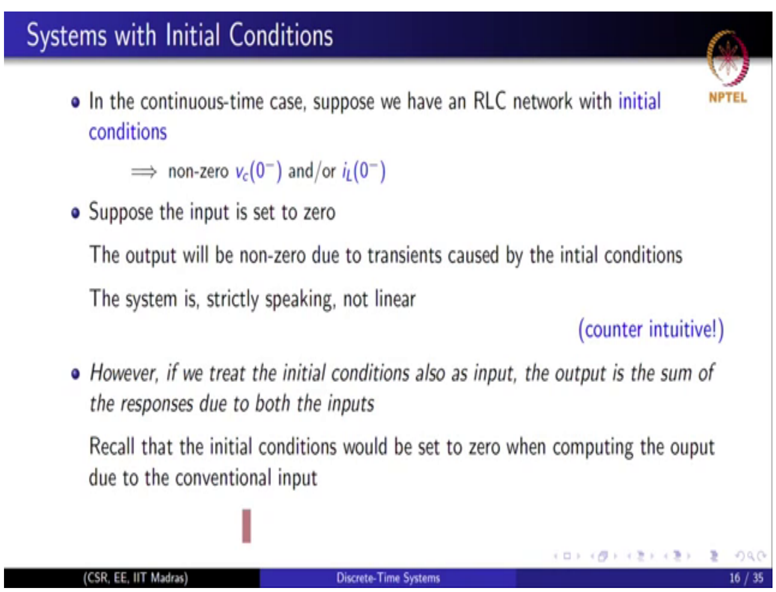

And, one of the consequences of this is if you recall your RLC network with initial conditions, your capacitor had non-zero voltage or inductors had non-zero currents. Suppose, the input is zero, the output will still be non-zero, because of the transients caused by the decay of these initial conditions, capacitor voltages and inductor currents.

So from the strict definition, the system is not linear because zero input must produce zero output, and an RLC system not being linear seems counter intuitive. So, what is happening here is, if you treat the initial conditions also as an input, then the overall response is sum of the outputs due to initial conditions and the output due to the input with the initial condition zeroed out, right.

So, when you want to compute the output due to conventional input, you will zero out the initial conditions, ok. If you take that approach, then systems with initial conditions are indeed linear.

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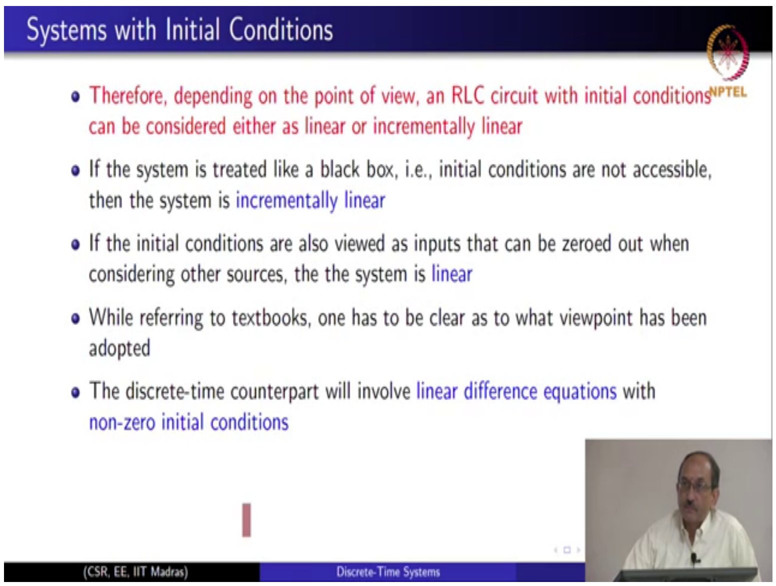

Again, the reason I am mentioning this is some books treat this two things differently. If, you treat the system like a black box, then the initial conditions are not accessible, which means a system is incrementally linear, that is the initial conditions are not accessible then you cannot go in and zero them out, if you want to find the output due to the conventional input. If the initial conditions also are viewed as inputs that can be zeroed out, then considering the other sources then the system is linear.

So, when you are referring to textbooks, you have to be clear as to which approach you are taking. If I am not mistaken, I think Lathi follows the first approach in which he assumes the initial conditions are accessible which means you can zero them out and then you can apply your conventional input. Therefore, as far as Lathi is concerned, this system is linear. Oppenheim treats the input as a black box which means strictly speaking, you do not have access to the initial conditions, which means in those cases, the system is incrementally linear.

And, this is as far as RLC is circuits are concerned, it is linear constant coefficient differential equations. The counter part for discrete-time systems, it is linear difference equations with non-zero initial conditions. And in the case of systems with initial conditions, you solve them using unilateral Laplace transform handed over whether this was part of your earlier course.

You can use unilateral Laplace transform with initial conditions and then solve in a very simple manner of the circuit. And the corresponding counterpart is linear constant coefficient differential equation.

And, you can solve them using unilateral z-transform whereas in this course, we will be considering only the bilateral z-transform and then we will draw parallels to the bilateral Laplace.

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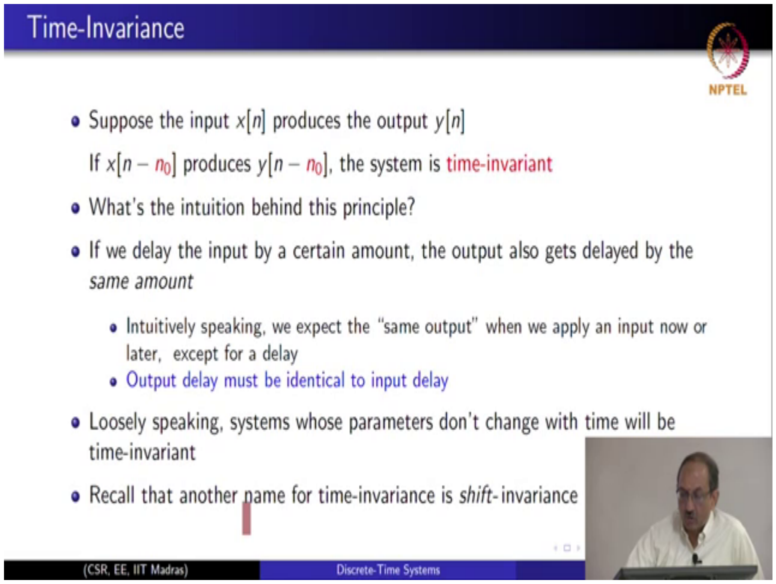

Now, let us move on to the next property, namely time-invariance. Suppose, the input $x[n]$ produces $y[n]$. If $x[n-n_0]$ produces $y[n-n_0]$, this is the system is time-invariant. So, this is no different from what was happening in continuous-time case. And the intuition behind this principle is if you delay the input by a certain amount, the output gets delayed by exactly the same amount.

For example, if you apply $x[n-n_0]$, if the output is not $y[n-n_0]$, but it is $y[n-n_1]$, then this system is not time-invariant. The delay has to be identical. So, intuitively you expect the same output to appear when the input is applied at a later time. So, the only difference is the delay in the output. And the delay is identical to the delays suffered by the input. So, output delay must be identical to the input. And loosely speaking, if system parameters do not change with time, the system is time-invariant.

For example, if we had a system that had a time varying gain, then if you apply an input later, the gain of the system would have changed. So, you cannot expect the same output except for a delay. So, that is the intuition behind this time-invariance property and time-invariance also is called shift-invariance.

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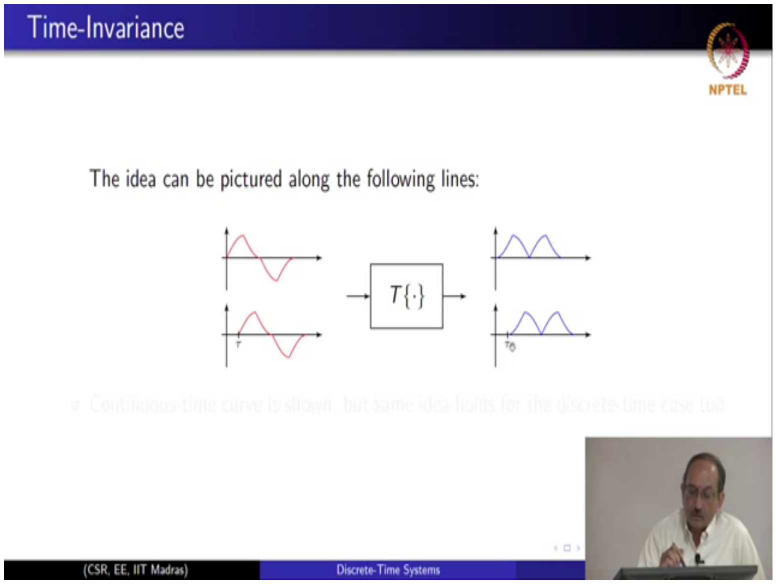

And in terms of pictures I have shown, this in terms of continuous-time plots. The same thing applies in terms of continuous-time as well. So, here you have the certain input producing certain output and when you delay it by τ you will get exactly the same output except for this delay and the output delay is identical to the input delay.

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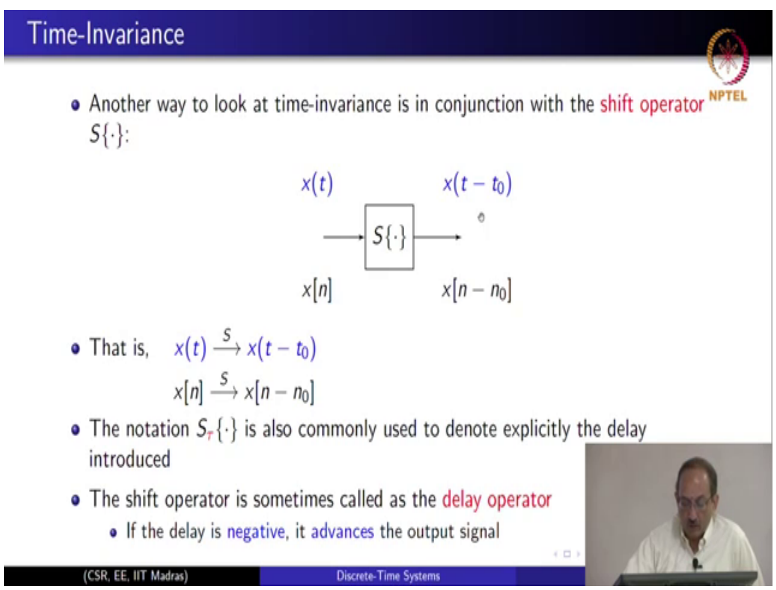

And, another way of looking at time-invariance is in conjunction with the shift operator, which we call it as S, the system operator is called as T. So, if you had $x(t)$ applied, the shift operator will shift it by t_0 in the continuous-time case and n_0 will be discrete-time case. So, these are the corresponding shifts that the system produces when the input is applied. And, you can use the suffix τ if you want to explicitly mention the delay. And shift operators also called as the delay operator. And if the delay is negative, it advances the output signal.

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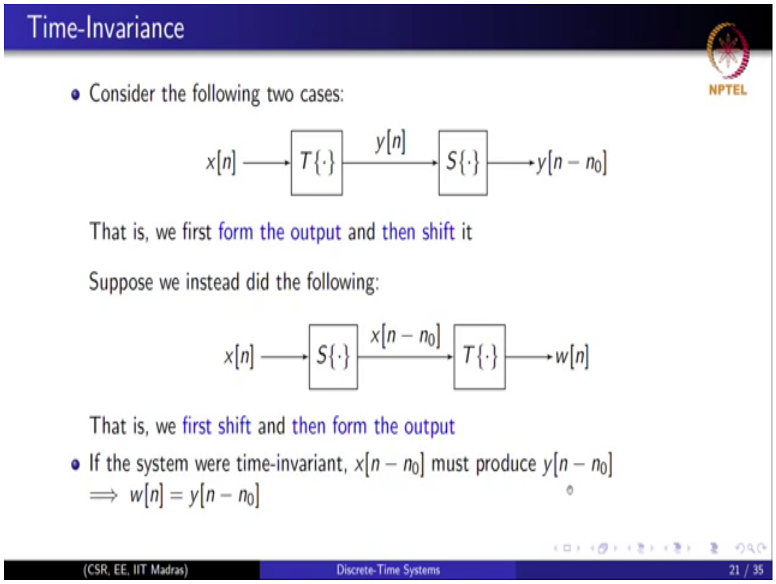

So, for time invariance, again this parallels exactly what you have learnt for continuous-time. You apply the input to the system, you get an output and you shift, you get $y[n-n_0]$. So, this is one part for the test. The other part for the test is you take the input, first shift it, you get the delayed input, then apply to the system, then you examine the output. If $w[n]$ which is in the second case same as the output that you got in the first case, then the system is time-invariant i.e., $w[n] = y[n-n_0]$. Again this is no different from the CT system test, take the input apply to the system, delay the output, that is one part. Take the input, delay it and then apply to the system, check the output if these two are equal, this system is time invariant.

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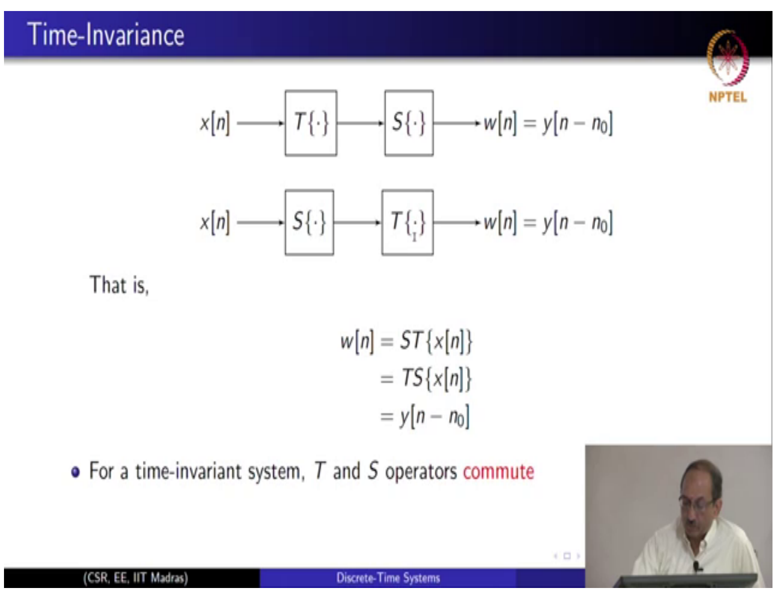

So, these are the two branches, first you operate on the system and then shift whereas, the second case you shift and then operate on the system. And, system has time-invariant if these two operators commute. If ST equals TS, the system is time-invariant. Again, continuous-time example is easy to see even if you have seen this already.

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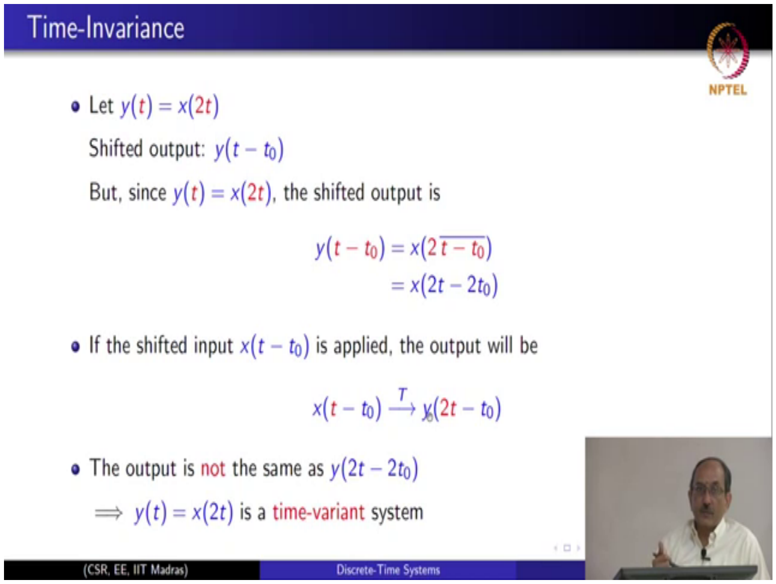

If you take the output, this system is $y(t) = x(2t)$, you shift the output, replace t by $t - t_0$ and wherever t is there if you replace by $t - t_0$, you get this. So, the output is $x(2t - 2t_0)$. On the other hand, if you shift the input and then apply to the system, all the system is going to do is wherever t is there, it is going to replace t by $2t$ i.e., $y_0(2t - t_0)$.

So clearly, these two are not the same and hence the system is time-invariant. The intuition behind this is if you take the output and then shift in the first case, you have a certain shift whereas, when you take the input, shift it and then apply to the system, because the system replaces t by $2t$, the shift reduces by factor of 2, that is why the shifts are not equal.

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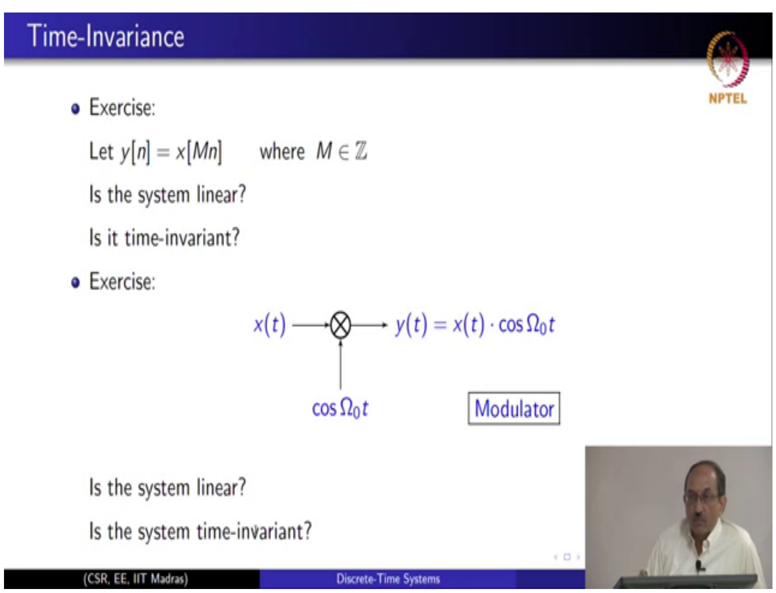

And, you can easily verify the same is true for the discrete counter part, $y[n] = x[Mn]$. So, check for linearity and time-invariance. And here is what is called as a modulator; so, $x(t)$ multiply by $\cos(\Omega_0 t)$ is this.

Otherwise, the system linear, the system time-invariant. Based on what I had already mentioned, if the system has a time varying gain, you can expect that the system to be time-variant. And, here $\cos(\Omega_0 t)$ can be thought of as a time varying gain and hence if you check for time-invariance, you will find that this will fail the time-invariance test and this is indeed time-variant.

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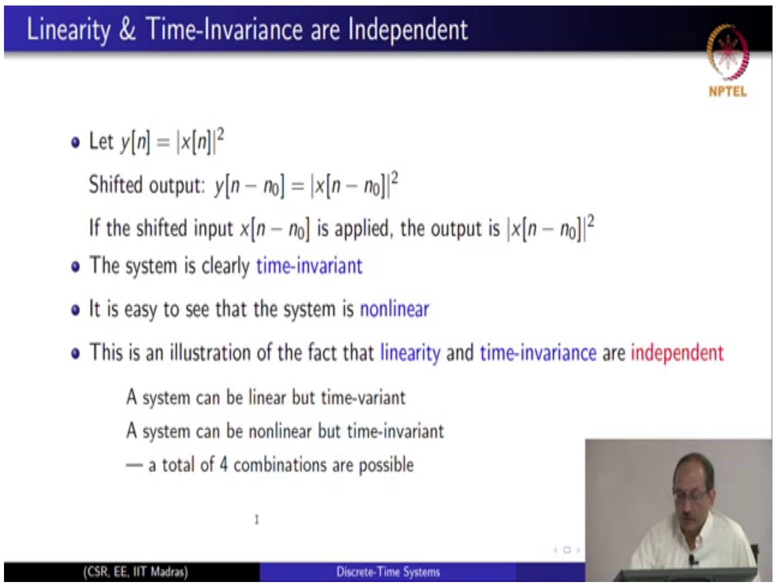

And, the linearity and time-invariance are independent. System can be linear and not time-invariant and so on. So, $y[n] = |x[n]|^2$; so, this is clearly time-invariant, because you shift the output that is one part, the other check is you shift the input and then apply to the system and then compare the outputs, clearly these are the same. So, clearly the system is time-invariant and very easy to see that this is non-linear.

Therefore, these are independent properties; system can be linear but time variant, can be non-linear but time-invariant and so on, so forth combinations are possible. So, later we will focus on systems that are both linear as well as time-invariant.

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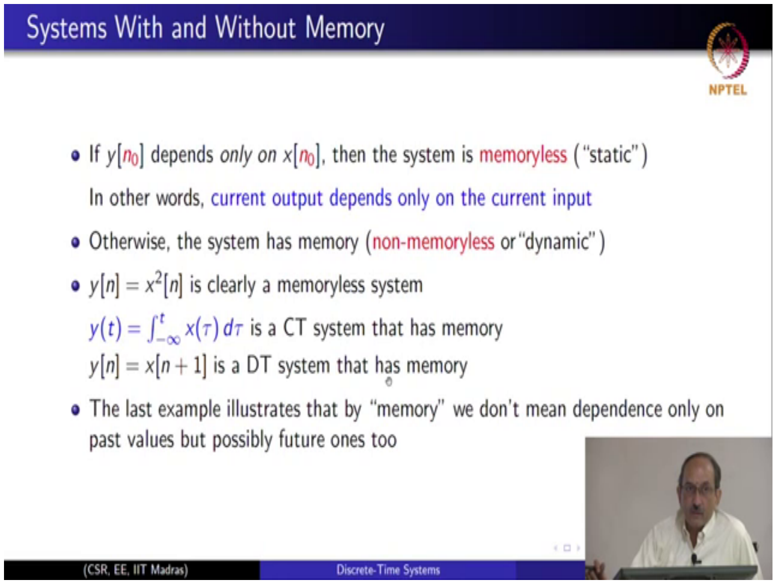

The next property as for a systems go is systems with and without memory. So, if $y[n_0]$ depends only on $x[n_0]$, then the system is memoryless the other name of course, is static.

So, the current output depends only on the current input or the present output depends only on the present input. Otherwise, the system is said to be having memory, it is non-memoryless or dynamic. So, $y[n] = x^2[n]$ is clearly memoryless system, on the other hand $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$ is the CT system that has memory.

And, here is the another simple example; $y[n] = x[n+1]$ is a DT system that has memory and the reason why this example is, usually when we talk of memory, in ordinary language, we are used to always memory of the past. Because in ordinary circumstances, memory of the future does not make any sense whereas, in this context $y[n] = x[n+1]$ is indeed a system that has memory.

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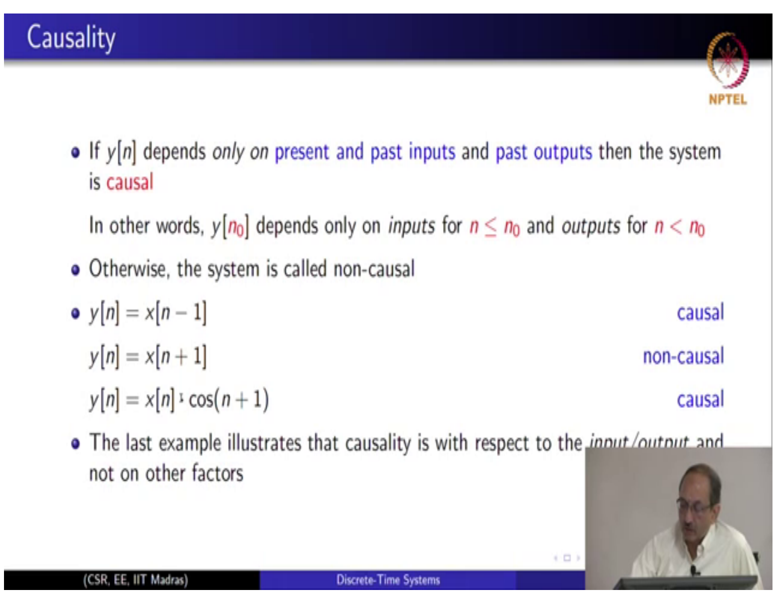

The next property is causality. So, if y of n depends only on the present and past inputs and past outputs, this system is causal. In other words, $y[n_0]$ depends only on inputs for $n \leq n_0$ and outputs for $n < n_0$. As far as causal systems are concerned, the current output can depend on current input and all past inputs and past outputs, if this is not satisfied, this system is non-causal.

So, $y[n] = x[n-1]$, this is a very simple system that is causal. $y[n] = x[n+1]$ is non-causal, because for example, $y[0]$ depends on $x[1]$, so called future input. And $y[n] = x[n].\cos(n+1)$, this is still causal, because as far as causality is concerned, it is the input that we are concerned with. You should not be fooled into thinking, you see an $n+1$ term here therefore, this might be non-causal, it is not right.

So, causality applies only to the input part. So, you are talking about the case where you are applying two inputs, all right.

Student: That can be (Refer Time: 16:31).

So, in general when you talk about an input, you assume an $x[n]$ to be a generic function, right? You do not talk about $x[n]$ taking specific force, when you talk about input-output relationship, some arbitrary input is there and then you find out what the output is. So, it is in that context we are talking about causality here. So, this is general right, expected input is some arbitrary input, it is not tied to any specific signal.

Student: Sir, what if $y[n] = x[n].z[n+1]$?

So, ok.

Student: Where it would be non-causal.

Right; so, now, what is $z[n+1]$?

Student: $z[n]$ is the (Refer Time: 17:27).

Ok. So now remember, we are now talking about single input single output systems. So, what you are raising is something that is related to MIMO; Multiple Input Multiple Output. So, that theory surely can be addressed in that context, but as far as this is concerned, we are only worried about single input single output, all right? So, does that answer your question?

Student: Yes sir.

So, system can be causal or non-causal based on these definitions.

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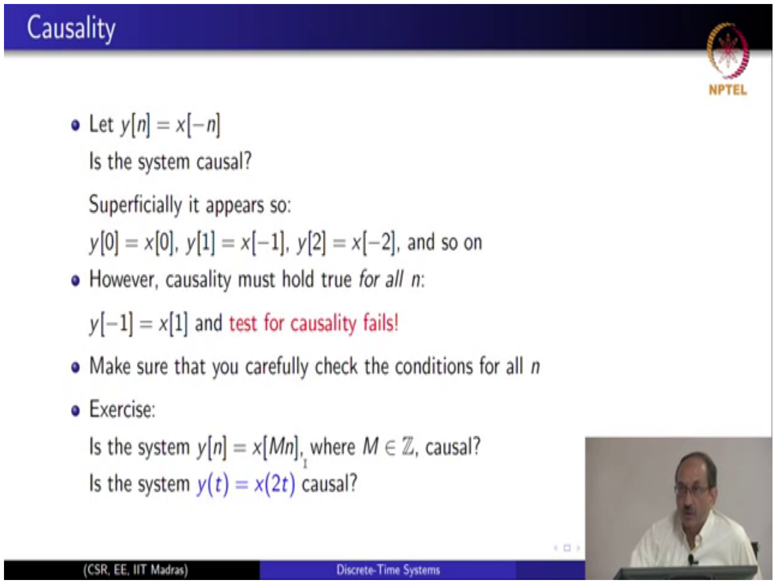

And, again if $y[n] = x[-n]$ as an example so, if you look for test for causality, superficially it appears to be causal because $y[0]$ is $x[0]$, $y[1]$ is $x[-1]$ and so on. But, remember, causality must be true for all n, $y[-1]$ is $x[1]$ and the causality test fails. So, these simple things might trap you into giving the wrong conclusion. So, make sure you test these things completely.

So, as an exercise a very easy to do, $y[n] = x[Mn]$, is this causal? $y(t) = x(2t)$, is this causal? Again this is no different from what was happening earlier, you would be given an input-output relationship and then you have to test for various system properties. So, in this case, we will focus on discrete-time systems. And, you should be able to see the parallels to continuous-time systems, you may have done some of these for discrete-time systems as well.

So, that is fine. So, this is just recall, some overlap is fine. So, it strengthens your understanding.

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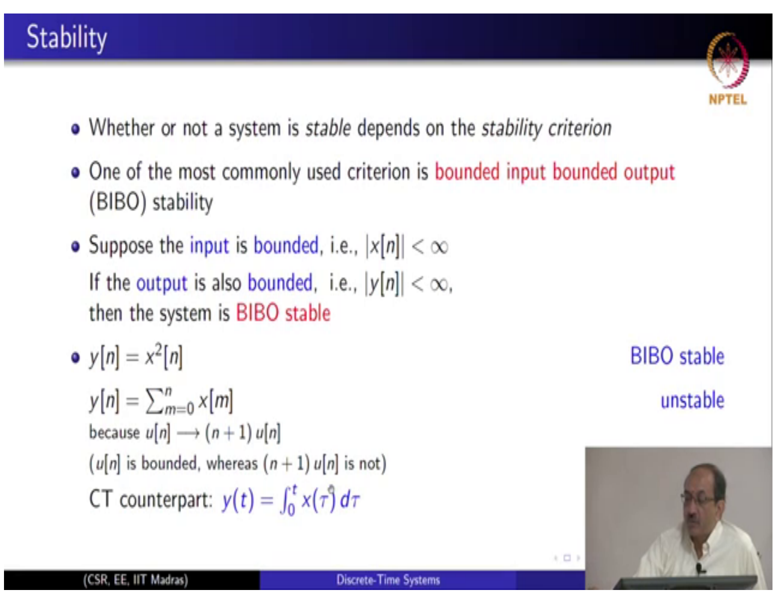

The next property we look at is whether or not the system is stable, whether the system is stable or not, depends upon the stability criterion. There is no one universal stability criterion and the most common stability criterion is bounded input bounded output (BIBO), that is if you give an input that is bounded, then for stable system, you expect the output also to be bounded. If for a bounded input, you get an unbounded output, then the system is not stable. Of course, if you give an unbounded input, then all bets are off.

So, the input is bounded by B_x then if the output also is bounded say B_y, then the system is BIBO stable. So, $y[n] = x^2[n]$ is clearly a stable system, because if the input is bounded, output also is bounded, because all you are doing is you are nearly squaring the input. On the other hand, if you have $y[n] = \sum_{m=0}^{n} x[m]$, then if you give this system the unit step as the input, the output is $(n+1)u[n]$. Clearly, $u[n]$ is the bounded input whereas, $(n+1)u[n]$ is not.

So, here if the input-output relationship is this, then this system is not BIBO stable, because here is a simple example that blows up when you give it to the system. And, the corresponding counterpart for the continuous-time system is the running integral. Here is the running sum CT counterpart is the running integral. Here, if you give again $x(t)$ to be $u(t)$, the running integrals are ramp which is not bounded.

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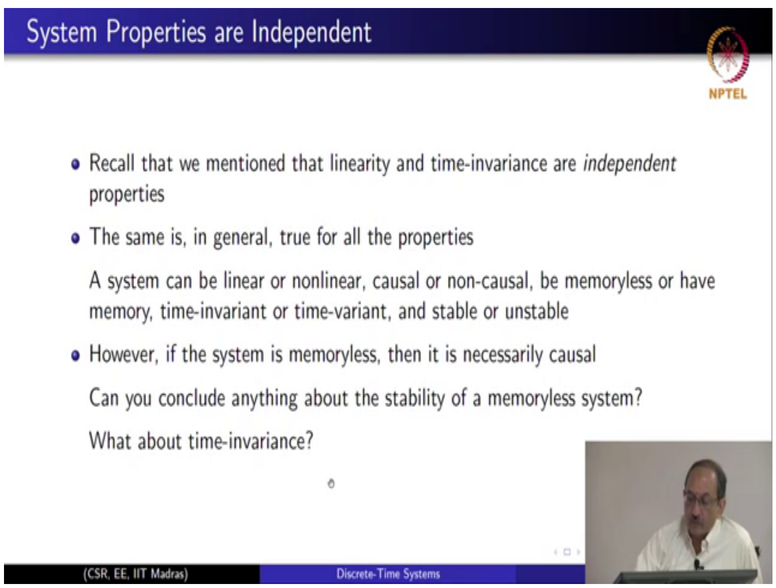

And generally, system properties are independent. We saw already, linearity and time-invariance are in general independent. So, you can have any combination of these properties. However, if the system is memoryless then it is necessarily causal. So, if the system is memoryless it cannot be non-causal. So, some properties have dependence.

So, again this is something about you to think about, what about stability of a memoryless system? What about time invariance of a memoryless system?

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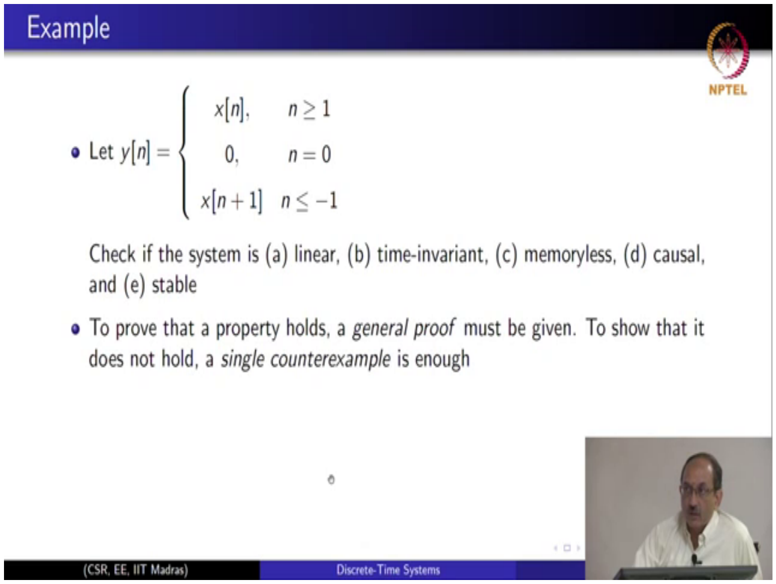

So, now let us take one concrete example and then check for all these properties. So, we want to check whether this system is linear, time-invariant, memoryless, causal and stable. Again as I mentioned earlier, if property holds, you have to give a general proof. If it does not hold, a single counter example is enough.

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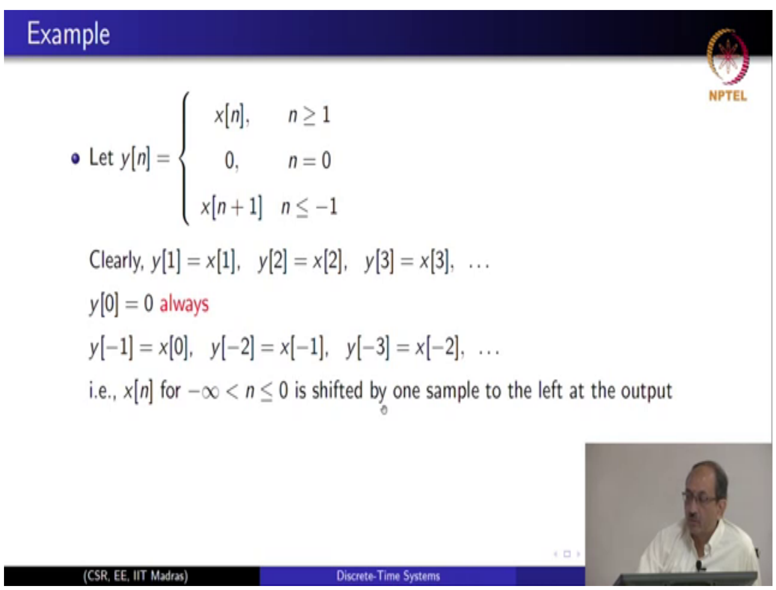

So, now let us look at this particular example. So, clearly $y[1] = x[1]$, $y[2] = x[2]$ and so on. So, $y[n] = x[n]$ for $n \geq 1$, $y[0] = 0$ always, that is how this input-output relationship is and $y[-1] = x[0]$, $y[-2] = x[-1]$ and so on. So, it is $x[n]$ shifted by 1 sample to the left.

So, this what you get as the output as far as this system is concerned. So, for linearity, if x_1 gives you y_1, x_2 gives you y_2, then $a_1x_1 + a_2x_2$ should give you $a_1y_1 + a_2y_2$. So, that is the check you need to make.

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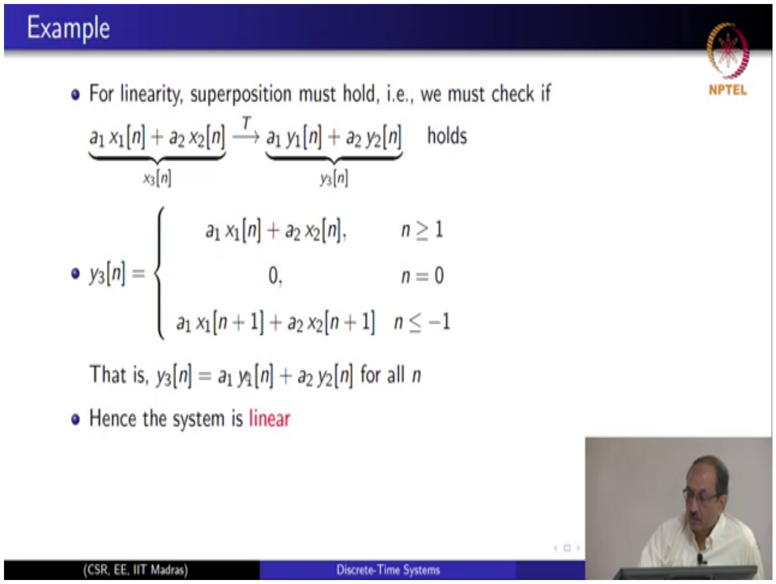

So, let us assume the input is x_3. So, x_3 gives me y_3, x_3 in turn is $a_1x_1 + a_2x_2$. So, since x_3 is the input, this exactly the input-output relationship. All I have done is, I have replaced y by y_3 and x by x_3. Otherwise everything is the same, but now we know what x_3 is. x_3 is nothing, but $a_1x_1 + a_2x_2$.

Student: (Refer Time: 23:24).

So, for various indices I have replaced x_3 by the corresponding definition. Therefore, it is easy to see that y_3 is indeed $a_1y_1 + a_2y_2$ for all n. Therefore, the system is indeed linear.

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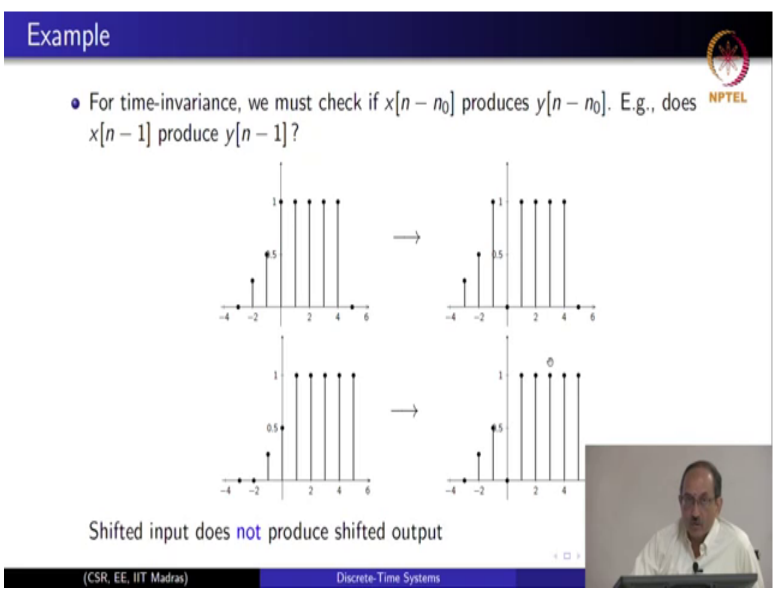

The, next check is for time-invariance. One intuition you can have is remember, $y[0]$ is always 0 for this particular example, the ways being defined. So, we want to check whether $x[n-n_0]$ produces $y[n-n_0]$.

In particular, does $x[n-1]$ produces $y[n-1]$? So, here is one input I have taken and remember $y[0]$ is always 0. And, for all positive indices, the output is the same as the input, for negative indices all you need to do is shift the input by 1 sample to the left. So, that is exactly what is happening here because $y[-1]$ is $x[0]$ and so on, all right?

So, this is indeed the output for this given example. Now, the first step is to shift this by 1 sample to the right. If I shift this, the entire curve shifts by 1 sample to the right. In particular, $y[0]$ after the shift by 1 sample will no longer be 0. I have not shown this shifted curve here, but easy to see that you can shift this by 1 sample to the right. So, this 0 will come here. Now, let me shift the input by 1 sample to the right and then check the output. So, this is the shifted input by 1 sample to the right and here I get the output and notice that the output $y[0]$ will always be 0. So, this is shift by 1 sample to the right, by definition I will get 0 here.

So, this and this shifted by 1 sample to the right or not the same. Therefore, this system is time-variant. So, shifted input does not produce shifted output.

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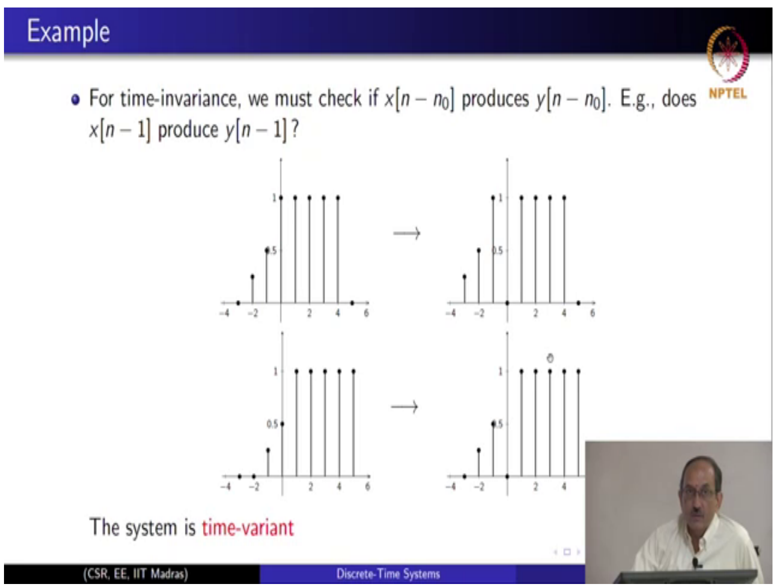

Therefore, the system is time-variant.

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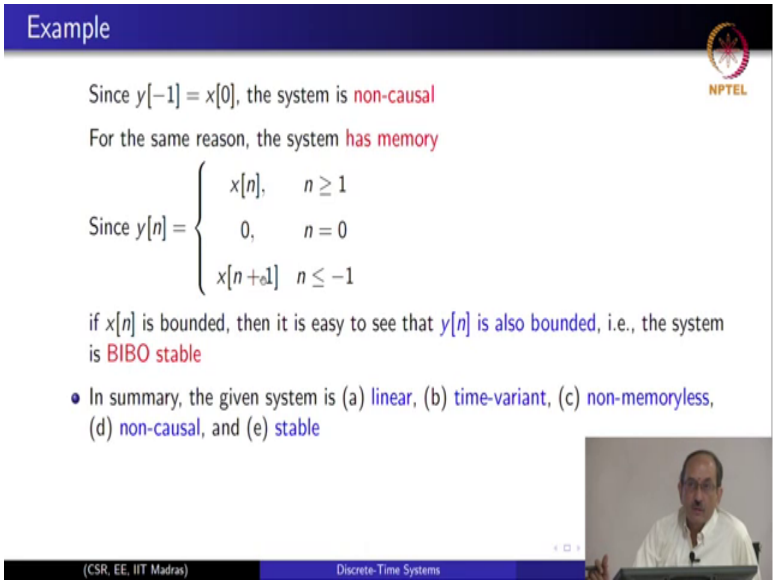

Since $y[-1] = x[0]$, this system is non-causal. And, for exactly the same reason, this system has memory. And if the input is bounded, remember for $n \geq 1$, the output is the same as the input, at $n = 0$ the output is always 0, for $n \leq -1$ you will shift by 1 sample to the left. So, if the input is bounded, the output always will be bounded. Therefore, the system is BIBO stable.

So, system is linear, time-invariant, non memoryless, non-causal and stable. So, this is the typical kind of exercise that you should be familiar with in terms of question in the quiz or the exam. Given a certain input output relationship, are you able to test for each of these properties clearly without any confusion? There should be clarity in your steps, that is all.