Digital Signal Processing

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Lecture 10:
Elementary Signals (4), Systems and their Properties (1)
Independence of sinusoidal signals

Welcome to lecture 5. Last class we started looking at continuous-time sinusoids and discrete-time sinusoids. And we saw that two main differences. The two main differences are as you keep on increasing the frequency, unlike in the continuous-time case the discrete-time case, the repetitive of the oscillations reaches some maximum, and then the signal starts to slow down. The other important difference is not all frequencies give rise to sequences that are periodic.

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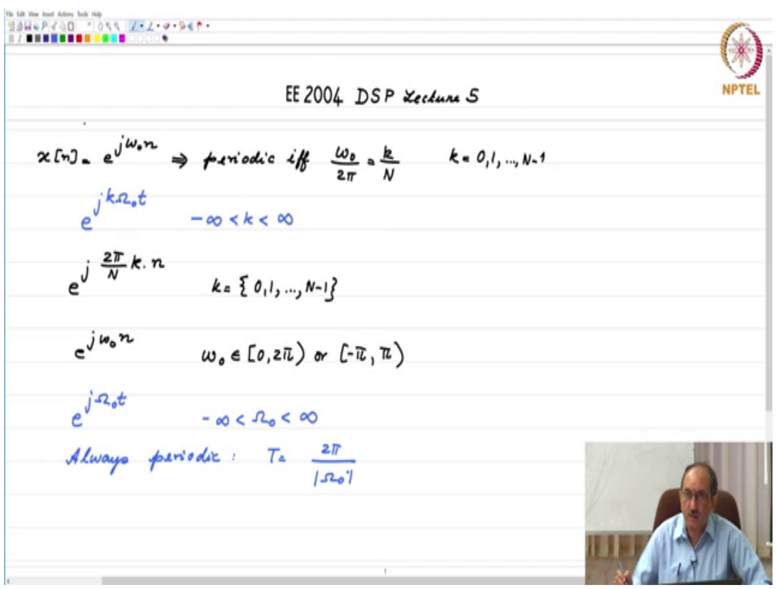

So, the discrete-time sinusoid $e^{j\omega_0 n}$, so this if it is periodic, then if and only if $\dfrac{\omega_0}{2\pi} = \dfrac{k}{N}$. And then we also saw that you need to restrict yourself only in the range $0, 1, \ldots, N-1$, so k is actually $\langle k \rangle_N$. Any value of k that is outside this gives rise to sinusoid whose k value falls within this range. So, just to recap some of the consequences that I had mentioned because of this towards end of the last class, if you consider $e^{jk\Omega_0 t}$, as you let k vary in this interval, you get countably infinite number of sinusoids that are harmonic in the continuous-time case.

Whereas in the discrete-time case, if you had $e^{j\frac{2\pi}{N}kn}$, k takes on the values $0, 1$ up to $N-1$. Therefore, k here belongs to the set $0, 1, \ldots, N-1$. So, this is a finite set. So, in the discrete-time case, the number of harmonic related sinusoids is a finite set, whereas in the continuous-time case, the set of harmonically related sinusoids is countably infinite. So, this is one consequence that follows.

The other consequence is if you had $e^{j\omega_0 n}$, ω_0 belongs to the interval 0 to 2π or equivalently $-\pi$ to π. As ω_0 varies from 0 to 2π continuously, you get an uncountably infinite number of sinusoids that are generated, of these only N of them or periodic. And those N set of sinusoids that are periodic have to satisfy $\frac{\omega_0}{2\pi} = \frac{k}{N}$. And in those instances, the period T is N.

So, even though you have uncountably infinite number of sinusoids as you vary omega naught from 0 to 2π, only N of them are periodic. On the other hand, if you had $e^{j\Omega_0 t}$, in this case Ω_0 can take on values in the interval $-\infty$ to ∞. For every single value of Ω_0, the continuous-time sinusoid is periodic; for every single value of Ω_0, as Ω_0 goes from $-\infty$ to ∞, every single sinusoid is periodic with period $T = \frac{2\pi}{|\Omega_0|}$. So, this is always periodic, period is $\frac{2\pi}{|\Omega_0|}$.

So, these are the two main differences that you see between continuous-time and discrete-time or the set of harmonic related sinusoid is finite whereas in the continuous-time case, the set of harmonically related sinusoids is infinite. And in the continuous-time case, it is always periodic, whereas in the discrete, time it is periodic only if this condition is satisfied. Now, let us look at another aspect of sinusoids again to reinforce some of the ideas that you may have already learnt from signals and systems.

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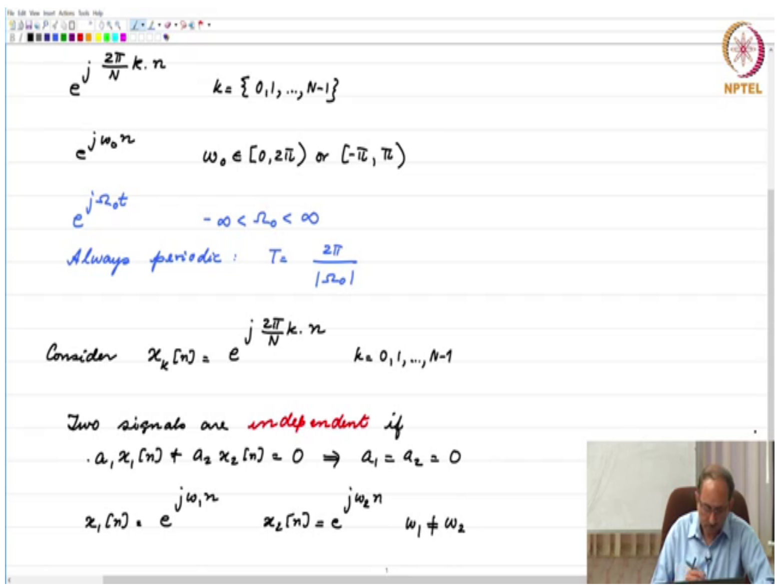

So, let us consider $x_k[n]$ of the form $e^{j\frac{2\pi}{N}kn}$. So, these are clearly harmonic related sinusoids and k of course, goes $0, \ldots, N-1$. And these are periodic, and there are N such sinusoids here. So, as k varies, you get distinct sinusoids. For each value of k, you get distinct sinusoids. And distinct is kind of the term that we understand, but we need to make it precise mathematically. So, when I say as k goes from $0, 1, 2 \ldots N-1$ for each value of k, you get distinct sinusoid. If you want to make it precise, what would be your way of making it mathematically precise? We know loosely what it means to say that these are distinct, we kind of have a picture, but.

Student: (Refer Time: 08:21).

The frequencies are harmonic all right, but then remember here we are trying to make precise the notion that these two sinusoids are distinct. I mean is there any equation that you can put down to capture this?

Student: Orthogonality.

Ok, orthogonality is a good guess. So, if two sinusoids are not orthogonal, then does it mean they are not distinct, I mean orthogonality is very good guess, I mean say this definition and then see whether you can pin it down to an equation. The notion that I am after is I want two sinusoids to be independent. The reason why I was little hesitant about orthogonality is, when it comes to vectors, there are vector that can be independent but not orthogonal.

So, here drawing upon your knowledge of two vectors being independent, if you want to talk about two signals being independent, and how would you quantify that notion? So, two signals or sequences in this case, two signals are independent if

$$a_1x_1[n] + a_2x_2[n] = 0 \quad \Longrightarrow \quad a_1 = a_2 = 0.$$

So, this is the notion of independence of signals.

Now, let us considered the general case, here we have consider the case where we have $x_k[n] = e^{j\frac{2\pi}{N}kn}$. So, if $x_1 = e^{j\omega_1 n}$ and $x_2[n] = e^{j\omega_2 n}$, and we are given that $\omega_1 \neq \omega_2$.

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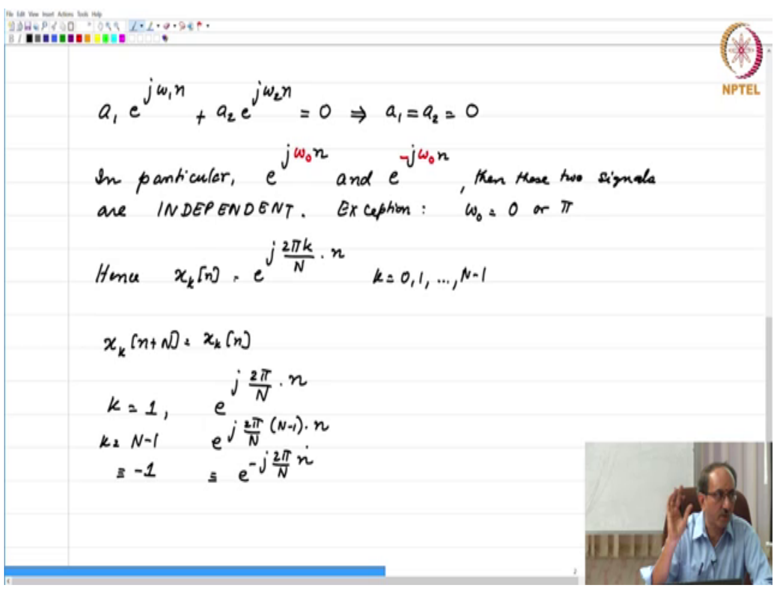

Then you can actually verify that $a_1e^{j\omega_1 n} + a_2e^{j\omega_2 n} = 0$ will imply $a_1 = a_2 = 0$, all right. So, if you have two frequencies, ω_1 and ω_2, these are independent. So, this is an easy thing to verify. What is the intuition behind this, you can also gather an intuition from this equation. If two signals are dependent, then one can be expressed as a constant times the other. So, this what dependence means. And this constant in general can be complex.

Therefore, it is easy to see once you have this intuition associated with dependent signals. We have two distinct frequencies, no way can you take one sinusoid multiplied by a complex constant and get the other signal which is a different frequency. So, it is very clear that if two frequencies are different, they cannot be a scalar times each other, all right. Now, in particular, once you have this definition in place, $e^{j\omega_0 n}$ and $e^{-j\omega_0 n}$, that is if the two frequencies are ω_0 and $-\omega_0$, then these two signals are dependent or independent? Now, these two signals dependent or independent?

Student: Independent (Refer Time: 13:26).

Independent? Some of you think, it is dependent?

Student: Suppose even (Refer Time: 13:31).

Ok, why is $\omega_0 = \pi$, a special case?

Student: These are the same thing, I think it is a same thing.

Yeah, that is a very good observation. So, if you have, this is a special case in which $\omega_1 = \omega_0$ and $\omega_1 = -\omega_0$, right. So, this is true for all $\omega_1 \neq \omega_2$, where $\omega_1 = \omega_0$ and $\omega_2 = -\omega_0$. So, they are independent. And as observed correctly, the only exception to this is when $\omega_0 = 0$ or π. If $\omega_0 = 0$, then $e^{j\omega_0 n}$ and $e^{-j\omega_0 n}$ are the same. When $\omega_0 = \pi$, $e^{j\pi n}$ and $e^{-j\pi n}$ were exactly the same signal.

Therefore, exception to this is ω_0 being either 0 or π. So, this is the general case. So, this also applies to the case when the signal is harmonic. So, hence $x_k[n]$, yeah, go ahead.

Student: (Refer Time: 15:30).

Ok.

Student: So, (Refer Time: 15:37).

Let us examine $\omega_0 = \dfrac{\pi}{2}$. So, I then saying $e^{j\frac{\pi}{2}n}$, and $e^{-j\frac{\pi}{2}n}$ are the same, one can be express it as.

Student: (Refer Time: 15:50).

No.

Student: (Refer Time: 15:52).

Hm

Student: (Refer Time: 15:53).

Ok.

Student: (Refer Time: 15:54).

So, when you are taking two, when you are considering real valued signals, we will come to real valued signals after this.

Student: I was talking about the linear combinations.

Ok, but when you are talking about the linear combinations, you are considering the special case of $e^{j\theta n} + e^{-j\theta n}$ of that form, am I understand you right? So, the real valued case will come now. So, $x_k[n] = e^{j\frac{2\pi}{N}kn}$ as $k = 0, 1, \ldots, N-1$, so you have N independent signals. So, we have $x_k[n+N] = x_k[n]$, and there are N independent signals that are harmonically related.

Now, let us come to the real valued case. Now, the $e^{j\omega_0 n}$ and $e^{-j\omega_0 n}$ are independent. Applying that to this specific case, if you put say $k = 1$, you get $e^{j\frac{2\pi}{N}.n}$. So, this is one signal. And if you put $k = N - 1$, you have $e^{j\frac{2\pi}{N}.(N-1).n}$. So, here is an example within this set of two complex exponentials that are independent. And remember $k = N - 1$ is also the same as k equal to?

Student: -1.

-1.

Therefore, this is the same as $e^{-j\frac{2\pi}{N}.n}$. So, here are two specific examples of this set in which the two sinusoids complex exponential are independent. So, we will use the term sinusoids, complex exponentials interchangeably, even when the signal is the cosine we will refer to that as a sinusoid.

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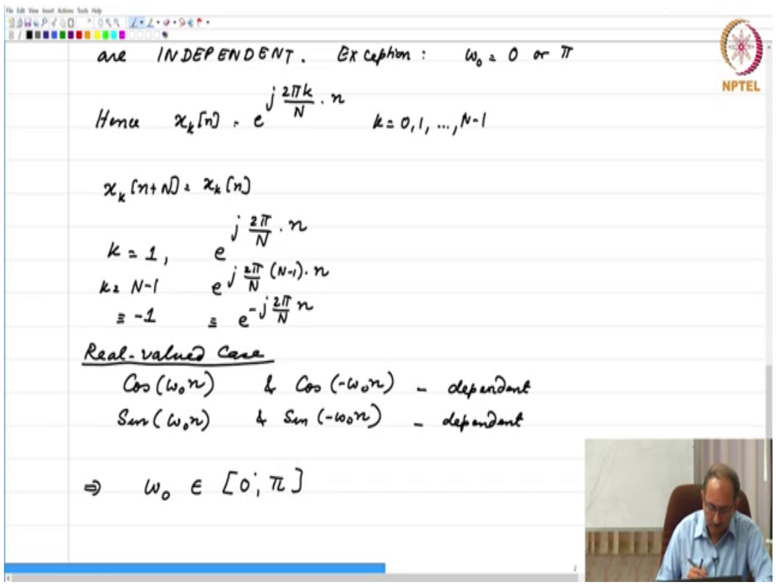

Now, coming to the real valued case, suppose you have $\cos(\omega_0 n)$, and if you consider $\cos(-\omega_0 n)$, remember in the complex case $e^{j\omega_0 n}$ and $e^{-j\omega_0 n}$ are independent of each other, which means you cannot take $e^{j\omega_0 n}$ multiply by a complex constant and hope to get $e^{-j\omega_0 n}$, these are two completely independent signals.

For the real valued case, what about $\cos(\omega_0 n)$ and $\cos(-\omega_0 n)$? They are exactly the same signal, therefore this is dependent. And the constant associated with them is unity. We can go from one to the other by multiplying by 1. $\sin(\omega_0 n)$ and $\sin(-\omega_0 n)$, again they are dependent. And the constant that you need to use to go from one to the other is -1, all right, therefore, these are dependent signals.

So, one fall out of this is that this implies you need to restrict yourself as far as ω_0 is concerned only in the interval, what is the range of ω_0 you need to consider when you consider real valued sinusoids, both cosine and sine.

Student: 0.

0 to π.

Student: π.

0 to π, both points included, because any frequency that is in the interval π to 2π can always be generated as a frequency in the interval 0 to π.

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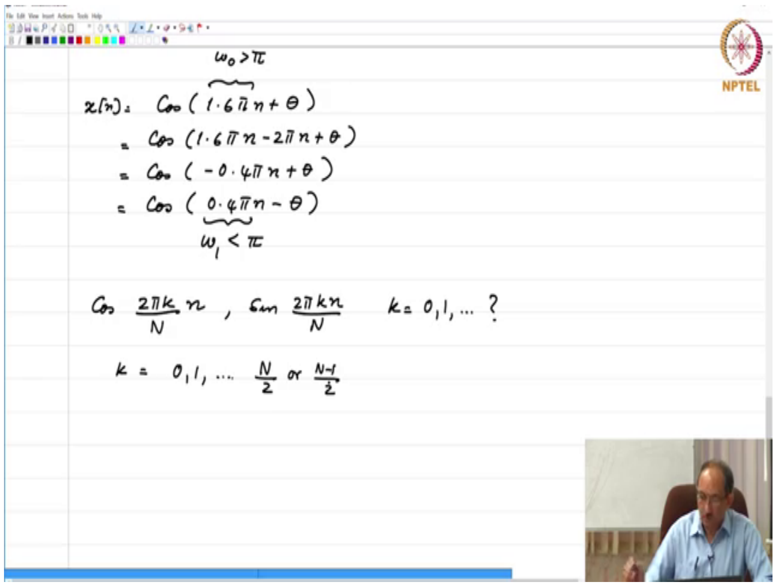

And as an illustration of this, suppose you have $x[n] = \cos(1.6\pi n + \theta)$. So, here is an ω_0 that is greater than π. So, this can be written as $\cos(1.6\pi n - 2\pi n + \theta)$, nothing has changed. So, this intern is $\cos(-0.4\pi n + \theta)$. And since $\cos(-\theta) = \cos(\theta)$, I can also write this as $\cos(0.4\pi n - \theta)$.

And now if I focus on this by call this as ω_1, now this is less than π. This illustrates the fact that, any real valued sinusoid whose frequency is greater than π or in particular it is between π and 2π can always be transformed to another sinusoid whose frequency is between 0 to π. Whereas, you cannot do this in the case of a complex sinusoid; in the case of complex sinusoid, you need to consider ω_0 in the range 0 to 2π. And in terms of harmonic sinusoids, suppose if you consider $\cos\left(\frac{2\pi}{N}kn\right)$ and $\sin\left(\frac{2\pi}{N}kn\right)$, so these are real valued periodic sinusoids. So, what is the range of k you need to consider; what is the range of k you need to consider for these two specific real valued sinusoids?

Student: (Refer Time: 23:25).

Ok, so those are correct answers. So, you need to consider that value of k between $0, 1, \ldots, \frac{N}{2}$, where N is even or $\frac{N-1}{2}$ when N is odd. Because any value of k that exceeds this can always been mapped to an index k that is between 0 to $\frac{N}{2}$ or $\frac{N-1}{2}$. Now, recall this particular thing that is, the set of complex sinusoids that are harmonically related, $e^{j\frac{2\pi}{N}kn}$, k varies from 0 to $N-1$, and there are N independent sinusoids in the set; for each value of k, you will get a sinusoid that is independent of any other value of k. So, totally N independent sinusoids are there.

Now, let us come and look at this. Now, let me look at this particular set. So, k goes only from 0 to $\frac{N}{2}$ or $\frac{N-1}{2}$. How many independent sinusoids will be there, when you look at cosine and sine? So, this is an exercise that I want you to try out later. So, how many independent signals are there in this set?

So, all you need to do is you need to consider the case N odd and N even separately. And whether it is N even or N odd, you have to convince yourself the total number of independent signals in this set when you consider both cosine and sine together, the total number is indeed still only N, all right.

Even in this case, the number of independence signals only be N. But for that, you have to enumerate all the signals considering N odd and N even separately. So, the simple exercise, but make sure you do it and you are comfortable with the answer that you get. You have to convince yourself that, indeed even in this set also there are only N independent signals.

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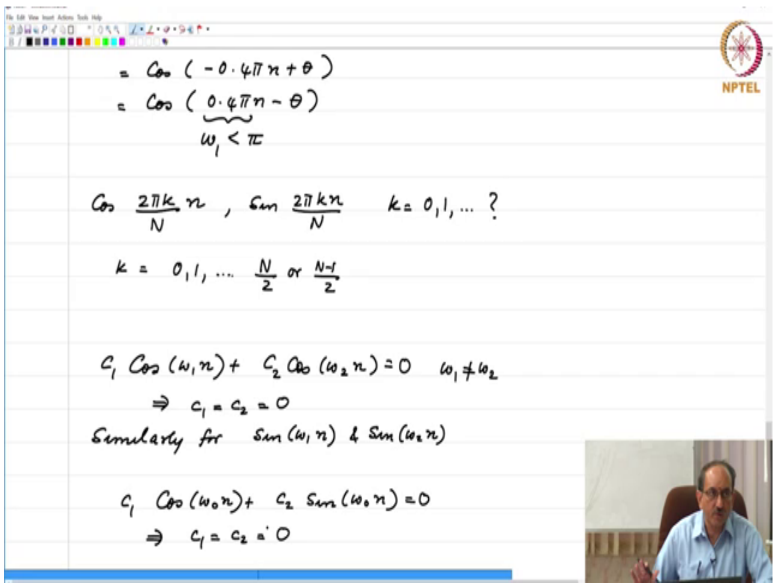

So, again continue on this independence, just to wrap up with the couple of more points. If you have $\cos(\omega_1 n)$ and $\cos(\omega_2 n)$, $\omega_1 \neq \omega_2$, again these two signals will be independent or dependent? It will be independent.

Therefore, $c_1 \cos(\omega_1 n) + c_2 \cos(\omega_2 n) = 0$, then $\omega_1 \neq \omega_2$ will imply $c_1 = c_2 = 0$. Similarly, for $\sin(\omega_1 n)$ and $\sin(\omega_2 n)$, these two are independent when $\omega_1 \neq \omega_2$.

Then finally, suppose the frequency is the same, I have $\cos(\omega_0 n)$, and I also have $\sin(\omega_0 n)$. Now, these are independent or dependent? Again this should be very easy. All you need to ask at an intuitive level is can I take cosine multiplied by a constant which can take on even complex values. Can I take cosine multiplied by some constant and get sine, is that possible? No. Therefore, for the same frequency, $\cos(\omega_0 n)$ and $\sin(\omega_0 n)$ are independent. Therefore, $c_1 \cos(\omega_1 n) + c_2 \cos(\omega_2 n) = 0$ will imply $c_1 = c_2 = 0$. So, this kind of, ties up of all the things related to discrete-time sinusoids.