Linear Systems Theory
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Module - 02
Lecture - 09
Math Preliminaries: Linear Algebra 3

Ok, in this lecture, we will talk a bit about linear maps or linear transformations essentially between two vector spaces.

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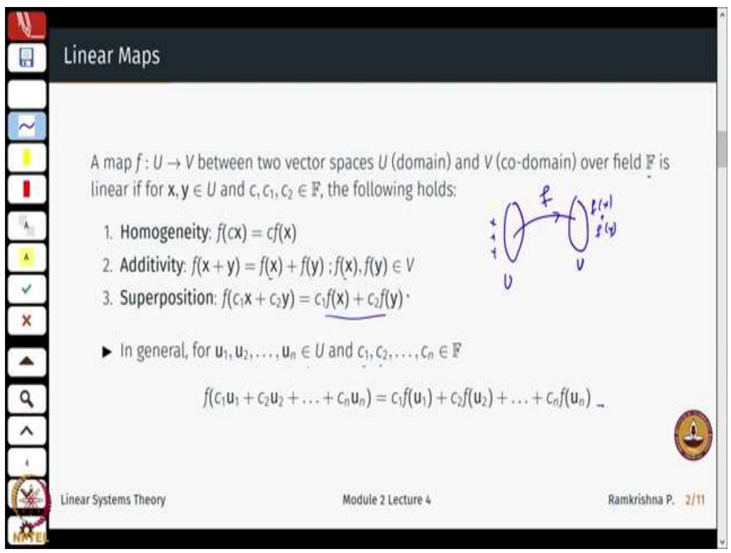

So, to begin with let us let us start with a map say I have a space U, I have a space V and there is a map f between a vector space U and V and both could be defined over the same, same field; they did not be of the same dimensions and so on. So, this map F is linear if and only if the following definition have the homogeneity, you have additivity and the superposition things something like this.

So, for examples, if x is an element here, f (x) is an element of over here. So, if you add two elements here x and y, they will be added like in this way, so in f (x)+ f (y). So, is the same as adding. So, this plus, this is the same as adding this and this in the in the in the co-domain space. So, in general if I have vectors U_1 till U_n and number $c_1$to c_n belonging to F. So, something like this feels like a generalization of what we have written over here. So, this is something which defines what are linear maps and linearity we would have studied in linear circuits. And an obvious there are thing to check for is the principle of superposition that is exactly holds here also.

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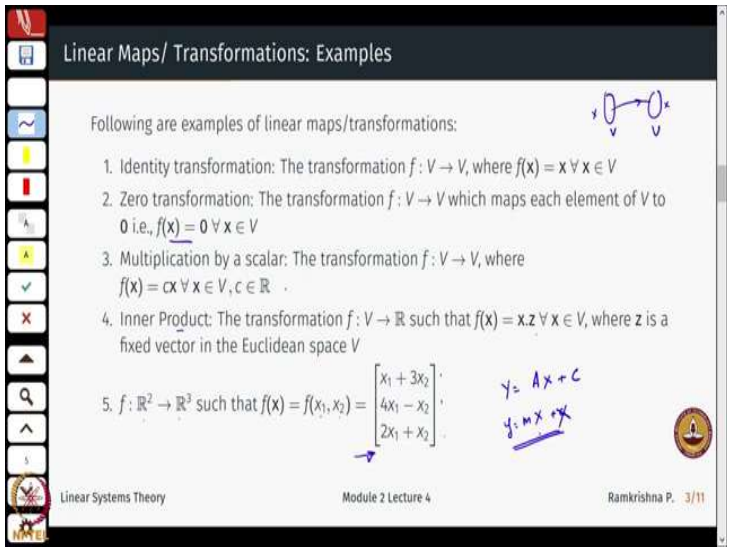

So, what are few basic examples, one is the identity map, where I just start from x. So, I have say both my domain and co domain are V. So, I start from an x and then I come back to x then it is an identity map. And it is it is it is very trivial to check the properties of superposition. Second is a zero transformation it is that f of x actually maps to 0. Multiplication by scalar is also an easier thing to check. The transformation defined by the inner product such that f (x) =xz is also a linear transformation.

Similarly, now if I if I look at different dimensional subspaces, a map going from R^2 to R^3 such that f (x) which has 2 takes two elements and gives me a three-dimensional vector defined by such a map is also a linear transformation right. So, in instead of I can write this as say some Y =AX. However, what is not a linear map is if I just put some constant vector here or some constant number right. So, so if I just talk in terms of real line I have this is a linear map y = m x, but as soon as add c, this is no longer a linear transformation especially it because of this term here ok.

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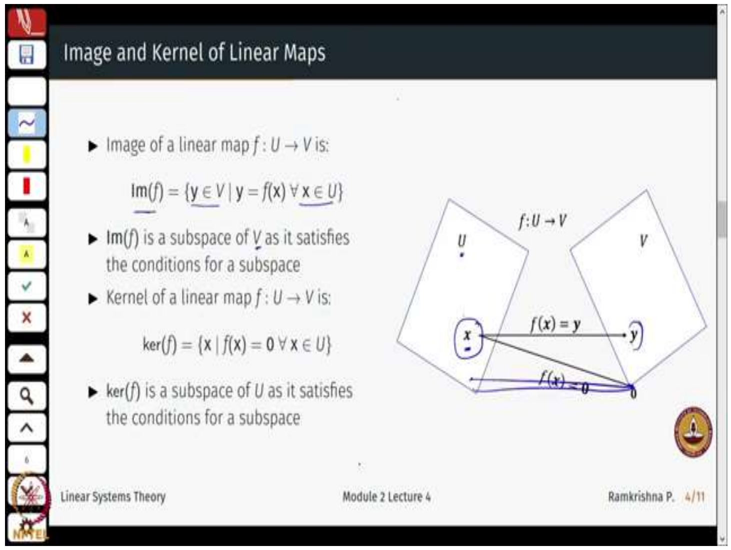

So, once we have defined the map f, it is it is it is important to define two other concepts. So, first is called the image of f. So, image of f. So, again let us draw picture of its. So, pictures are here. So, the image of f is the set of all y's here which come from V such that this y is comes from some x in the space you write, and this should hold for all x belongs to U, so that is called the image of this map f. Similarly, the kernel of a map is defined as the set of all x's which map to the zero vector here, or the zero element in V right. So, for also the set of all x, so this here is just one.

So, there could be several of the x which can map here, this could be some here and so on, so that is called the kernel. And it is easy to check as it is a, it is a, it is a small proof which you can do by yourself is that the image is a subspace of V. And it is (Refer Time: 04:32) to check that it satisfies the conditions of a subspace that all the properties of V over f are carried on to these subspaces. Similarly, the kernel is a subspace of U and this is the set of all x's which come from U; this is also a subspace. And it is it is easy to check that it also satisfies the conditions for a subspace, where allowed the proofs for this, but this you can do as an as an exercise for yourself, ok.

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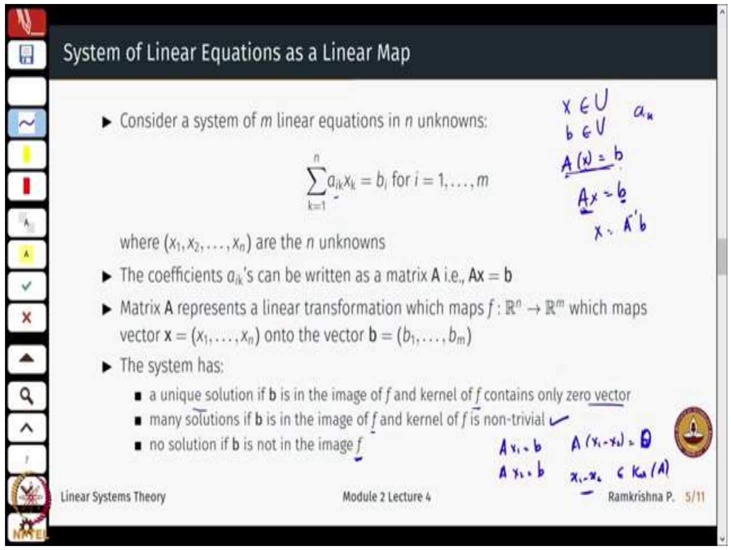

So, what for what are the properties are you know are there any applications for this of defining the image and kernel? So, let us start with system of m equations right. So, let us say that x belongs to some space U, b belongs to some space V, and A here defines the linear transformation. If I just denote it as a transformation A(x) will give me this number b, and a system of linear equations would solution would mean does there exist x such that which under this transformation gives me the number b ok.

So, what can we say about the solutions of this right. So, the solution, so in general if I if I look at as a set of linear equations written here A x = b you would say well invert that x equal to A^{-1}b is a solution, but A need not always b invertible. So, what can we say about existence of solutions of this right. So, first is for a solution to exist this b must be in the image of A, just for a solution to exist.

There can be multiple solutions also right. Just for the solution to exist this on it that the b is in the image of f ok. When will be, when, when will it be a unique solution, the unique solution would be when the kernel of this map is the zero vector right. And there is no solution if b is not in the image of f. Again I will not do proof. So, this, but let us assume that there are two solutions right Ax_1 is gives me b; Ax_2 also gives me b ok. $A(x_1 - x_2)$ will be 0, sorry this is the zero vector right.

So, here I can say that $x_1 - x_2$ is in the kernel of A ok. Now, from here you can kind of verify this statement that the solution is unique if and only if the kernel of f or the A in this case consists of the zero vector right. So, x 1 and x 2 should be unique, then x 1 minus x 2 is actually 0, then they will be equal. So, a little verification of this, but you can write down the details for yourself and check ok.

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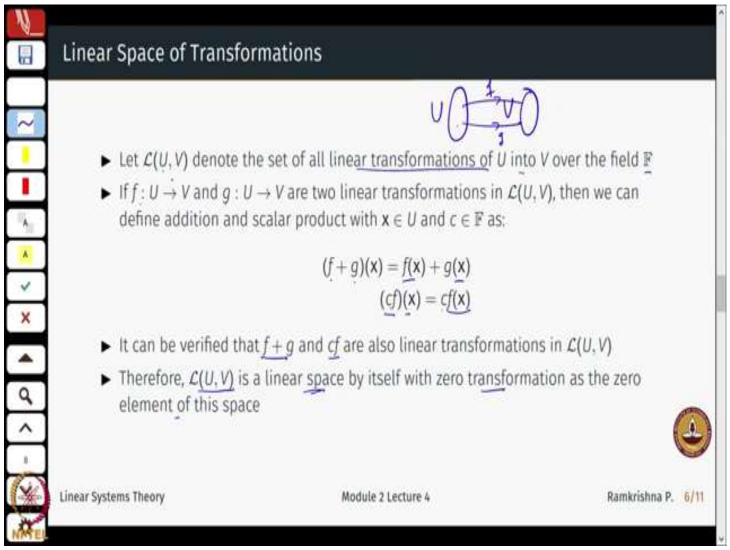

So, let us look at the linear space of all transformation. So, let us I just take U and V, the usual subspaces, and I clubbed together or set of all linear transformations of U into V, again defined over a field F. So, say for example, if f() and g() are two linear transformations say I have a f here, I have a g here, then we can define things that we said (f +g)(), yes, this is f this is g. So, (f+g)(x) will be similarly will be the same as adding f(x) and g(x) same that (Refer Time: 08:26) the function spaces things like that we saw earlier. Similarly, I can do it with the scaling right. So, (cf)(x) acting on the x will be same as c(f(x)).

So, it can be verified that f plus g and this c times say for also linear transformations on F very simple to verify the properties that we had seen over here. I will not go into the details of that, but I think it should be a kind of kind of obvious now. And therefore, the space L U, V which is the set of all linear transformation of V into U V of sorry of U into V is by itself a linear space with the zero transformation as the zero element that is a kind of a concluding trivial statement to show that L U, V is actually a linear space of transformations ok.

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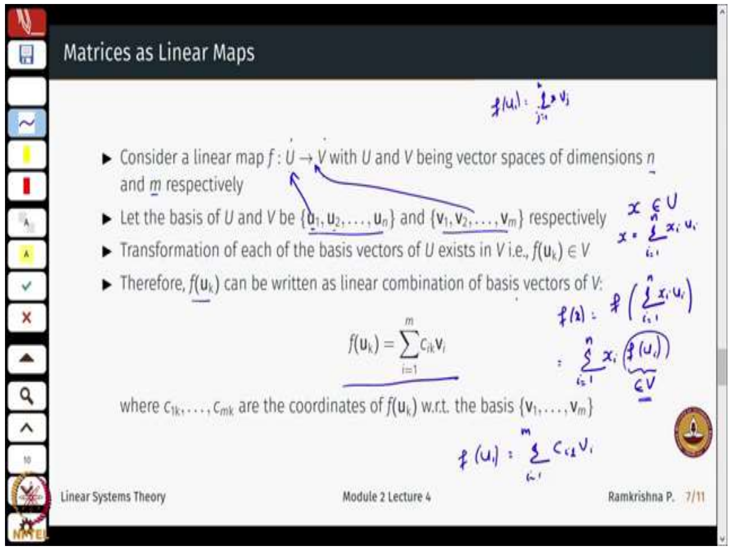

So, in the in the linear case, so what are the representations of this, this is A here or in general the fs here. So, we will look at when where is special class or which will also be the general form of it is to look at a matrices and its elements as linear maps. So, (Refer Time: 09:48) is something like this right. So, this map essentially from A x =b was essentially a map where all this a's where defined in the as elements of a matrix A ok.

So, let us consider now linear map from f going from U to V, and let us assume that they are of dimension n and m not necessarily of equal dimensions. Each space is equipped with the basis the basis need not be unique, but let us assume that there is a set of basis u_1 till u_n which belongs to U, and v_1 till v_m which belongs to V ok.

So, any element x belonging to U can be written as x = $\sum_{i=1}^{n} x_i u_i$, it is just come from the definition of the basis. Now, if I take this map right, so let us say f (x) is f(u i). Now, what I know that I am looking here as linear maps right. So, so this will be summation i equal to 1 to n, and because of the linearity this will be f(u_i) ok. Now, this f (u_i), what does f do it takes elements from U and gives me elements v, so f(u_i) will necessarily belong to V right.

So, what does this mean that each of this basis elements u 1, u 2 until u n transforms to some vector V under this linear transformation f right. And therefore, this f of u_i should have a unique representation in terms of this basis v_1 till v_m. So, f(u_i) should write in some terms of sum summation of sum numbers times v_j, j equal to 1 to n and that is true for each of each of this thing say. Ah

So, in general f (u_i) can be written as a linear combination of basis vectors of V and there is v_1 till v_n. As I said for each k I can write it is something like this if I just say f(u_1)ould be i equal to 1 till m $c_i v_i$ and so on for everything right. So, f (u) can be written because f of u_i is a vector in V, and therefore, it can be written as a linear combinations of basis vectors of V ok. And let us denote those coefficients as c_i k as set here ok.

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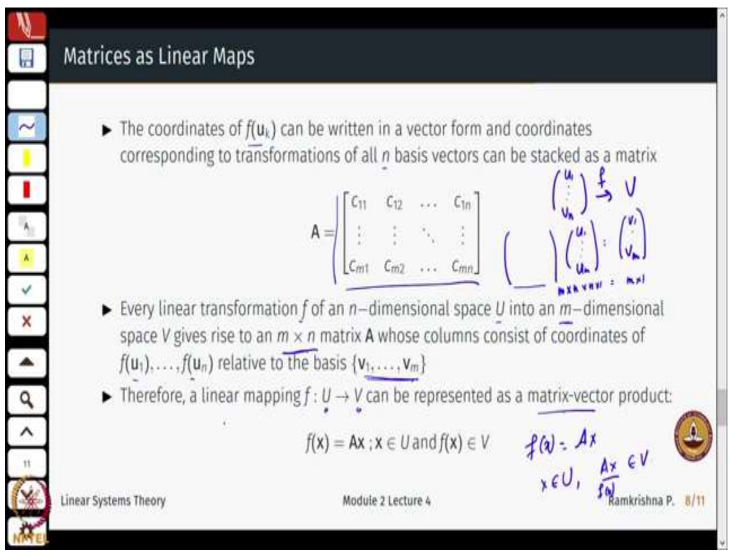

So, the coordinates of f (u_k) can be written in a vector form. And if I just generalize this to all n basis vectors u_1 till u_n, and then I can just write it as that the map which transforms all these vectors u_1 till u_n into V via the map f by just this matrix representation ok.

So, every linear transformation f often n dimensions of space U into and n-dimensional subspace V gives rise to an m x n matrix A whose columns consists of the coordinates of f (u_1) until f(u_n). Again with respect to the basis v_1 till v_m ok. So, so it is it is it is it is, so I will have u_1 till u_n. So, the f here is a matrix should give me v_1 till v_m. And therefore, this is a m x n matrix. So, m x n multiplied with n x 1 vector will give me a vector which is of dimension and m x 1, and if I just were to match the dimensions ok.

So, therefore, this transformation is a m cross n matrix. And therefore, a linear map from f, so a linear map f from a vector space U to another vector space V can be written can be represent represented as a matrix product. So, this f(x) is simply Ax right, there x belongs to U, and Ax will be in V at is the f(x). So, the conclusion here is that the linear map can be represented as a matrix or as elements of matrices ok.

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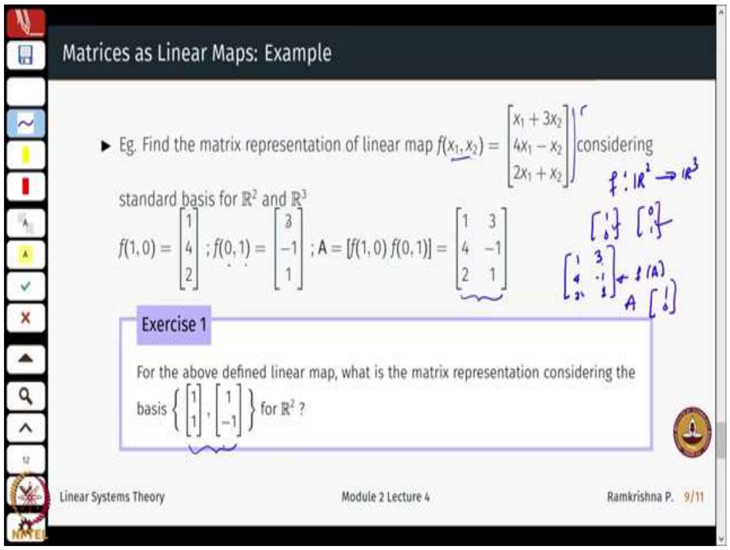

So, some real simple examples here. So, I have a linear map f (x_1, x_2). So, this is like a map from R^2 to R^3. And I want to find out how the standard basis in R^2 transform under this f here ok. So, what are the standard basis in R^2, I have $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and I have $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ok. Now, what is the matrix representation of this, the matrix representation of this is simply $\begin{bmatrix} 1 & 3 \\ 4 & -1 \\ 2 & 1 \end{bmatrix}$ here ok.

So, this is the transformation f or the corresponding matrix A, which takes elements in R^2 and gives me a vector in R^3 ok. Now, how does f 1 comma 0 translate in this side. So, I just look at this and just multiply this matrix a by the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and I will get the first column of this 1 4 2. Similarly, f(0, 1) will transform this way, and this is get the second column $\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$ ok.

So, so a little example of how the linear transformation essentially is related through elements of a certain matrix A. Similarly, you can do it for some other basis. So, here I just showed you illustration for the standard basis in R^2 which is $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$. You can similarly do it for other basis and check how these basis transformed to under this map A ok.

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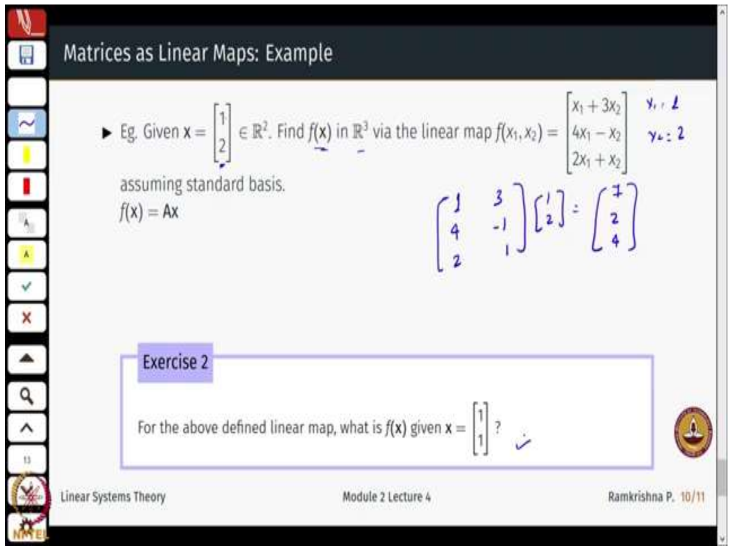

So, one more example so given a vector x in R^2, find what is f (x) in R^3 via the linear map which is again given by the same transformation as here as I that is again like a trivial stuff just substitute x_1=1, and x_2 = 2. So, this vector 1 2 will transform via this linear map which is which has a matrix representation $\begin{bmatrix}1 & 3\\4 & -1\\2 & 1\end{bmatrix}$ into, so this will be 1 and 6 is 7, 4 - 2 is 2, this is 2 plus 2 is 4 right. So, this is how this vector 1 2 we will look in R^3 via this linear transformation that I and then you can still check some other, some other properties also ok.

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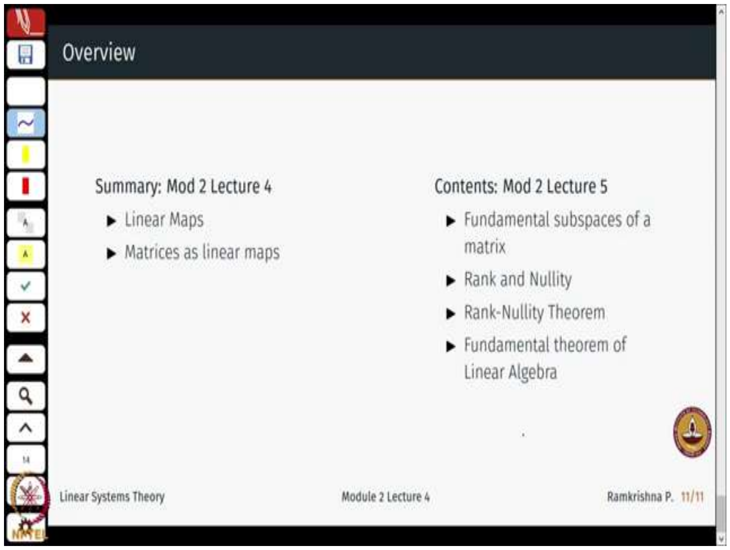

So, what we just saw was about linear maps and how linear maps are essentially defined by elements of f matrices. And we will do in the next lecture a little more on the properties of this of this subspaces what are the more are some general results about rank and nullity, and then we conclude with the fundamental theorem of linear algebra.

Thank you.