Linear Systems Theory

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Module - 02
Lecture - 03
Math Preliminaries: Linear Algebra 2

Hello everybody. So, in this in this short lecture, we will continue our discussions on Vector spaces and look at the little more more properties of the vector space. Beginning with what is called span of a vector space.

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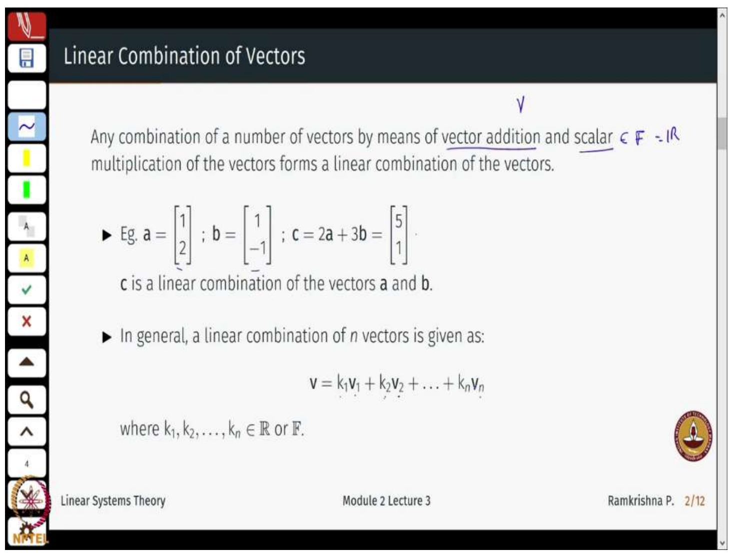

So, say I have say a couple of vectors like denoted by in coordinate representation by $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. So, any combination of a number of vectors or a linear combination by means of vector addition. So, then I say vector addition; so, these vectors will belong to the vectors space V and scalar multiplications of these are the guys which will come from the field F of which the vector space v is defined and in most of most part of the course, the field will simply be the real line R ok.

So, I take these 2 vectors a and b and I can write it as a linear combination say 2a + 3 b will gives the another vector. So, c is said to be a liner combination of vectors a and b and see this these numbers 2 and 3 can be any arbitrary number from the real line. And in general given n vectors v_1 to till v_n, I can write in any linear combination of $k_1 v_1 + k_2 v_2 \ldots + k_n v_n$ and this k's need not and they can some of them can be 0, some of them can be non-zero and so on ok.

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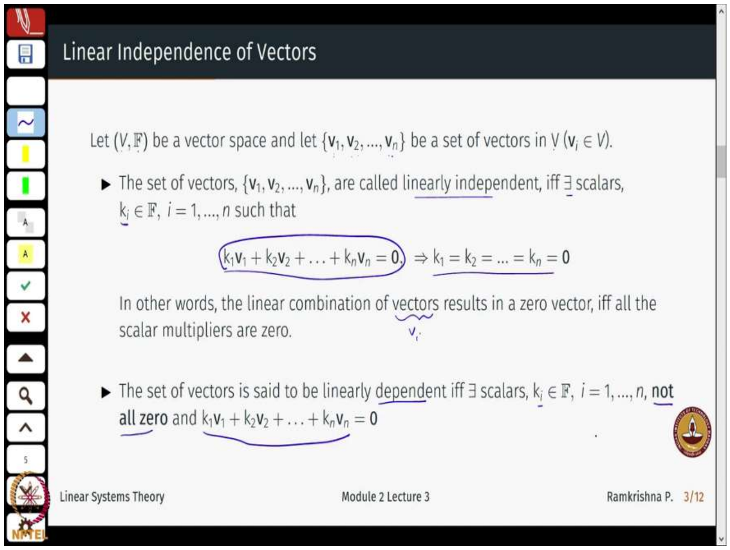

So, this leads us to something called a linear independence of vectors. So, I am writing down linear combinations of vectors and so on. So, let us start again with vector field V defined over a field F and just let me just randomly choose n vectors from v_1 to v_n from this vector space V.

So, these set of vectors are called linearly independent; yes, this is the key words. If there exists scalars again this scalars come from the field F, again would be the real line such that if I write it as $k_1 v_1 + k_2 v_2 = 0$ this necessary implies that all these numbers are 0 right. So, again let us let us repeat this again right.

So, I have the set of "n" vectors and these vectors are called linearly independent, if and only if there exists these scalars k_i; i going from 1: n such that whenever this relation holds it will hold if and only if these numbers k_1 to k_n are 0 or just to say in words linear combination of these vectors, I told this v_i results in a zero vector if and only if all these numbers are all this scalar multipliers. The k_1, k_2, k_n which are the scalar multipliers for the vectors v_1 v_2 until v_n respectively or all zero.

We will do some examples as we go through the lecture. So, the set of vectors is set to be linearly dependent ok, if there exists scalars K_i; i again from 1 to n not all zero. So, here everything was supposed to be 0. Here you can have say k_1 is non 0 k_2 is non zero; all of these can for example be 0 such that I can write these vectors in this way ok. So, here the necessary or if and only if the condition was all these k should be zero, here well the k's all of them need not be zero; all of them can be non zero also right ok.

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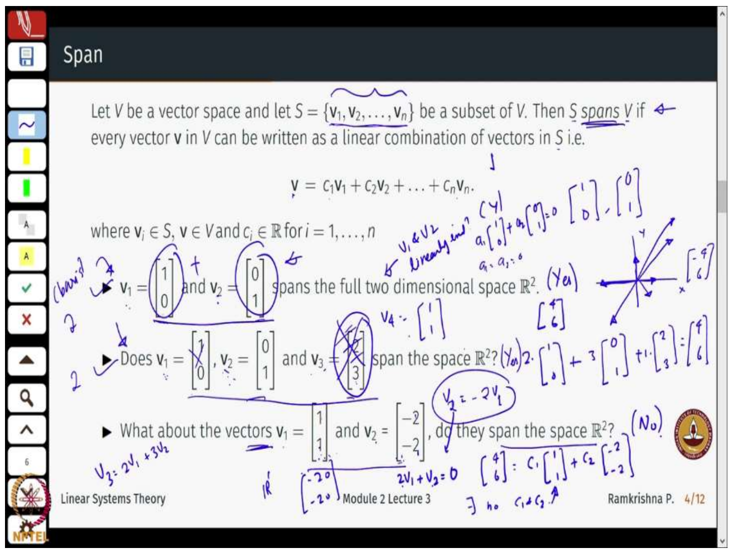

So, now this needs to what we do with these n vectors, if they are linearly dependent, if they are linearly independent and so on. So, is there some more information inside this or why do we need the notion of linear independence or linear dependence ok.

So, these are the properties called the span ok. So, let again start with a with a vector space V and again I just collect these set of n vectors could be a sub set of V ,a proper sub set of V also ok. It cannot be a proper sub set of V. I am just take taking some n vectors of V , then the set S spans V if every vector can be written as a linear combination of vectors coming from a set. Any arbitrary vector V can be written as a linear combination. What did we mean by linear combination is something like this, just multiply these vectors by scalars and sum some scalars and add them up.

So, if every vector v can be written as a linear combination $c_1v_1 + c_2v_2$ + until c_nv_n n right, then this what we what we say that S spans the vector space V ok. So, say look at this right. So, I have in two-dimensional space, I have $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$, as 2 vectors ok.

Now, well does this span the full R^2 right. If I just draw them in coordinate set, we are familiar with say I call them the x and y axis. $\begin{bmatrix}1\\0\end{bmatrix}$ would be the unit vector in the x direction; $\begin{bmatrix}0\\1\end{bmatrix}$ would be the unit vector in the y direction or give me any vector say something like this or something like this or something over here or something just say something like this. Can I write any vector as a linear combination of these 2 vectors? The answer is yes that is what high school coordinate geometry teaches us all right. So, the answer here given v_1 is $\begin{bmatrix}1\\0\end{bmatrix}$; $v_2$is $\begin{bmatrix}0\\1\end{bmatrix}$. Does it span the full dimensional space R^2? The answer is an obvious yes ok.

Now, similarly look at the second one all right. So, I have vectors $\begin{bmatrix}1\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\end{bmatrix}$ and another guy sitting here $\begin{bmatrix}2\\3\end{bmatrix}$. Does this span the space R^2? Well, I can similarly draw this is my as usual $\begin{bmatrix}1\\0\end{bmatrix}$; this is $\begin{bmatrix}0\\1\end{bmatrix}$. $\begin{bmatrix}2\\3\end{bmatrix}$ might just be somewhere here. So, can I write any vector in R^2as a linear combination of this? Well, just let me just look at 4 and 6 right. I can write this as 2 with v_1 that is $\begin{bmatrix}1\\0\end{bmatrix}$ plus say 3 time v_2 and v_3; 1 times v_3 1 times v_3 that is 2 3. So, this will give me $\begin{bmatrix}4\\6\end{bmatrix}$ ok.

So, I can write this vector $\begin{bmatrix}4\\6\end{bmatrix}$ and similarly with all other vectors I can use these 3 vectors or some combinations of them to write any vector in R^2say I can even write say vectors like minus $\begin{bmatrix}4\\6\end{bmatrix}$ and so on. Its everything else can be can be written in terms of these 3 vectors v_1, v_2 and v_3. So, answer here is also an obvious yes ok.

Now, what about the vectors v_1 and say if I have 2 vectors which are like $\begin{bmatrix}1\\1\end{bmatrix}$, $\begin{bmatrix}-2\\-2\end{bmatrix}$; do they span the space R^2? So, let us say can I write say for example, the vector 4 comma 6 as some linear combination c 1 with 1 plus some scalar c 2 with $\begin{bmatrix}-2\\-2\end{bmatrix}$. So, you can just do some solve some simultaneous equations here and find that there exists no c_1 and c_2 such that this the such as the above expression holds So, what about the vectors $\begin{bmatrix}1\\1\end{bmatrix}$, $\begin{bmatrix}2\\2\end{bmatrix}$ $\begin{bmatrix}-2\\-2\end{bmatrix}$; do they span the space R^2? The answer is no ok.

So, does this span some lower dimensional space; does this for example, span say R^1? Well the answer is yes, but we will come to that a little little later right. So, what does what if it does not span the space R^2, does it span a lower dimensional space? Well, the answer here is yes. But we will look at it a little later. So, here we was looking at what are what does span mean and then, what is the role of these vectors v_1 till v_n in determining if they if they span might would n dimensional sub space in general or two dimensional sub space in this 3 examples ok.

So, this leads to something called a basis of a vector space. So, if I look at; so something is not is does not does not fit here well right. So, here I have these 2 vectors span R^2, I can say will these 3 vectors also span $R^2$and if I write say 1 more vector v 4 called say some say $\begin{bmatrix}1\\1\end{bmatrix}$ for example, right. So, this will also span $R^2$and so on right. However, if I add another vector here which is called say $\begin{bmatrix}-20\\-20\end{bmatrix}$, this may not span $R^2$and so on.

So, what is what is going on here, so does it really mean that if I add just that if I add vectors in this problem arbitrarily, does it lead to R^2? Well, it may answer may be yes or no and what happens? So, why do we need 3 vectors here when I can just do the entire span with 2 vectors? So, that leads to some kind of compact definition of the span and these vectors from v_1 till v_n.

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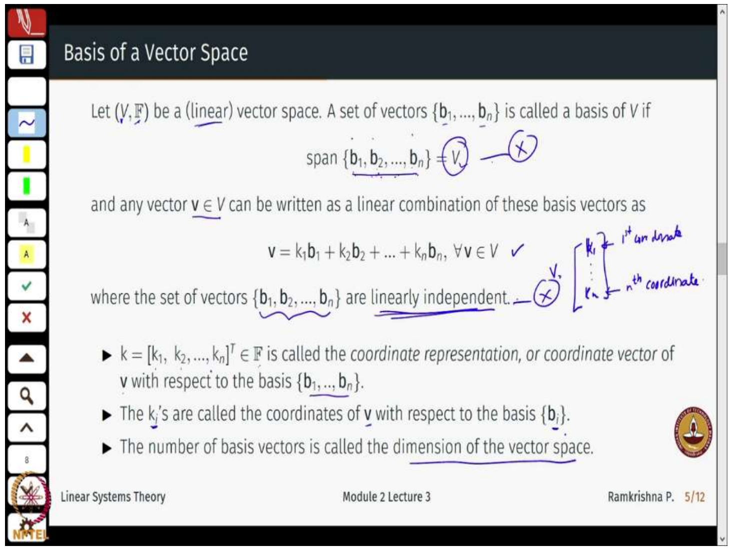

So, again we start with the vector field vector, sorry vector space v over a field F. So, this is a vector space and this is also initially called as a linear space. So, we will use this word inter interchangeably and they would mean the same. So, the set of vectors b_1 till b_n is called the basis of V, if b the set of n vectors span V or that is that is this is same as see here right. So, we this statement and the one here are same. However, there is something important to add here right. So, again the from the definition of span any vector v which comes from this vector space capital, v can be written can be written as a linear combination of these basis vectors right.

So, going back to this example; well, this is also the span, this is also the span. Now, are which of them are what are called as the basis vectors? Does this v 1 constitute basis vectors or does sorry v_1 and $v_2$constitute basis vectors or v_1 v_2 and v_3 constitute basis vectors? So, in addition to this definition of span to define a basis, what we also need is that these vectors are linearly independent. This is important because here I just find are there n vectors that span the entire space could be R^2, R^5, R^6, R^7 or whatever.

Now, for example, here this plus this is enough to span R^2; whereas, this is like unnecessary right. It is not required. I can still I will I can do away with this right and still have that v_1 and v_2 span the entire of R^2ok. So, in additional condition that we impose such that this set is not too large, but still large enough to span the entire space is that they should be linearly independent ok.

Now, the point of this is. So, here you can just look at it right. Here, are $v_1$and v_2 linearly independent in the first case right that is we need that one right. So, they are so the answer is here is yes. So, here are v_1, v_2 and v_3 linearly independent? Well, the answer is no. You can check that you can always write v_1 sorry v_3 as 2 v_1 + 3 v_2. Therefore, you can simply just through this out and say will v 1 and v 2 span my entire of R^2. You can also alternatively through v_1 out and v_2 and v_3, this will also span the entire R^2.

Whereas, here I just have 2 vectors, but they still do not span the entire R^2. That is because v_1 can be written as a linear combination of sorry v_2 can be written as minus 2 time this. This is a linear combination right. So, what the linear combinations as we saw earlier. So, if I if I rewrite this again. So, I have $2v_1 + v_2 = 0$ and here not all the coefficients are 0 right. So, that is what the definition says right of linearly. These 2 vectors v_1 and v_2 in this case are linearly dependent because there x's coefficient k_i which are not 0, such that this relation holds here.

Whereas, in the same thing in the first case, I can never write say some constant a 1 times 1, 0 plus a 2 times 0, 1 equal to 0. When is this possible? This is possible if and only if a_1 = $a_2 = 0$. You can solve them as simultaneous equations if you are if you do not understand what I am trying to say over here right. So, these are a couple of examples of linear dependence and independence, how they are related to the span and from the span how do we define the basis of a vector space ok.

So, these numbers k_1 till k_n which come from the space from the field which again will be the space of real numbers in through of the course will be are these are called the coordinate representation or the coordinate vector v with respect to the basis b_1 till b_n which means that I can write v as just this vector sorry k_1 till k_n ok.

So, these k_i's are called the coordinates of v with the respect to the basis b_i. So, this will be called as the first coordinate and similarly, this will be called the nth coordinate ok. So, the number of the basis vectors b_1 till b_n is called the dimension of the vector space right. So, we go back to this examples here. I have 2 vectors v_1 and v_2 these are enough to define R^2. And therefore, these 2 are; so the dimension here is just two.

Similarly, even though there are 3 vectors here, I just take the take the minimum number of linearly independent vectors and the dimension is still 2. Whereas, here well it is not true right. So, the number of the basis vectors b_1 till b_n which span v and which are linearly independent, this two conditions, so this condition here and this condition here are important and this will be called as the dimension of the vector space ok.

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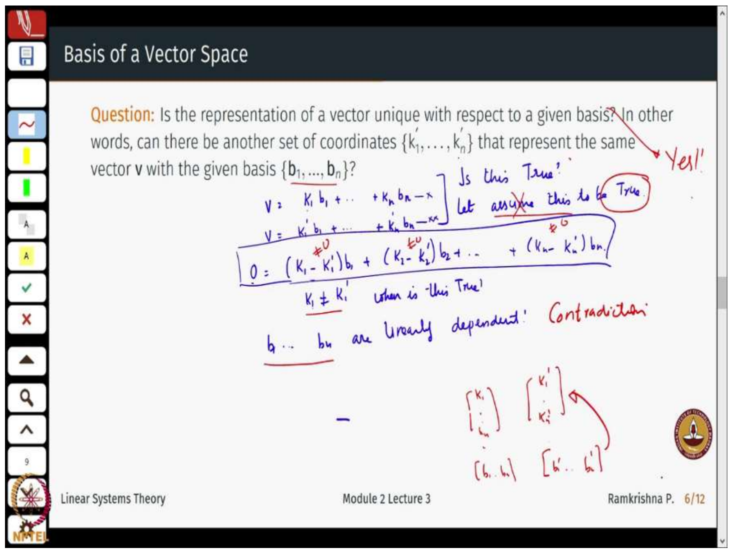

So, an important and interesting question to be asked is the representation of a vector unique with respect to a given basis. So, what is it mean that can there be another set of coordinates k_1' till k_n' that represent the same vector v with this basis. So, what is given to us is the following. So, I have a vector v which can be written as $k_1 b_1$ until k_n b_n right with respect to the basis b_1 till b_n and then, the coordinates of the coefficients are k_1 till k_n.

Now, the question here is can I write this in vector v as some other coefficients $k_1'b_1$+ $k_n'b_n$? Right. So, the question that we are that we want to answer is do there exist 2 distinct coordinate representations for the same vector defined with respect to the same basis of a given set of basis ok. So, is this true?. So, this is what we want to answer. Well, I do not know; so let us assume this is to be true ok.

So, if this is true that v can be written in this way and v can also be written in this way. So, this is true, then I can subtract and get the following as 0 is $(k_1 - k_1')b_1, (k_2 - k_2')$ b_2 all the way till $(k_n - k_n')b_n$ ok. Now, what do I know here is that if I assume this to be true, then $k_1 \neq k_1'$ ok.

Now, I have 0 here on the left-hand side and some a set of some new vector here right plus b_n ok. Now, let us us go back few slides and so, if $k_1 \neq k_1{}'$, then let us use a different colour here ok. Then, this will not be 0. This guy will also not be 0 until none of these coefficients are 0 right because we assume that that these are distinct right. This is this is assumption ok. Now, let us go back to few slides and look at the definition right.

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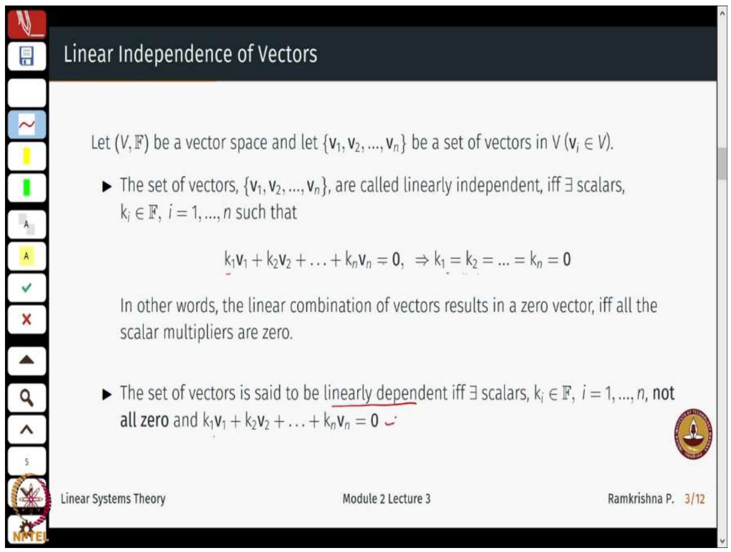

So, of a linear independence. So, I start with; so if v_1 till v_n are linearly independent, then this is 0 if and only if these coefficients are 0 ok. On the contrary, a linear dependence would tell me that well for some non zero vectors. So, sorry from now for some non zero coefficient k_1 till k_n something like this holds true.

Now, here now the question I will ask here is well. So, when will this hold true and that $k_1 \neq k_1{}'$? But this entire expression will be equal to 0. Well, from the definition of linear dependence and linear independence, it will suggest that these are this is possible if and only if b_1 till b_n are linearly dependent ok. What does the definition of span say? Well, the definition of span or the basis is that this b_1 till b_n are such that they must be linearly dependent and therefore, sorry linearly independent. And therefore, if this assumption word to be true, then b_1 till b_n are linearly dependent which is the contradiction right.

Therefore, I cannot assume this to be true because then, it violates a definition of the basis and therefore, the answer here is the representation of a vector unique with respect to a given basis. Well, the answer is yes it is it is unique. So, I cannot have vector representation

k_1 till k_n and the same vector being represented by k say k_1' till k_n' with the same set of basis ok. Now, the question is well can there be some other representation from a different basis; is the basis unique? We will try to answer that.

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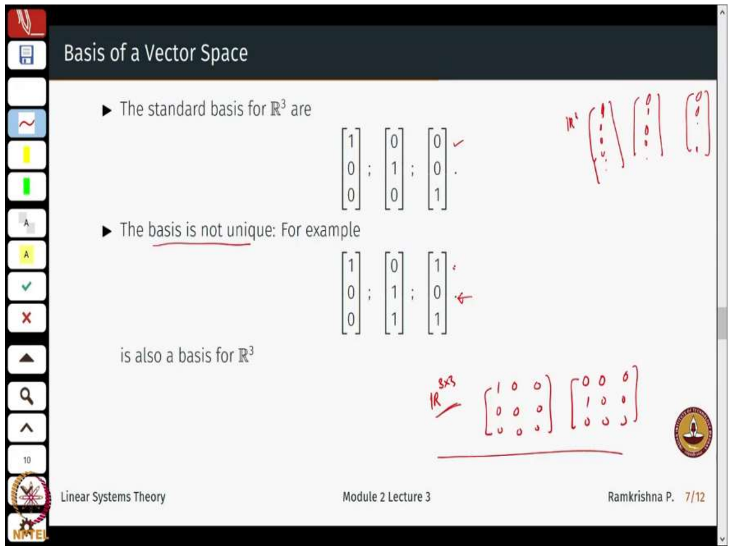

So, if I look at say R^3, these are the simples of all the spaces of what we also spend a lot of time in a in our earlier calculus courses, standard basis is some like this right and this basis is not unique for. For example, this is also a basis for R^3 and so in the same way there can be infinite basis right. So, I can represent; so this there could be several other combinations which could which should give me a infinite amount of basis right. So, so this basis is not unique. So, this is what we should remember. So, what is unique is given a basis the coordinate representations are unique.

So, this k_1 we have till k_n will be basis for say some basis say b_1 till b_n. However, there could be other basis; b_1' till b_n' for which the coordinates will be k_1' till k_n' ok. And similarly, I can write down basis for say the set of all 3 cross 3 matrices right. So, the basis would be 1 0 0, 0 0 0 all the 0s here.

Similarly, 0 1 0 all 0s here. And all possible and all the 9 combinations with 1 being interchanged nine times right. So, this will be basis for all the space of 3 cross 3 matrices. Similarly, we can write down 2 cross 2 matrices and in general, I can write on basis for R^n say 1 0 0, then I have 0 1 0 0 and so on until the nth vector will be 0 till 1 and of course all combinations of them in this way. So, what did we learn is that the basis is not unique, there could be infinite amount of basis and given a basis, there is a unique representation of a vector for that particular basis.

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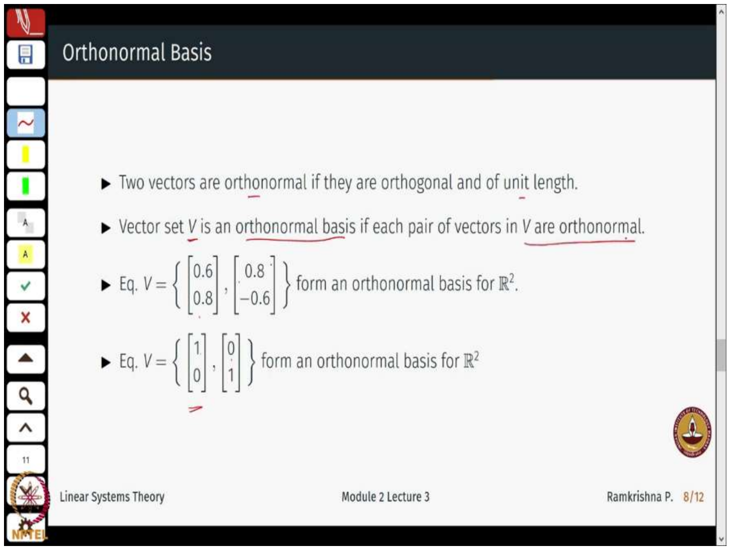

A important class of this basis are what I call the orthonormal basis. So, 2 vectors are orthonormal if they are orthogonal and also of a unit length. For example, say these 2 vectors this and this that is then, actually see that these are ortho orthonormal or orthogonal to each other that the dot product is 0; not only these 2 vectors here all these are of unit length. So, a vector set v is an orthonormal basis if each pair of vectors v are orthonormal.

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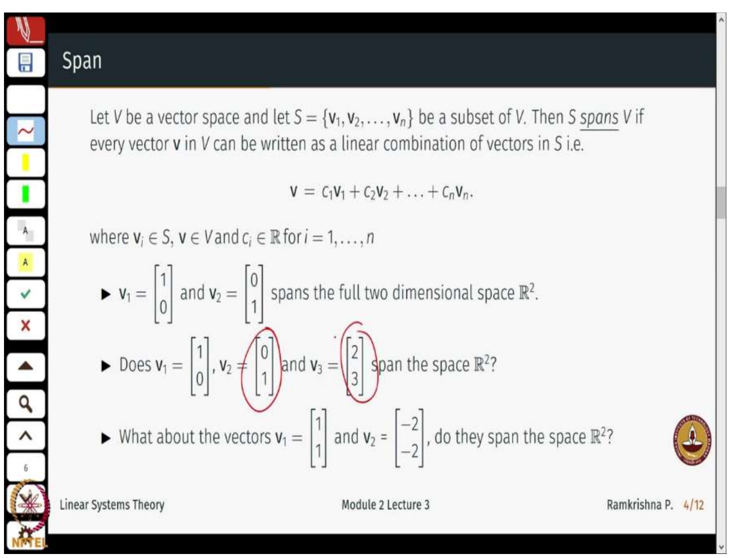

Like for example, if I choose this and this, well that is not a orthonormal basis right because the magnitude there here is it is not 1 and so on ok. Now, why is this useful? Right.

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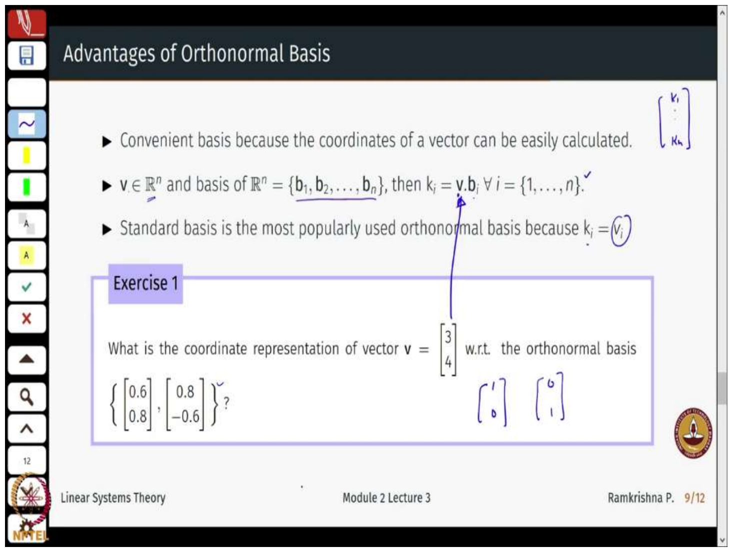

So, it is useful because I can easily find out the coordinates. So, what is my coordinate representation of a vector? My coordinate representation of a of a vector is just given by these numbers right from the earlier representation k_1 till k_n ok. Now, this can be easily calculated. So, if say for example, I have v as my vector in R^n and in a given set of basis right, then each coefficient k_i will just be the vector v times the b_i column right.

So, that is the dot product of v with b_i ok. I do not really need to do this. I think this is kind of kind of trivial in its in its own way maybe you can just write down the details of this right ok. So, in the standard basis used as orthonormal basis because k_i is simply v_i right. So, vi here would be the ith component of this vector or so I was coming likely from here ok. So, as a little exercise, you can just find out what is the coordinate representation of vector b with respect to the orthonormal basis of this one right.

The trivial exercise would be to do what is the coordinate representation with $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$. We can just write try to do this its it is should give you some good answers. The hint is you can just follow this rule here and ok, these are all these are orthonormal basis because they are they are and then they are of also of magnitude 1. So, that kind of concludes the discussion on what is meant by basis representations of vectors, uniqueness of basis functions or not and among several others. So, continuing on that that aspect an important topic would be a vector subspaces ok.

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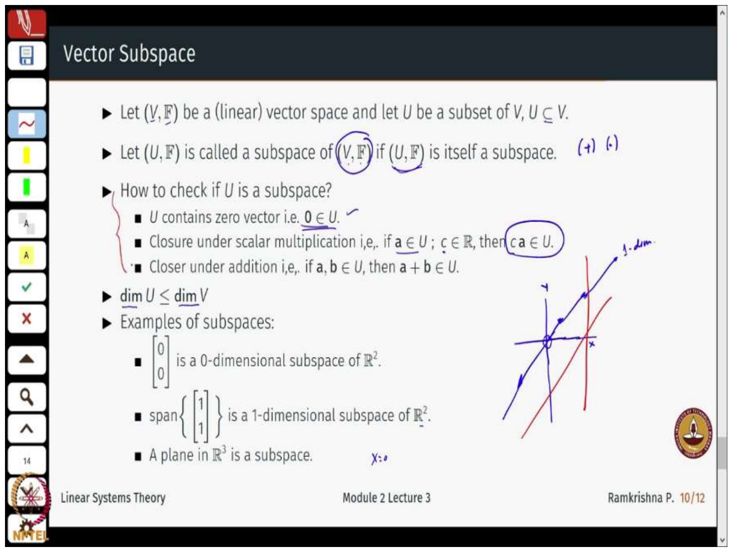

What is a subspace or a vector subspace? So, again we start with vector space V defined over a field F and let U be a sub set of V, maybe proper sub set in general. It is also be something like this ok. Now, when is U which is a subset of V called a subspace of V over F ok. So, the answer is or by definition that this is true if U ,F again U defined over the field F is itself a subspace ok.

Now, how to check if it is a subspace well. So, loosely speaking all properties of this V in terms of the addition operator, in terms of and the multiplication operator and the distributive property of the multiplication operator over addition and so on. All everything should be retained in U right. So, the first thing is well, it must contain the 0 vector and the important property is the closure under scalar multiplication right if a is in U and I just take some number c in R, then this should also be in U it should not be in V right.

So, I am just restricting everything to U. It should also be closed under addition right. So, and in general, it could happen that the dimensions. So, now, we know what was the dimension starting from here, the dimension was defined with respect to the basis of a vector space. So, the dimension of U is typically less than or equal to the dimension of V.

Well, examples could be that this is the vector 0 is a 0 dimensional subspace of R^2 if I just look at 1 comma 1 this will just be one dimensional subspace of R^2, you can just try to just plot what are all the possible combinations of this span in R^2. So, let me just do it here right. So, if I have x and y, I have (1,1), so this is my basis vector and every vector, if I were to write whether be here or here right. So, all vectors which I can represent by this span would just be on a on a straight line right and this therefore, is a one dimensional subspace or in other more generally a plane in R^3.

So, just impose a condition say x=0 that will be subspace. The important thing is that so, the subspace that should pass through the origin and this kind of captured over here like if I just draw another line say for example, where as this blue line was a subspace, where this red line is actually not a sub space. Similarly, with this one is also not a sub space of R^2, because this checks will failed ok. So, what are the good properties of this? So, I will prove one of them and leave the second one to you ok. Check if there are 2 subspaces U 1 and U 2 is the union subspace ok.

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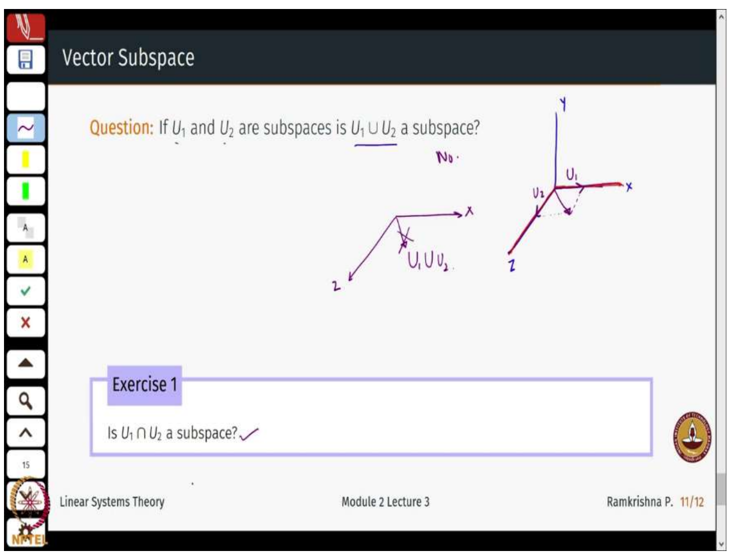

So, in much of mathematical proofs it is sufficient to give one counter example to show that this is not a subspace right. So, that is also one very useful proof techniques. So, earlier we saw a proof by contradiction we just we will just show one counter example. So, let me just take R^3 right. So, I have the x axis y axis and the z axis ok. So, individually this is a subspace right. So, the entire x axis is the subspace of R^3 Similarly, the entire z axis is a subspace ok.

Now, just do check this conditions ok, it has the 0 vector is it closed under addition ok. So, let us quickly check that let me take a vector which is let me use a different colour ok, this makes visible ok. If I take a vector here and a vector here well, is this. So, this is somewhere here right.

So, this vector which comes from, so let me call this U_1 let me call this U_2 and the intersection or so, the union is just these 2 straight lines right ok. I must not talking of the span of these, I am just taking that the union of $U_1 + U_2$ is just these two lines here this along the x axis and this along the z axis, x and z there is no y component; whereas, this vector which is a result of adding these 2 vectors. This is not in the union of two.

Therefore, is U is U_1 union U_2 a subspace; well, the answer is no and this is a counter example to that this is easier to show. So, I will leave the proof for you, is U_1 intersection U_2 a subspace and so, these are little interesting things that you can you can like work out by yourself and these properties will be more then useful in the coming parts of the course.

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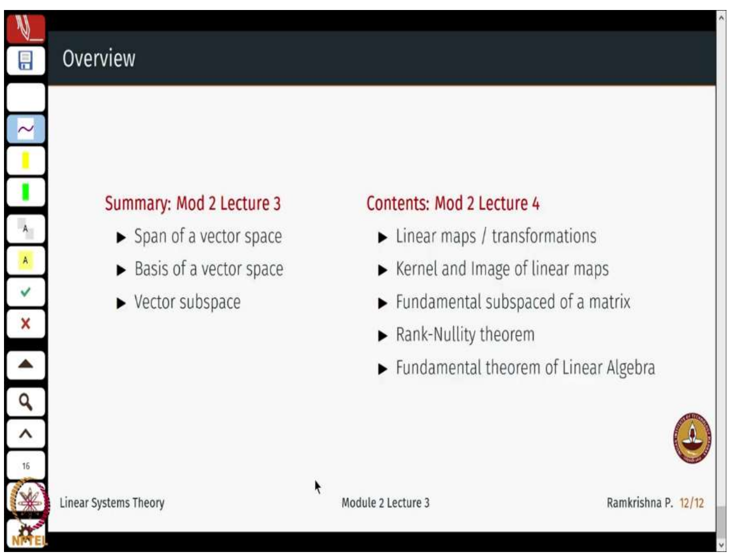

So, just to conclude what we learnt were the span of a vector space, the basis of a vector space and a certain deference between the span and the basis all right and that is I think should be in your mind and in a very (Refer Time: 34:30) kind of always remember it by heart and of course, the notion of a of a vector subspace and this we will use this concepts to build up on things related to linear maps and other properties of linear maps called the kernel, the image and so on. And so that will be the part of the next course of the next lecture.

Thanks for listening.