Linear Systems Theory
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Module - 02
Lecture - 02
Math Preliminaries: Linear Algebra 1

Hello everybody in this brief lecture, we will slowly introduce the concept of a vector space, the properties associated to vector space. We define the idea of a norm and also the idea of a metric on a set or a space.

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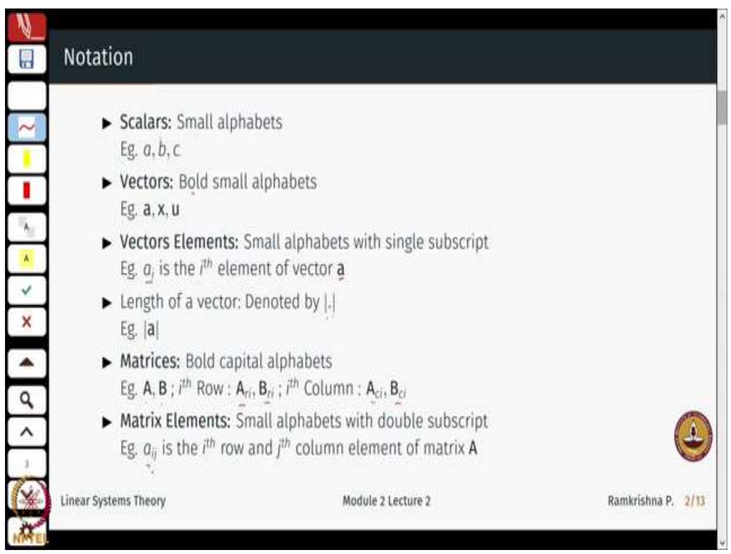

So, to begin with let us just have some small slide on the notations right. So, the way we distinguished between scalars and vectors is just with the bold and then the regular letters. So, for example, a I would be the i^{th}element of a vector a, length of the vector is simply denoted by the by the modulus of it.

Similarly, for matrices we have notations of capital letters bold A and bold B. Similarly, we have notations for the i^{th} row and the i^{th} columns and small a is for just denote the i j th element of a matrix A ok. Sometimes we may not follow this very strictly whenever it is it is obvious ok.

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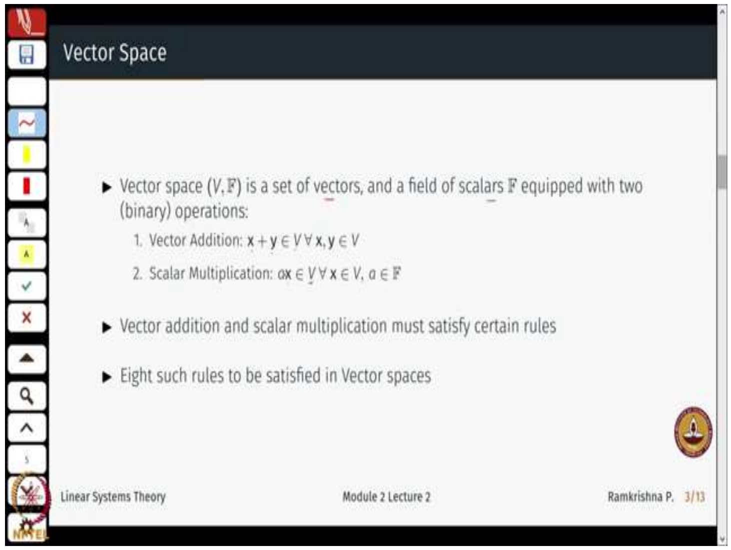

So, to begin with well begin with on the idea of a vector space. So, by definition it is a set of vectors and a field of scalars equipped with two binary operation set so, the vector addition which is defined on the elements in V in such a way that it is closed under addition that x + y will give me another vector which is also in V so, this is closed under addition.

It is also closed under scalar multiplication in a way that if I take an element from the field f in this case or simply see the real line r multiplied with it with x, I still get an element of V. So, it is closed under vector addition and also closed under scalar multiplication ok. Are there a little more rules that these two operations need to satisfy ok.

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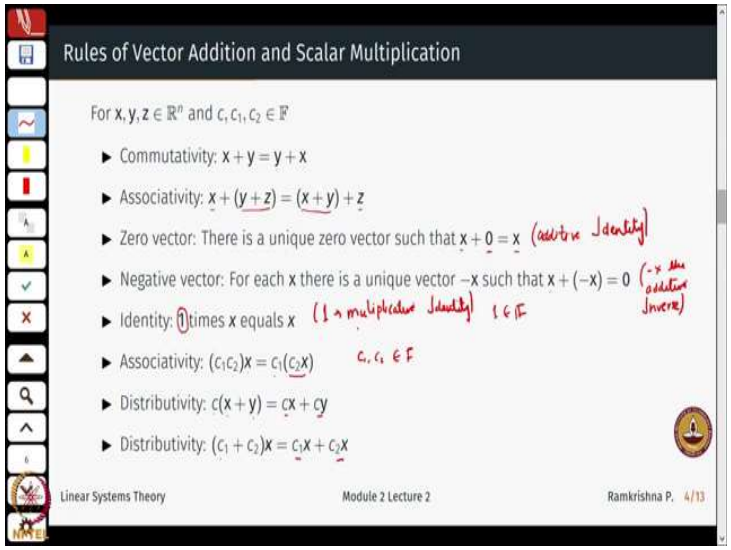

So, first is yeah this may look pretty simple, but it is nicer to do it again. So, commutativity, I add two vectors x and y, it is same as adding y and x in the reverse way. Associativity, if I have y+z and then add it to x is the same as to I do the operation on x and y first and then go to z. The zero vector there is a unique vector which is which we call as a zero vector such that if I add 0 vector to the vector x, it will give me the same vector x again there is also. So, this is called the additive identity ok.

Similarly, there is a negative vector such that if I take x, the negative of it is minus x I do the vector addition and I get the 0. So, this minus x is called the additive inverse then there is also something called a vector 1 such that 1 times x will gives me back x. So, this vector 1 is called the multiplicative identity.

Let us see other property which follow if I take two elements c 1 and c 2 belonging to the field F then I that the multiplication of c 1 with c 2 times x will give me the same as if I do this operation first and then multiply by c 1. Then you have the distributive property, c acting over x + y will give me cx + cy right and similarly if I take two elements from the fields c 1 and c 2 acting on x, it will give me c_1x +c_2x. ok.

So, these are the eight properties that we need to check given a certain space is a vector space or not. So, by the way so, this 1 is just it is not a vector 1, but it just comes from the field right from F. So, this is a little typo here, but the multiplicative identity like still holds right. So, it is a consequence of this scalar multiplication stuff ok.

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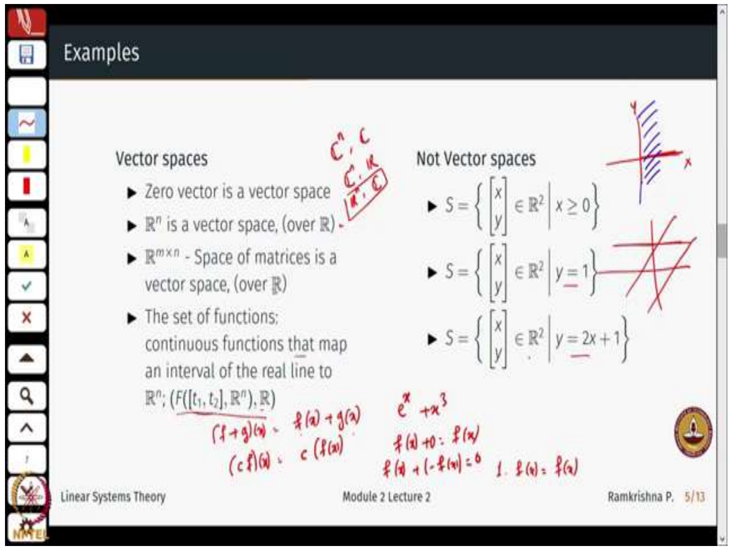

So, some examples of a vector space the zero vector is definitely a vector space you can check all these operations, C^n is a vector space over R. Similarly, I can also say that C^n is a vector space over C; C^n is the set of complex numbers or the space of complex numbers. Interestingly, C^n is also a vector space over the real line, you take a number from the real line, multiply it with the complex number you still get an element which is in C^n.

However, interesting is a following say if I take R this is not a vector space over C right. So, if I take C multiply with element in R, I get an element outside R^n, I get an element in C^n. So, this is like not a vector space. Similarly, I have R^{mxn} in the space of matrices is also vector space over R, I can define I can take a scalar, multiply it with a matrix I can add two matrices and so on.

All these eight rules can be verified easily. An interesting example is also the set of functions right continuous functions that map an interval of the real line this could be the entire real line also to R^n over the field R right. So, a mathematically or you can say this continuous functions that map an interval of the real line to R^n is mathematically written in the following way.

So, what are the properties I need to check for this take two functions f plus g acting on x, I will still get f(x) plus g(x) and this both are in R so, I am just adding up those numbers in the real line say as good as take $e^x + x^3$ this could be a simple example of this. Similarly, I take a scalar c f over x is the same as c f(x) I will also have this 0 function f (x) + 0 will give me f (x).

Similarly, I will also have the negative of function f (x) + (- f(x)) = 0 ok. Now, I will I will have the number 1 multiplied by f (x) will give me the same function. So, I can just write down all the set of rules and verify that the set of functions that map say the interval of the real line to R^n is actually defines a vector field.

Things that are not vector spaces are the following so, if I take x and y in $R^2$and restrict x to be positive. So, the area of interest here is this one, you can easily see that this will not constitute a vector space because -x is not an element of this vector space and so on so, some of those properties will fail.

Similarly, if I take in the in the second example say y =1 say this real line, this line here is also not a vector space; obviously, because some of those properties like would fail. And, I can then keep on drawing this line say for example, lines which go this way or this way, they will also not constitute a vector space essentially because they are do not also pass through the origin. So, the zero element is not part of this and it is neither a part of this space defined by this expressions.

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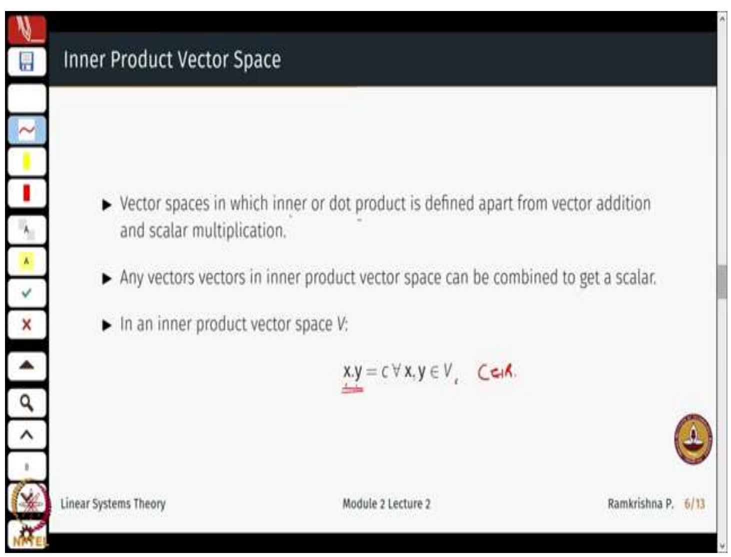

So continuing to introduce few more concepts so, define the notion of an inner product vector space which we also learnt as the dot product right. So, it is so, vector space is in which inner or dot product is defined apart from vector addition and scalar multiplication as we saw in the definition of a of a vector field.

So, any vectors in a inner product space can be combined to get a scalar right so, if I take dot product of two vectors x and y, I get some number c where c is in R ok. So, this we know this little from our high school calculus. So, any vector space equip with an operation like this, the dot product is called an inner product space. Say, $R^n$is or R^3 is a very simple example of that.

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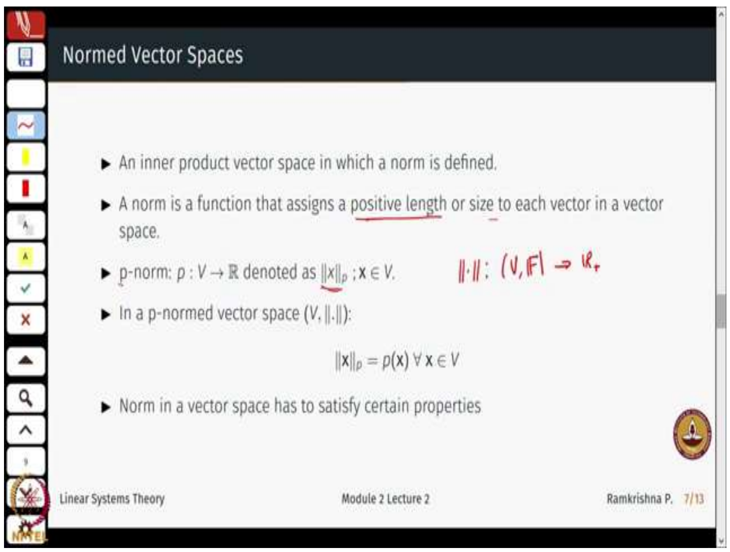

The next thing that is important is the idea of a normed vector space we are just introducing these concepts and as we go through the lectures, we will be a kind of obvious of why we are defining so many so many quantity. So, we are just like learning tools which will be useful to us little later ok.

So, a norm is a function that assigns a positive length or size to each vector in a space speaking also talk about as in R^2 say the length of a vector so to speak right. So, just say some p-norm we will denote this as x x this two parallel bars with x and two parallel bars and p right ok.

So, this norm I can look at it as like again as a map which goes from the vector field sorry vector space V defined over a field F and gives values in R^+ this is what is meant by positive length here ok.

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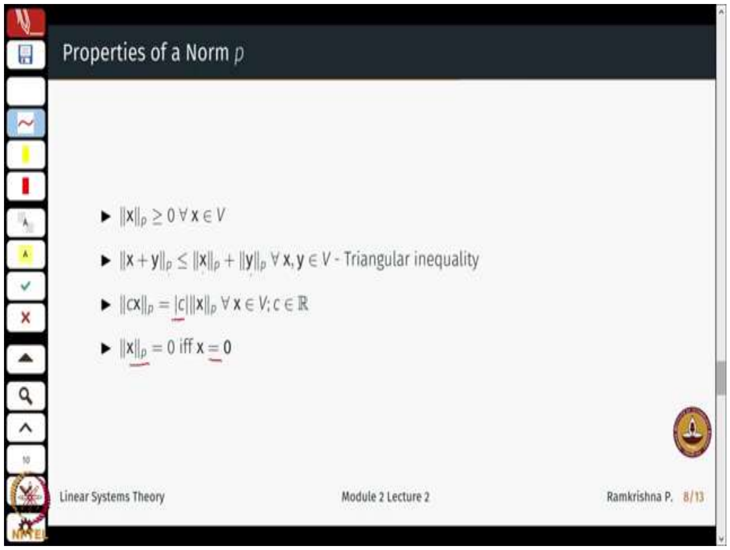

So, this norm has to satisfy some properties. So, the norm is usually greater than or equal to 0 and it is 0 if and only if x= 0 right. So, that is right, if you just add the point, the length is 0, if you just at the at the origin right so, that is the zero vector. Similarly, there is something called the triangular in inequality, I take two vectors x and y the norm of this will be less than or equal to the $||x||_p + ||y||_p$. So, we will we will derive this in one of the tutorial classes which my TA will be will be doing it shortly.

Similarly, take a scalar c multiplied by x, the norm would be the modulus of $c||x||_p$. So, this is not always necessary that c should belong to R, c should can also belong to the set of complex numbers where then the mod of c would just be the magnitude of the complex number ok.

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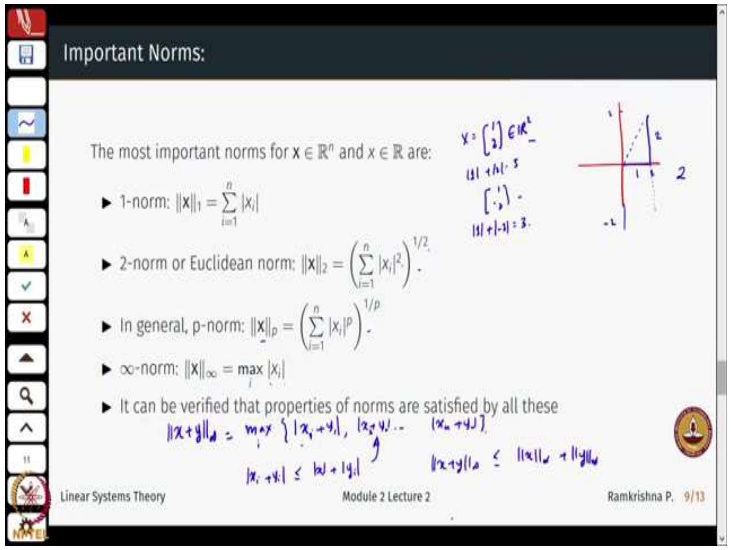

So, just to give some examples of what are what are norms. So, some of the important norms are like the so we start with the 1-norm I just draw some pictures to make it a little clear ok. So, the 1-norm is summation of all the magnitudes of this individual x_i. Say, x is equal to say 1 and 2 this is in R^2 so that will be say a 1 here and a 2 here so, this is let us say this is of length 1, this is of length 2 here.

So, the norm here would be 1 +2 and it will be equal to 3. Similarly, if I have 1 minus 2 which will mean that this stretches till here until minus 2 then the norm will have 1 plus minus 2 it will be 3 and that is that is the definition of a of a 1-norm, I just taking the absolute values of each components and adding them up.

2-norm is what we know as length of a vector in from coordinate geometry. So, in this picture it will be like is so, this is the 2-norm. So, this is like you take ${x_1}^2 + {x_2}^2$and then do the square root of it. I can generalize this 2-norm to something called p-norm, let us say here I have this 2, I just replaced this 2 by p and I have what is called a p-norm.

And as I extend this p or take the limit as p goes to infinity, I have something called the infinity norm what is the infinity norm, I just take the maximum of all these values right maximum of all the $|x_i|$. So, for example, in this case, the infinity norm would be this two in both the cases right. So, its maximum value is 2 and the maximum value here is a mod of minus 2 is still two and I just take the maximum of this values.

So, are these norms are, is it obvious that that these are these are like norms right so, we have just defined it right. So, what should these norms satisfy our properties like this ok. So, as a little exercise you can just do it for yourself to check that all these norms 1, 2, the p-norm and the infinity norm actually satisfy the properties of the norms.

So, the most difficult one is to tricky one is to prove the triangular inequality. So, I just do it for one of the cases and the remaining should be like obvious. So, let us take the infinity norm. So, what I have to prove is a triangular inequality for the infinity norm. So, what is x plus y the infinity norm of it by definition is max over I and so, what are the components I am looking at x i plus say max $|x_1 + y_1|$, $|x_2 + y_2|$, we will say if I have n dimensional space I am looking at $|x_n + y_n|$ ok.

Now, what do I know is if I take this magnitude of these are these simply numbers right x x_i and y_i. So, this is pretty straight forward to verify that this will be less than or equal to $|x_i + y_i|$ when we do it for these two numbers here and check that ok. So, now I get put it back here and what I would find is x plus y of infinity is less than or equal to x of infinity plus y of infinity. Similarly, you can check for the p-norm, the two-norm and the one-norm and so on ok.

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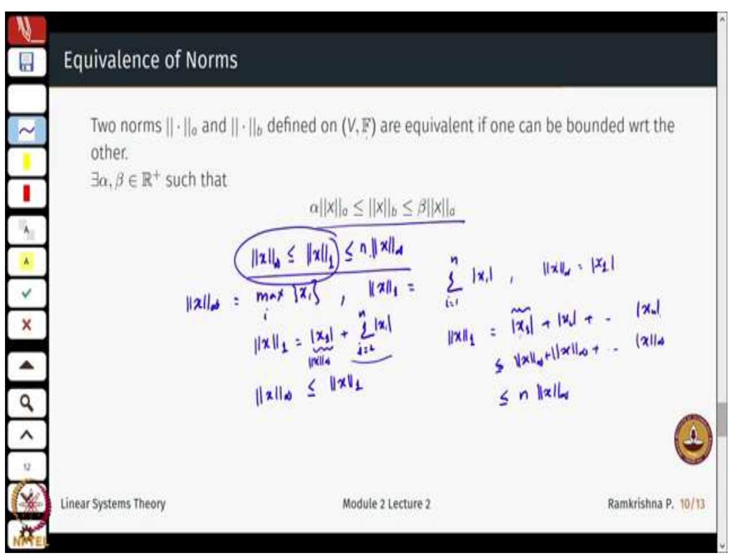

So, what is nice about this norms and why I say nice is because this will be very useful to us in the future lectures when we are talking of stability and another properties of systems. So, two norms are equivalent right so, again defined both are defined on the same V and the same F are equivalent if one can be bounded with respect to the other. So, let us just take this two norm, norm a and norm b so, so equivalence means that mathematically, it satisfies the property like this right where α and β are just positive numbers. So, this is this is obvious so, I will do an example and the rest you can check by yourself.

So, let us do this for the infinity norm; so, if I just look at the infinity norm and the 1-norm is less than or equal to n times right. So, this equivalence means you know something like this holds between the infinity norm and 1-norm where α is 1 and β is n. So, let us verify this equalities one by one. So, x of infinity by definition is the maximum over i of all these x_i 's and what is $||x||_1$, $||x||_1$ is summation i equal to 1 to n each of the x i's and one of these x_i's is the infinity norm.

So, let us with let us assume with a lot of general generality that the infinity norm sorry, the infinity norm is just the first element ok. So, this can be second, third, whatever, just for simplicity I am writing this way. So, therefore, x of 1 is now should do this way. So, i equal to 2 to n x i. Now, this is the infinity norm and then you have some other terms here and therefore, I can write that the infinity norm here is less than or equal to the 1-norm, it is right?

So, this is the 1-norm and this is the infinity norm and this one ok. So, because this guy here is my infinity norm ok. So, so the first part of the equality is done ok, the second part should be should be straight forward again right. So, if I look at the relation between $||x||_1$ and $||x||_\infty$. So, $||x||_1$ is again I will have the magnitude of $|x_1|$ plus $|x_2|$ until $|x_n|$ and again so, I assume that this is the infinity norm.

So, therefore so, I can just write this as $||x||_\infty$ and what I know that this is the highest or this is the largest or the maximum value therefore, all others will be less than or equal to the infinity norm ok. So, therefore, this will be less than or equal to again for everything will be just x of infinity and therefore, I have this is less than or equal to $n||x||_\infty$ right.

So, similarly I can show the equivalence between the 2-norm and the infinity norm, the 1-norm and the 2-norm, but in general in any finite dimensional spaces or where the dimension is finite that are n is finite, all norms are equivalent and this is this will be a very beautiful property as we will see little later.

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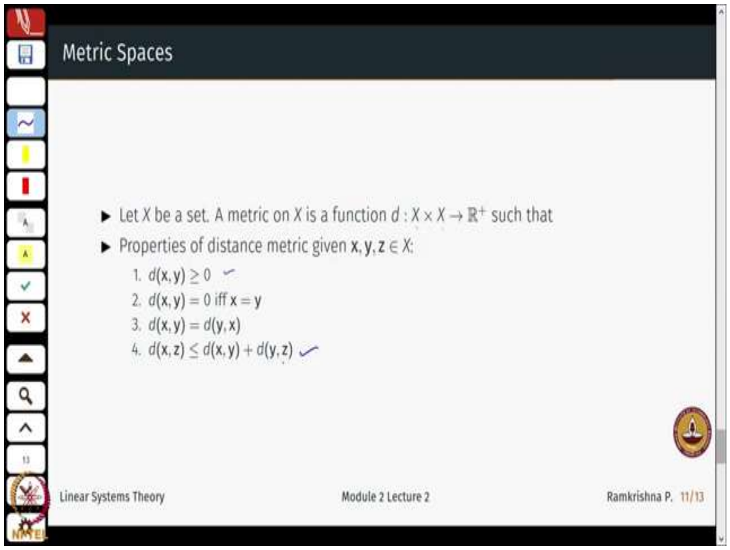

Something which is a generalization of a normed vector space and this we if some somebody would be familiar with the course on topology; they would have encountered things like this of metric spaces. So, let us have with the set X right a metrics a metric on X is a function defined on X x X again to R^+ such that the properties of the distance are maintained.

So, I am just defining a distance between two points. So, first point is obvious and the distance is usually between two points is greater than 0 and it is equal to 0 if and only if a two points are the same ok. Secondly, the distance between point x to point y will be the same as the distance between point y to point x that kind of obvious right and of course, something equivalent of the triangle in equality right. So, distance between x and z will be less than or equal to the distance between x and y and y and z ok.

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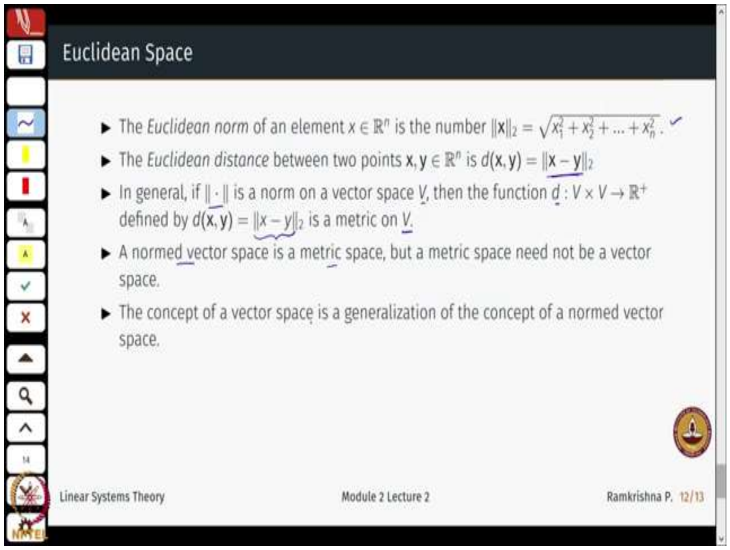

So, how does this relate to the Euclidean space? So, lot of time to Euclidean space how do normed spaces and metric spaces relate to each other. So, let us do something from what we know from our coordinate geometry classes. So, the Euclidean norm of an element is just this is norm, just the length of the vector right or the distance between the vector and the origin that is the Euclidean norm. Whereas, if I look at the Euclidean distance between two vectors, it is defined this way x distance between x and y is just the two norm of the difference between x and i.

So, we can say something in general based on these observations that if I define a norm on a vector space V then the function d which is defined in the following way the d(x,y) is the $||x - y||_2$ is a metric on V right. So, I have a norm the notion of a norm from the notion of a norm I can define a metric on V. And therefore, I can say in general that the norm vector space is usually a metric space, but a metric space need not have the same property I am not even talking of a set. So, when I look at the definition of vectors I just begin with the set, it has no properties, it is not a vector space or nothing it is just some set which has these four properties satisfied ok.

So, therefore, I can say that the concept of a vector space is a generalization of the concept of a normed vector space. So, we will do some problems on this in the tutorial classes.

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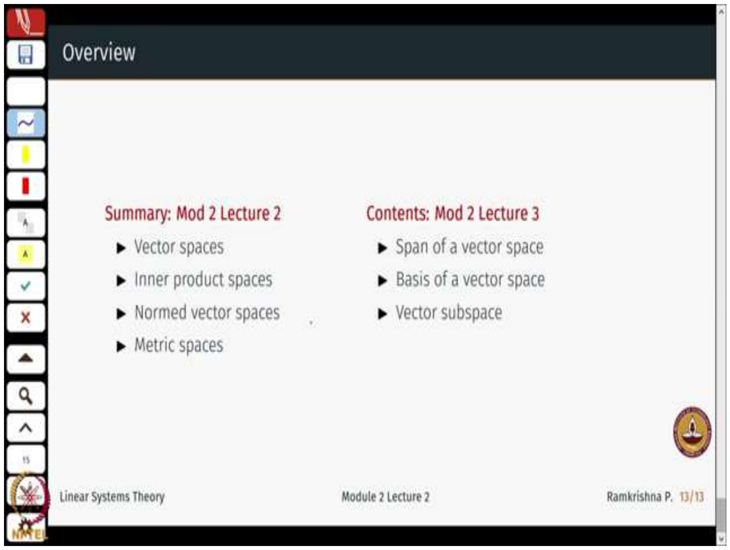

So, this was just to introduce you to the concept of a vector space, inner product, the idea of norms and the metric spaces. So, later on we will do something on what is the span of a vector space, the basis of a vector space, are these is the span and basis the same concept or different and then we will slowly introduce the idea of a vector sub space.

Thank you.