Linear System Theory
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Module - 02
Lecture - 01
Math Preliminaries: Sets, Functions and Fields

So, welcome to the second module on Linear Systems course So, I said earlier as much of the next two modules we will spend equipping ourselves with tools from linear algebra, some of the things which I am going to tell you in the next 20 minutes also might be a bit familiar some may not be. So, as said earlier, we will still try to make it as inclusive as possible, but it might help to have some references on early or later on as the lecture goes by ok.

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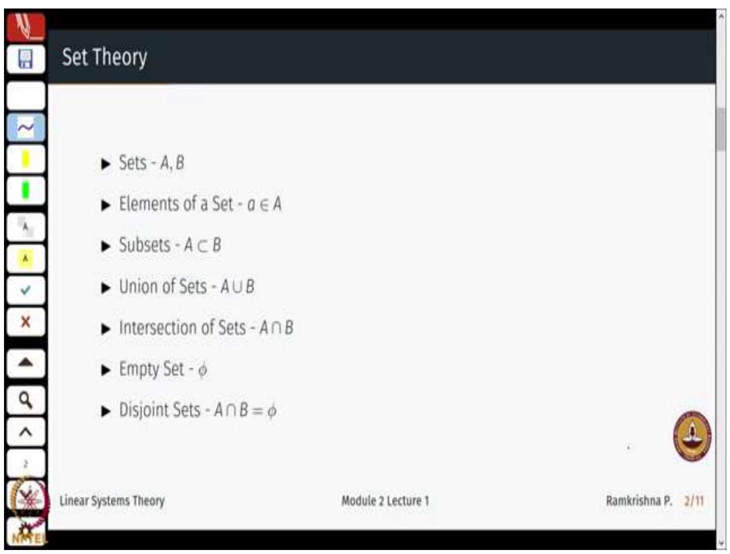

I guess we all know about set theory right. So, how to define elements of sets, how to define subsets, unions, intersections, empty set and disjoint sets. So, and if you are not familiar then maybe then you can just go back to your earlier maths course and then and maybe in first or second year of engineering So, I will skip the details of this So, what I will just introduce you to the terminology which I possibly think you may not be too familiar with or the kind of notations that we will use throughout the course which are a little more general than what you would have learnt in some earlier set theory class.

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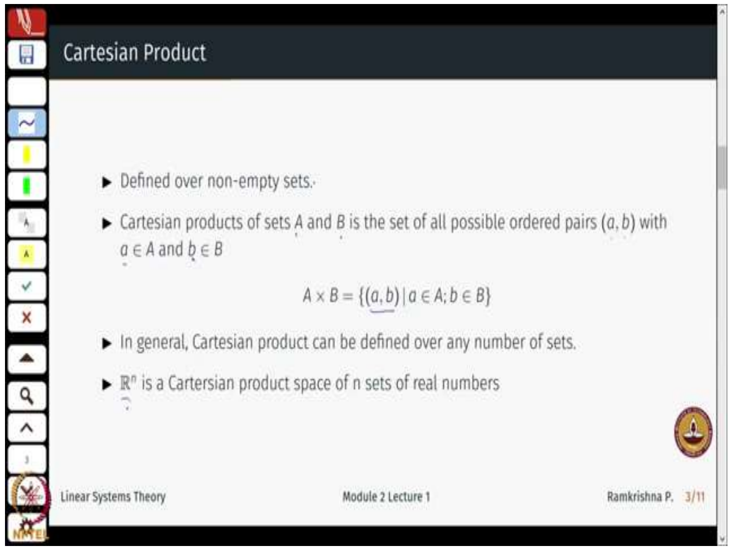

So, so the first thing is how do we define what is called a Cartesian product of course, we define it over non-empty sets while it is a the empty set becomes a bit trivial so, it does not make sense to define it over non-empty sets. So, just take two sets A and B So, the Cartesian product is a set of all possible ordered pairs certainly why I call this ordered so, take (a,b) with a belonging to A and b belonging to B such that, where A x B is represented as just this (a , b).

So, the order comes from the from the notion that a comes from small a comes from capital A small b becomes from capital B and then so on right. So, that is nothing really that is defined over two sets, you can still define it over any number of subsets. So, the simple example is the space R^n or the n dimensional space is a Cartesian product of "n" sets of real numbers. So, much of these things will be I mean cleared in detail as we as we go by.

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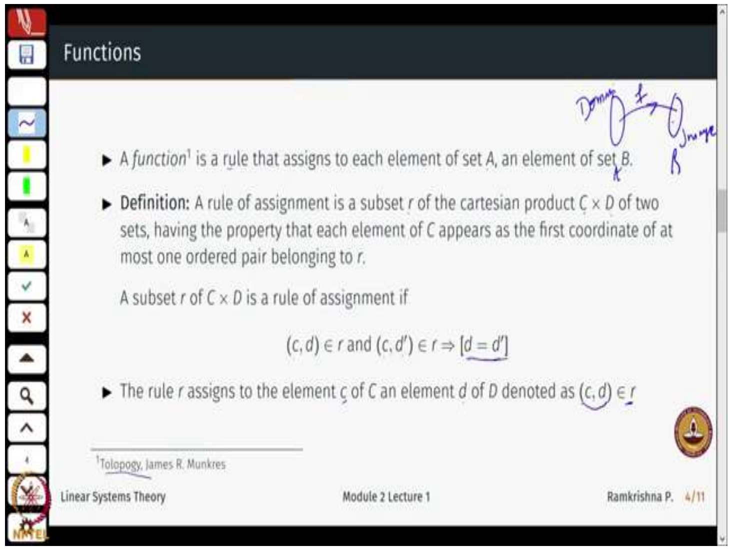

So, some other things which we would have learnt may be a bit in formally is notion of a function and function is you always or usually define between two sets at which in such a way that it is it is a rule this is a little abstraction, but things will be cleared soon. So, rule that assigns to each element of a set A and element of set B.

So, usually you know we talk of A this is a standard textbook notations you take a function f associates each element of A to an element of B ok. So, first a formal definition right so, I think. So, I just take this things from this famous book on topology by James Munkres So, by definition it is a rule of assignment is a subset “r” as assuming we know the notions of subsets of a Cartesian product C and D of two sets having the property that each element of C appears as a first coordinate of at most one ordered pair belonging to “r” ok, we will just see what this mathematically means so that the statements are a little easier to follow ok.

So, I look at you know (c, d) is assigned or it is it is associated with the rule “r” let us say that this is the same rule if I say (c, d’) it essentially means this one either d should be equal to d’. So, there is only one unique rule that defines c and d, c coming from set capital C, d coming from a set capital the D right. So, in general the rule is such this assigns to the element, but every element c of this set capital C and element d and denoted as the (c,d) is it is this rule “r” ok.

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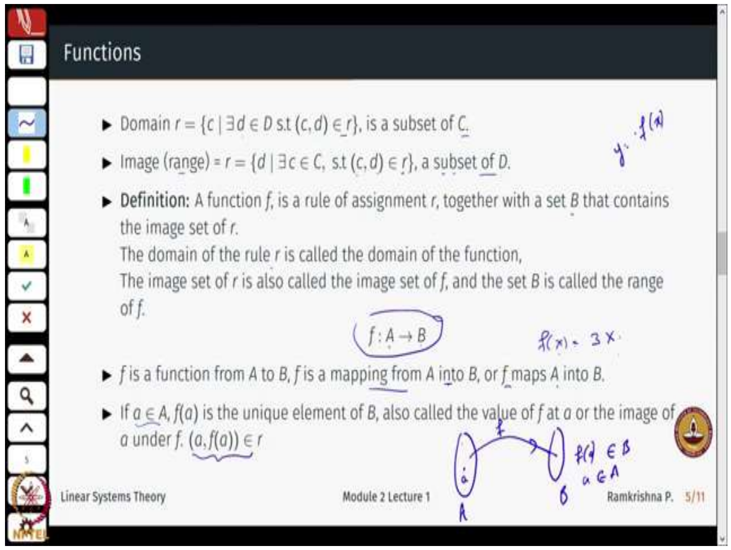

So, we will do it little more formally now right. So, then things you would have heard of it what is the domain of a set and what is or domain of a function and what is the range. So, the domain of this rule r is the set of all c's right such that, there exists a "d" right such that this pair (c, d) is associated to this rule "r" ok.

So, this is typically a subset of C, it could be a the entire set C also similarly, the image is also called the range is r again it is so, the so typically if I just look at this picture, I would associate the domain to the left hand side, and so, on the hand side is what I have will define shortly as the image. So, this image is the set of all points d here such that there exists a "c" which comes from C and this c and d together are associated to this rule "r" and not surprisingly the image or the range is a subset of D, it could be again as be an subset entire set D too ok.

Now, in general now how do we define a function so, far we said whether there is a some rule which associate elements over here to some elements over here ok. So, function is a rule of assignment together with the set that contains the image set of r. So, the domain of the rule is the domain of the function and the image of the rule also called the image set of f is called the range of f. So, I will define a function f as simply some like a something which associates elements of set capital A to set capital B ok, we would have done this earlier and this nothing really surprising here just that it it is a little more formal.

So, when I say f is a function, it also means f is a mapping from A into B or simply f maps A to B right. So, what does it mean in terms of the rules. So, if I take an element a of A, f(a) is the unique element of B so, if I just draw a little picture. So, I have a A here, a B, a little element of A under the function f match a some point f (a) A. So, f(a) is in B, a is an element of A right. So, this is this should be like in x.

So, a f (a) is a unique element of B and in some also called the value of the function f at the point a or the image of a under f right or simplify if I just say a rule based notation, it will just be something like this if I just take say for example, a vector "x" and the function which just multiply is it by is say a number 3 right so, this is a simplest example of a function right. So, f (x) is 3 x we will just and most of the times we restrict ourselves to linear functions, this are all in one dimensions, we will slowly generalize this to to R^n and so on right ok.

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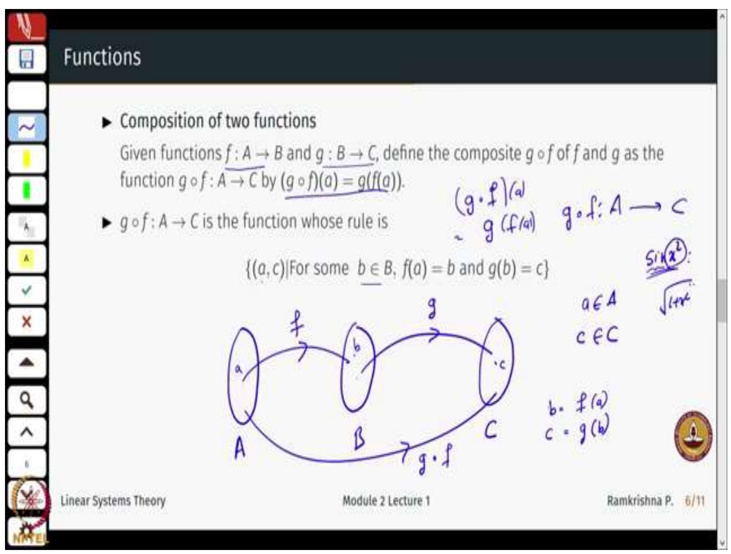

So, what is interesting so, before we go to composition is so, whenever I just say a function f (x)it is well, but the definition is not complete so if I just say y = f(x), I need to really specify what is that say the domain and the range. So, let us just do a quickly some kind of examples.

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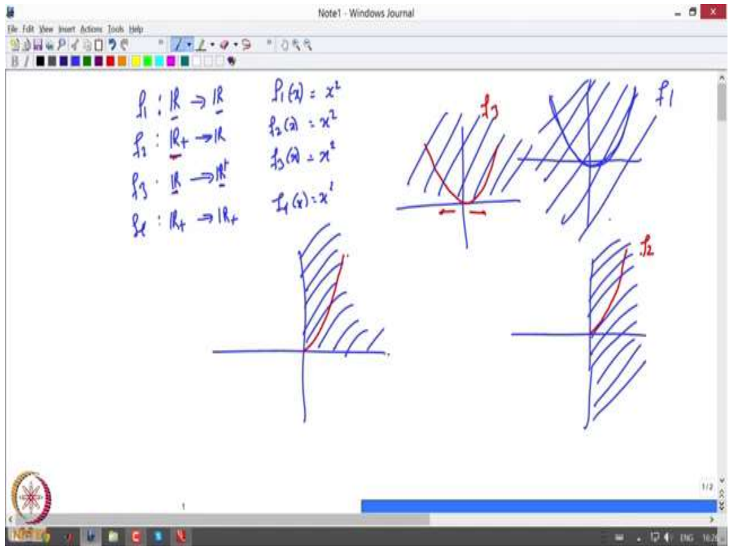

So, let us say I take function f_1 which is the R to R which is the entire space of a real numbers that is a that is a domain space, the range space is also the entire of R and it is defined is in such a way that f_1 (x) is simply x^2. So, if want to draw a graph of it.

So, not surprising it looks like this is a parabola so, I will just get this entire thing right. So, this is so; obviously, this entire space. So, my domain and range are this spend by this entire thing this is this is my function x square I can do something else also right. So, if I do define this as say $R^+$to R such that again same here the definition would still be same, f_2 (x) is x^2. So, I am restricting my domain now we will just so, when I say R^+you just I am just talking of numbers greater than or equal to 0.

So, essentially, I am so, this is my f_1 and if I look at f_2 we use a different color So, f_2 would look something like this so, this is my f_2 right. So, I just discard this negative thing because this is not in the in the domain. Similarly, if I define some function f_3 in such a way that it is goes from R to R^+ right. So, I am just restricting now the range of it to from 2 to R^+in such a way that again the function is defines in the same way x^2.

So, now, I have the graph would look something like this So, since this is the entire real line so, from positive, negative values and positive values this anyways just takes positive value so, this will just be this one. So, this is my f_3 and lastly if I look at f_4 which is again I just restrict them to just $R^+$for all numbers greater than or equal to 0 to R^+ and f_4 it is still remains the same and if I like draw a graph this would it is how would it look like, I will just this be this.

So, here the so, the domain and range is this entire shaded blue line. In the in the second case, my domain and range would just be say this one. So, I am just restricting to R^+ so this is this is my domain here which is all the shaded region. So, in the third case again so, this is the entire real line.

So, this would be my domain and in the fourth case, I am just restricting to positive (Refer Time: 11:31) right ok. So, just to give an illustration of how we define functions and why the definition or is not always complete unless you define the domain the domain and the range space ok, we will revisit this shortly again ok.

Now, something which is important is also look at so, compositions of functions say I have a function say let us say I define A to B, define a function f() which associates this A capital A to capital B and let me also talk of C which takes elements of capital B and gives me elements in capital C with say some other function g(). Now, can I use the standard definitions to just generally define this function here right, which directly goes from A to C. So, I have f() from A to B, g() from B to C, can I define something which associates A directly to C ok. So, one way to look at it is so, I just look at the composition of f with g. So, what is the domain of this composition of this big map?

So, this is f composed with g. So, its domain is A and the range is C. So, what is the rule of now for this? So, the rule if you see that the rule associates say a number from a which is a domain space then to the to the range space. So, here I am looking at a comma c right where a is in is in capital A, c comes from capital C in such a way that for some b this lies here. So, this here we call this a some point c here b here such that, f this a under the function f or the map f goes to b so, that b is now f (a) ok.

Now, what does the g do? The g takes this point b and gives me a point here. So, this g takes in its argument b and gives me a point c ok. So, this is this the general way, this (a,c) for some b in this capital B which is the image of this point f so, this point a under the map was function f and similarly, c is the image of this point b coming from the map or the function g. So, this is now the rule right.

So, g; so, f composed with g(a) I can write simply this as g (f (a)) and we have done several you know things like this without really understanding that we are actually doing a composition of function for example, say $\sin(x^2)$ at or even say functions like $\sqrt{1+x^2}$ and you know several things like that. These are essentially composition of functions one which takes the associates the so, this number in this argument to a sinusoidal value and one which just squares the number inside it. So, these are all composite functions and then we would remember from calculus of how we do differentiation of this by the chain rule and so on.

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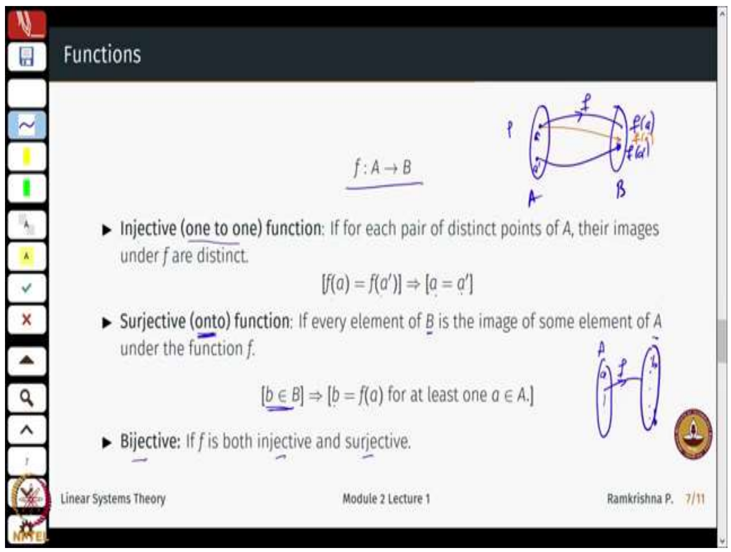

Something which we will which will be a little important and to again, we would have done this a little informally somewhere, but let us let us learn this in a different way. So, let us take a I just talk of this general functions f going from A to B So, this is this function is called one to one or more generally as an injective function, if for each pair of distinct points say that images under f are distinct.

So, say suppose I have A to B say this is f sorry this is the point a and say I have function f this is f (a). So, which means that a if f (a) = f (a') then, a=a' right. So, there is one which will always be distinct points so, it is a will go to f (a) if say some may if take some other point a', it will go to its own distinct point f(a').

So, this guys will never leave, this two are points are equal it means, say this two points are equal too right or in other words, I know maybe this a cannot map say into say some other point here. So, it cannot go this way right they cannot can cannot be any other a which associates which transforms under f to here. So, it will just be one to one that is that is we call it ok.

Now, similarly if I look at what is called an onto function so, this we would be we would be familiar with this kind of terminal which is said one to one and one onto. So, this means that if every element of b is in the image of some element of a under the function f. So, mathematically what it means. So, you look at b for all elements b belonging to B sorry all elements small b belonging to or coming from the set capital B, this b is equal to f of a for at least one a belonging to A which means. So, just take any point here say b_1 till say b_n, it should definitely come from some point.

So, a_1 or whatever right in from the set a right ok. So, now, a map is called bijective if it is both one to one and onto right, it is both injective and it is it is both surjective, some of this proofs I will leave to you to prove the following right.

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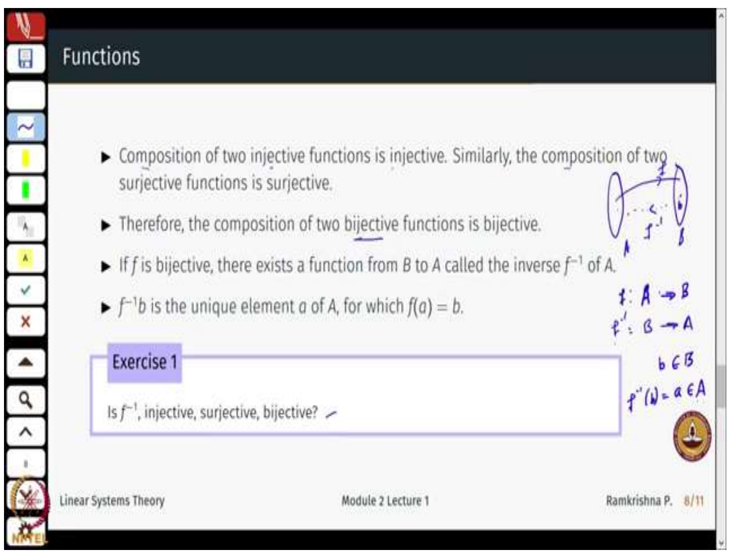

So, just a (Refer Time: 18:13) the composition of two injective functions is injective. Similarly, I can say that the composition of two surjective functions is surjective, proves are like this to and proves. So, I will just expect you to work out yourself if not then we could we could discuss this about the forum and therefore, we can also say that therefore, the as a consequence of these two statements that if I have two if I have composition of two bijective functions, the resultant function will also be a bijective. One interesting thing is of this map, if f is bijective it essentially means that there exists so, we are so far we have been talking of things going from A to B right ok.

So, what can the reverse excess right something here? So, that if turns out that f is bijective then there exists a function from B to A now, f was defined on B sorry A to B. Now, if f is such as it is bijective then there is something called f^{-1}, for which the domain is B and the range is A or a subset of A and this is called the inverse of a right.

Now, for a point b here so, this small b belonging to B the capital B, under the function f^{-1}(b), will give me an element "a" small a which is an which is a the member of capital A right. So, the notion of inverse so, again we would have done this in some settings some in some math course, but again just to make it a little more formal. A little exercise again for yourself just check if so, given that f is both injective and surjective, what is f^{-1}, is it do the same properties hold or not, again this is should be again like a two line proof ok.

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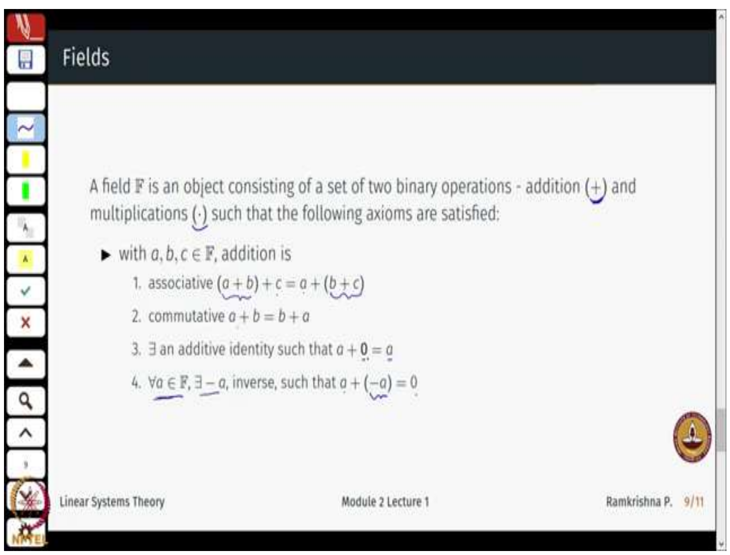

So, before we go little more formal, we will also define something called a field so, we so, if you would remember something from set theory, you would know something called groups and rings and so on, but what we will use throughout this course is the notion of a field, again it is a little generalization of what we possibly already know. So, a field is an object consisting of two binary operations one is in associated to an addition, second one you can associate to it to a multiplication. We just follow some axioms and again this axioms are nothing surprising, we would know this in one form or the other, but it is nice to write that write them down mathematically.

So, associative property is you just had up a plus b and then add c is the same as you first add up b and c and add a to it. Commutative well I just say $a + b = b + a$ it might be some trivial, but there are some interesting things that will go on over here. There will always exists an additive identity such that $a + 0$, is 0 is an additive identity will just give me the same point a. Now, for all elements in this field f there will exist an inverse $-a$ such that, $a + (-a)$ of this a this is the inverse will give me 0 and this 0 is the additive identity here.

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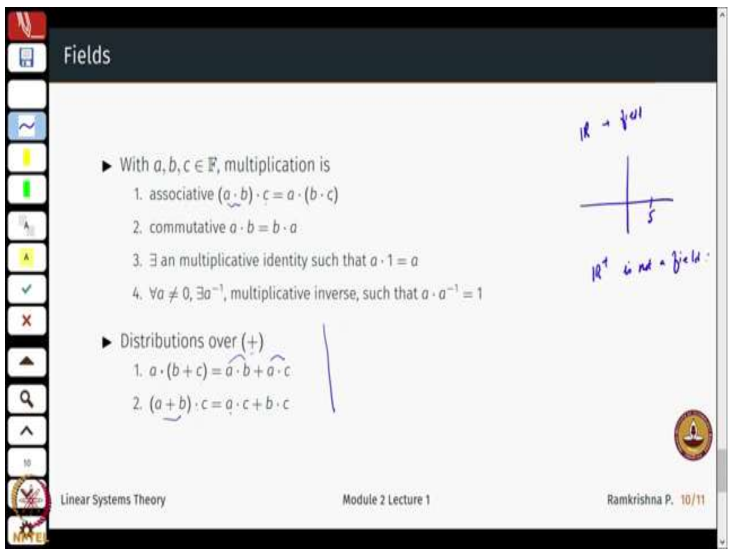

Similarly, we have rules associated with multiplication, same associative property a times b if you multiply these two first and multiply by c is the same as first multiplying b with c and a. Commutative, $a*b = b*a$ there always be a multiplicative identity which $a*1$ will always gives me a, there will be a multiplicative inverse and so on. Last interesting property is the distributions over the additive property. So, if I take say all these elements again come from the field f, a, b and c. So, if I take $a*(b+c)$ is same as $a*b + a*c$.

Similarly, if I do a a bit of the converse so, I just add up a and b first and multiply by c is the same as individually multiplying a with c and b with c ok, there is several examples here. So, if I just take the real line, it will have it is it satisfy all this all these properties, but if I just take the real line with which the positive say just the positive part. So, this may not be a field because if I take plus 5 there is no -5 associated.

So, R is a field whereas, R^+ is not a field and you can similarly construct several examples in higher dimension spaces and so on. So, this is like just a brief prelude to the things we will start in Linear Algebra in the next course.

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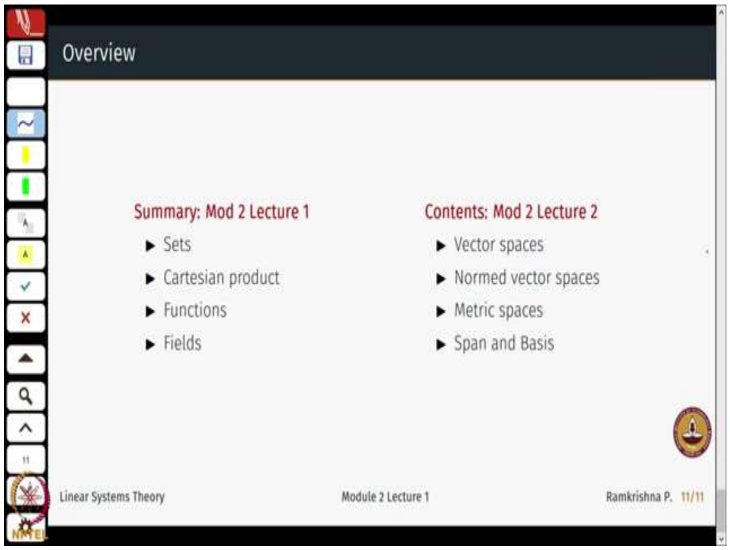

So, we will shortly look at vector spaces, non-vector spaces, metric spaces and then come with span and basis for a vector space. Again you would have done this in some form or the other maybe in your electromagnetics course or any other related course, but we will learn this a little more formally, generalize this not restrict ourselves to 2 dimensions or even 3 dimensions, but just go to some n dimensional spaces and see how do we formally define elements on those spaces and what are the corresponding operators or responding operations that we can perform on those. So, that will be in the in the in the next lecture.

Thanks for listening.