Linear Systems Theory

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Module - 12
Lecture – 02
Properties of LMIs and Delay LMIs

Hi, everyone. Welcome to this lecture number 2 of week 12 of the course on Linear Systems Theory. In lecture number 1, we or the aim was to introduce you to some computational tools that would help you solve some basic matrix inequality starting from the Lyapunov inequality in continuous time to discrete time, stabilizability and so on.

So, today what I will teach you is some computational tools that you would, that would be helpful in solving this kind of LMIs. I will also run you through few examples or through tests or few test cases of formulating problems as LMIs. I will briefly touch upon the linear matrix inequalities related to time delay systems. We will derive or we will see how stability of time delay systems can be formulated as equivalent LMI problems and see if the systems are independent of delay depending on delay and so on.

Similarly, with another concept called passivity of linear systems. Of course, a delay systems by itself is a vast course it has a bunch of books, bunch of literature on that. So, this is not a comprehensive or even a small introduction to delay systems. Neither is it a good introduction to passivity related concepts, but we will just learn little things that we may need and how we actually attempt to solve problems from different domains.

Neither is this weeks lectures and an extensive introduction to LMIs, right. There are also a bunch of books that are available lot of literature, lot of research work still going on, but it just to expose you to certain tools which might be useful for you if you are pursuing your research activity or reading some other papers or just if you want to look at some areas which are related to what we learnt in the course.

So, also the aim of the course or aim of this week's lectures is not really to do is to give you a comprehensive introduction on LMIs, is neither to do with delay systems, nor to do with passivity or the other things that we are; that we are about to talk in this lecture. So, we just to give you a brief idea on how to use tools of this form, ok.

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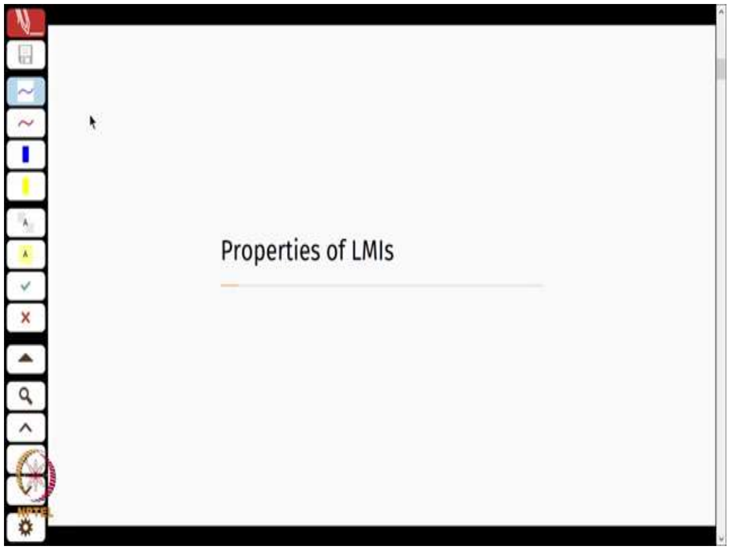

So, before we start, so just a little properties of LMIs that I would like to list out is the following.

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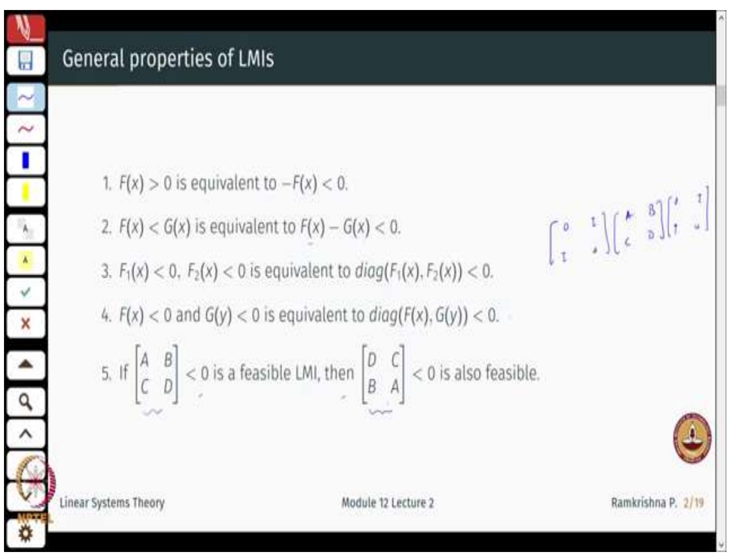

So, if I say that F(x)> 0 this is equivalent to saying -F(x) < 0. Similarly, F(x) < G(x) is equivalent to saying F(x) - G(x)< 0. If F_1(x) < 0, F_2(x) < 0 this is equivalent to saying that the diag(F_1(x), F_2(x)) < 0. Similarly, if I have LMIs in two different variables F(x)< 0, G(y) < 0 then I can say that the diag(F(x), G(y)) < 0.

Similarly, if this LMI is feasible that this LMI is also feasible. If you just check by pre and post multiplying in this way A, B, C, D with $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ that is this like. So, these two matrices this matrix and this matrix would have some kind of a equivalence relation as we talked about in lecture 1, while we were deriving Schur complements, ok.

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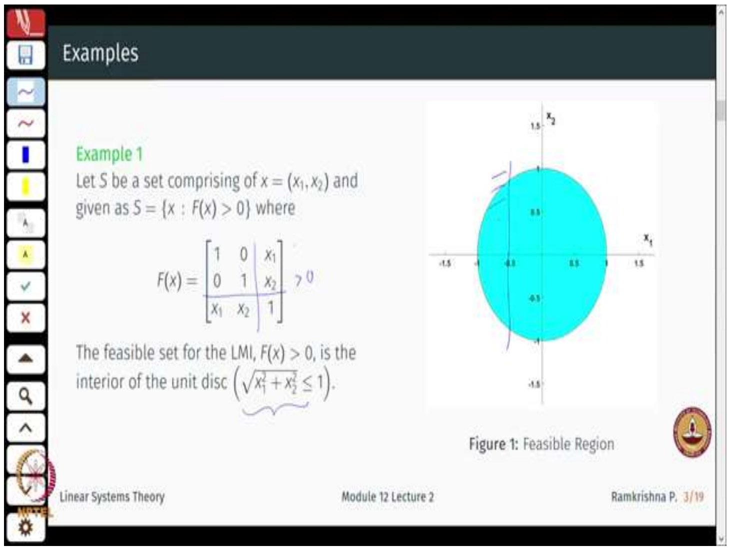

So, again start with the example that we did in the in lecture 1 using Schur complement of this block, ok. So, I am about to solve this LMI where I want to find what is the region for which F(x) >0, ok. So, based on 2 different methods that we learnt in lecture 1, we will see that the feasible region is just the interior of this unit disk and will look something like this and this is straight forward to check.

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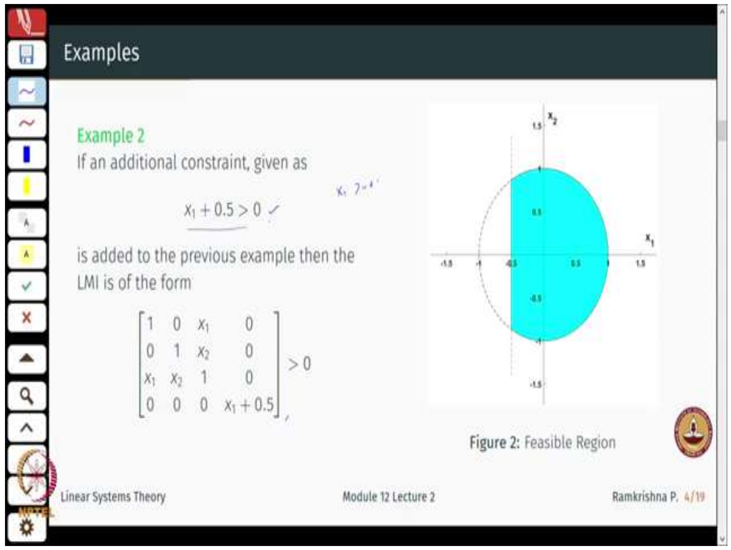

In addition, if I impose a constraint that, in addition to just F(x) being 0, I impose another condition that x_1 + 0.5 > 0, right. So, the LMI would change now to something like this, ok. So, it is easy to check right that I am only looking at regions where x_1 > -0.5, right.

So, this the region to the left of this will no longer be the feasible region and so, solving for this LMI now would give me an appropriate feasible region which in addition to this constraint also takes care of this particular constraint that x_1 + 0.5 > 0, ok.

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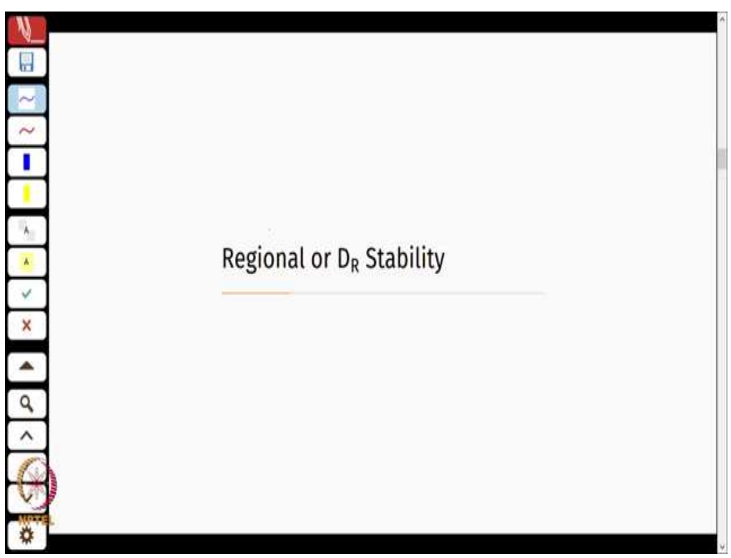

So, LMIs can also be used to solve more you know kind of interesting problems. Much of the design problems that we looked upon were, first stability was important and second was design in terms of pole placement, right. So, what we will do here is, again this is not really a comprehensive introduction to what is called in literature as D stability or even D R stability.

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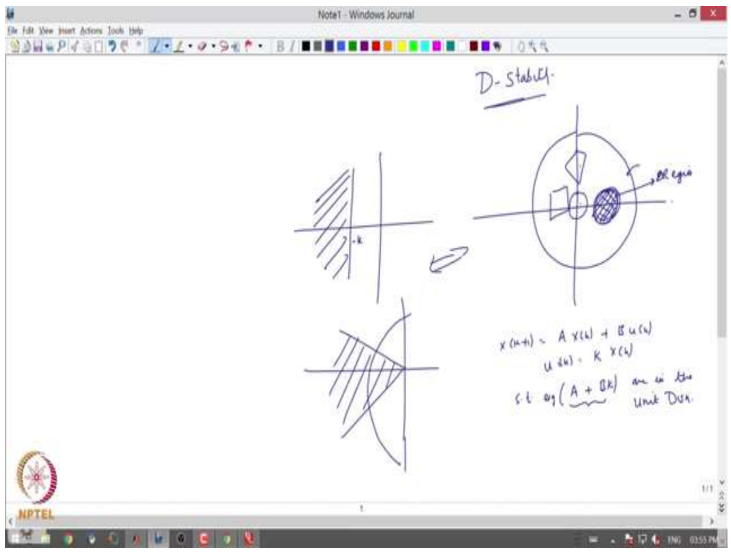

So, suppose, let us let us work in the discrete time case for the moment, that so I have x(k+ 1) = Ax(k) + Bu(k). The standard stabilization problem would be find control law u(k) = kx(k) such that the eigenvalues of A + BK are the eigenvalues are in the unit disk, right. And then we know now, how to first formulate this as an LMI that is what we learned in lecture number 1.

Again, though I am just dropping the case the -K here which we would usually use, but nothing really changes, ok. So, suppose now so my usually I am not really looking at perform as stability, in addition to stability I am also interested in performance and performance usually translates to placing poles of my closed loop systems at appropriate locations.

So, let us say can I say maybe I want my poles to lie within this region here or maybe some region like this or maybe like this or this several possibilities or say I just want my poles to lie within a circle of say radius 0.1 instead of 1, right. So, this these problems are called D stability regions.

Well, in the continuous time it could just translate to say your closed loop poles being say to the left of say -K for example. It could also translate to poles being say within this region where you put certain restrictions on the damping. If you put certain restrictions on omega n then you will have another region and so on, right.

So, equivalently like you learn concepts also exist LMI in the continuous time domain. So, let us look at for example, this particular thing in the discrete time case of how do I solve for my poles to be in one predefined region of the stable region, ok.

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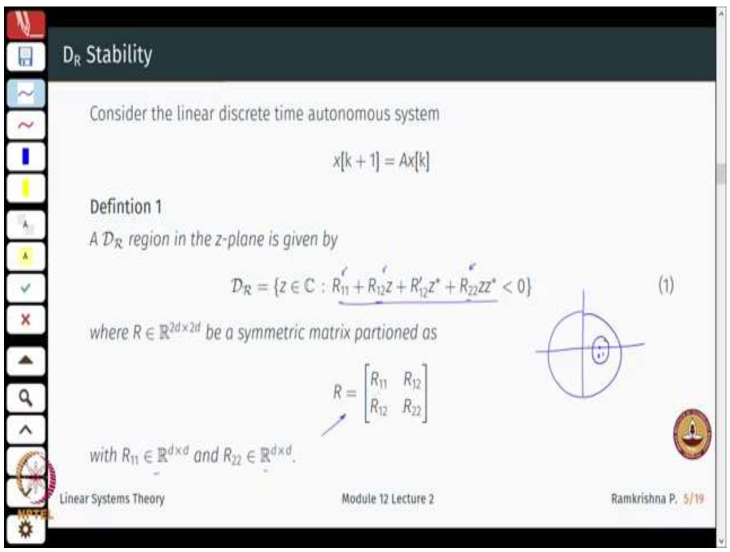

So, let us look at the autonomous system to begin with and ok. So, a D R region is defined by this following relation, right. So, in the z plane, so I have matrices $\begin{bmatrix} R_{11} & R_{12} \\ R_{12} & R_{22} \end{bmatrix}$ ok. The D R region is now represented by this particular LMI, right, where, ok. So, R_{11} and R_{22} are of these dimensions and overall I can write R as a partition matrix of this from R_{11}, R_{12}, its symmetric and you have R_{22} here right, ok.

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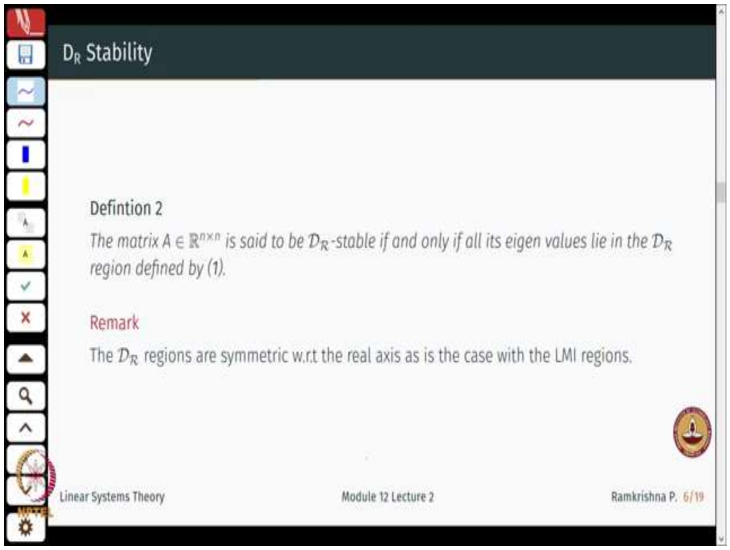

So, what is the definition of that? The matrix A is said to be D R stable if and only if all its eigenvalues lie in the D R region defined by 1. And how what defines these regions? Are this choice of matrices R_{11}, R_{12}, and R_{22}; I will show you, I will shortly show you some examples. So, if I say well this my D R region looks something like this, so this is my D R region, ok.

So, my system would be stable or my system I would call it D R stable if in addition to stability all eigen values are within this shaded region here. So, I am restricting my closed loop poles to certain sub regions of the stable region in this in the unit disk and the stable region is the unit circle, ok good. Some properties the D R regions are symmetric with respect to the real axis as this case within any general LMI regions.

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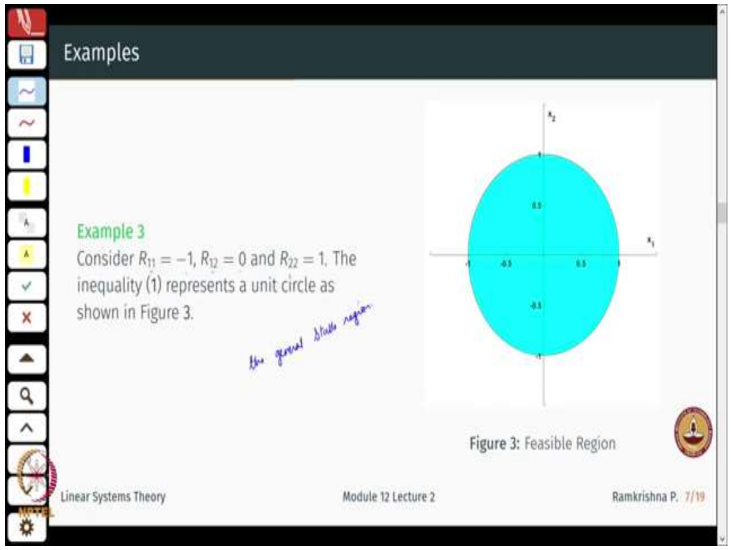

So, a few examples of what are these D R regions. If I take a very simple example of say R_{11} = -1, R_{12} = 0, R_{22} = -1, this will give me the standard unit disk or they just is the general stable region, ok.

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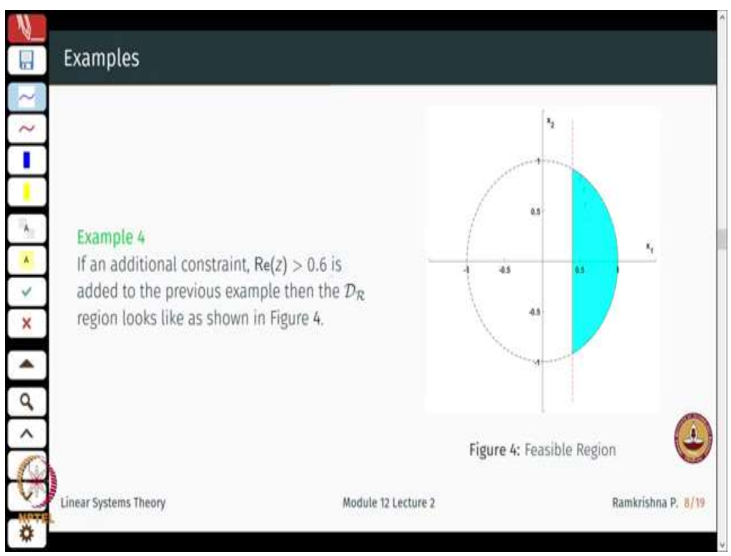

Some other example, so if I just add a constraint saying that the real part of z >0.6 then, if you just add it into in addition to the constraints here then I get that this is my new D R region, right the region in blue here.

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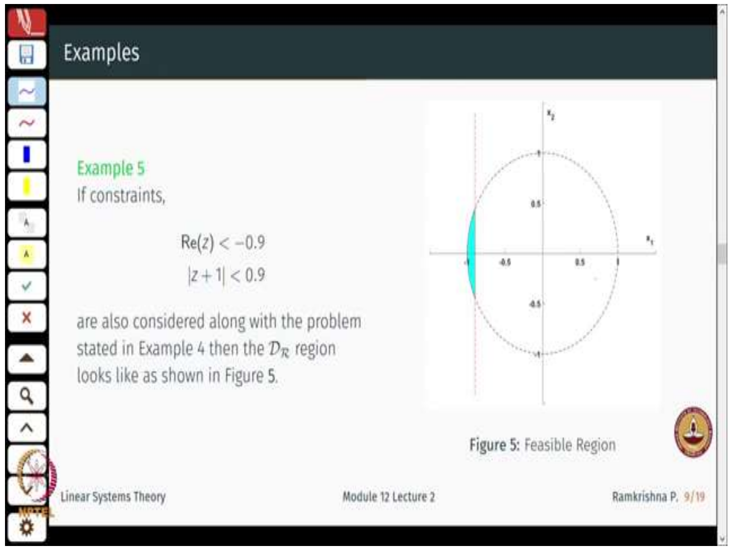

Similarly, I can restrict to several other regions of the z plane or of the unit disk, right. So, all these constitute D R regions. So, now, how to check; so ok. There are there are a bunch of results that you can obtain by choosing this appropriate Rs.

So, you could also say well can I have my D R region as a say as a circle centered at 0.1 with a radius of 0.1 and so on. So, the literature that is at the end of the; end of the slides will guide you through little more exposure to these kind of D R regions. But it just to give you an idea of what is what kind of problems then we can solve by formulating them as LMIs, ok.

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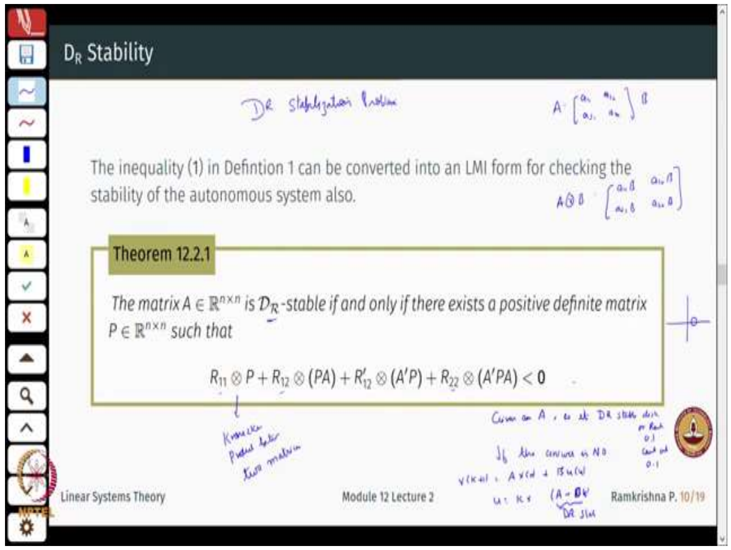

So, what is the LMI here? How do I check given a A matrix and given a certain D R region if that particular system represented by an A matrix is D R stable or not. So, one result that says, ok, this from a reference that is mentioned towards the end of the slides is that a matrix A is D R stable and this D R stable these regions are defined by this choice of matrices R_{11}, R_{12}, and R_{22}.

So, it is D R stable, if and only if there exists a positive definite matrix P also symmetric such that any relation like this and this is usually the Kronecker product, Kronecker product between two matrices, right. So, in general if I say A is a two-dimensional matrix of the form a 11, a 12, a 21, a 22 $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$and say B is also a two-dimensional matrix A (x) B would be something like this that you have $\begin{bmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{bmatrix}$ ok.

So, this is the LMI that we would be; we would be interested in solving. We can also formulate the problem of what if well what if this LMI if the answer to this LMI is no. So, for example, I ask a question, so given an A ok, is it D R stable. D R stable, say that I am only looking at not the unit disk, but say disk of radius 0.1 centered at 0.1, right.

So, I am not interested in the stability region, but say some region like this, ok. So, if the answer is no, then I will look at an alternate problem, right. So, I will look at say I have x(k+1) = Ax(k) + Bu(k). Now, can I find a u = k x such that A + B k is now D R stable? And these problems are called D R stabilization problems, ok.

So, I will not go into the details of that it is again an exercise which is similar to what we did for stability of stabilization of discrete and systems using Schur complements and so on, but I will again you can just refer to the literature at the which is mentioned at the end of the slides, ok.

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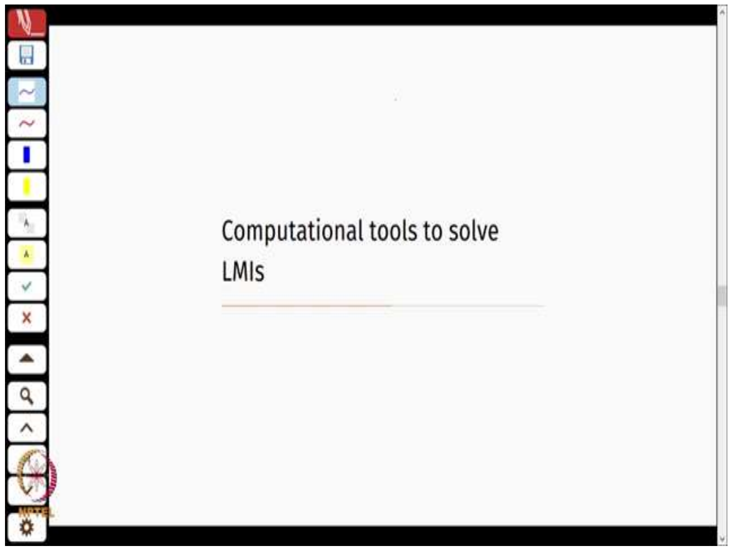

So, now, how do we solve this via using MATLAB?

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So, a couple of tools that would be useful, ok, there also this LMI toolbox in MATLAB, but I would prefer using these two you could you may just want to download these two software packages called YALMIP and Sedumi. Just add those folders, we have MATLAB file your MATLAB path and then things will be good, right, ok. So, the first thing, ok. Let us start by solving a very simple problem of stabilization or not stabilization, but even say stability problem.

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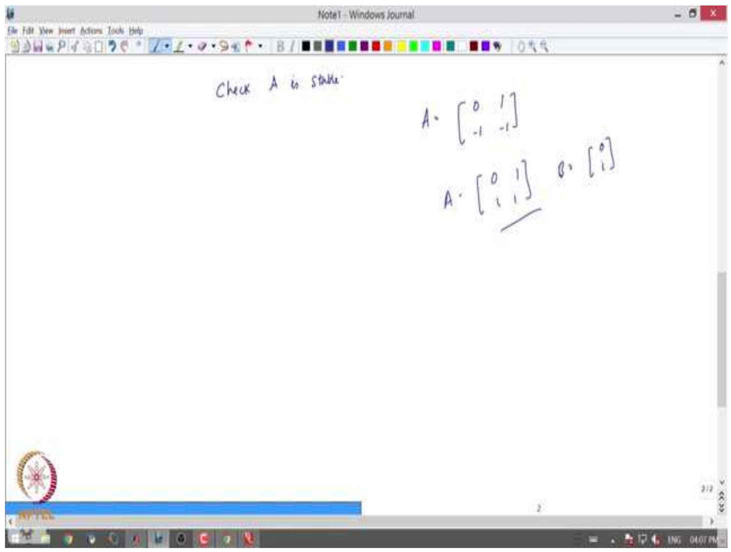

So, for example, check whether a given A is stable. Check whether A is stable, ok. It could be a continuous time and discrete time both, ok.

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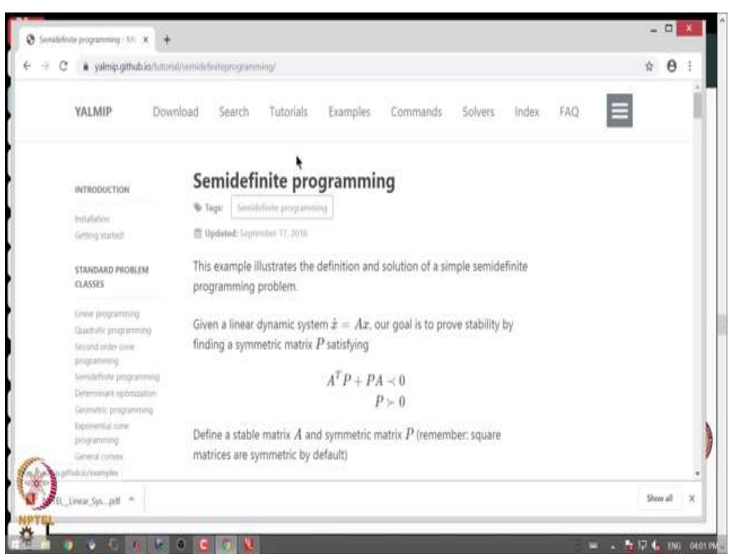

So, what, I am just at this year this YALMIP website. So, there are a bunch of tutorials which you can go through, so, ours comes in the category of semi definite programming. So, start from here. So, I have a linear dynamical system $\dot{x}$ = Ax.

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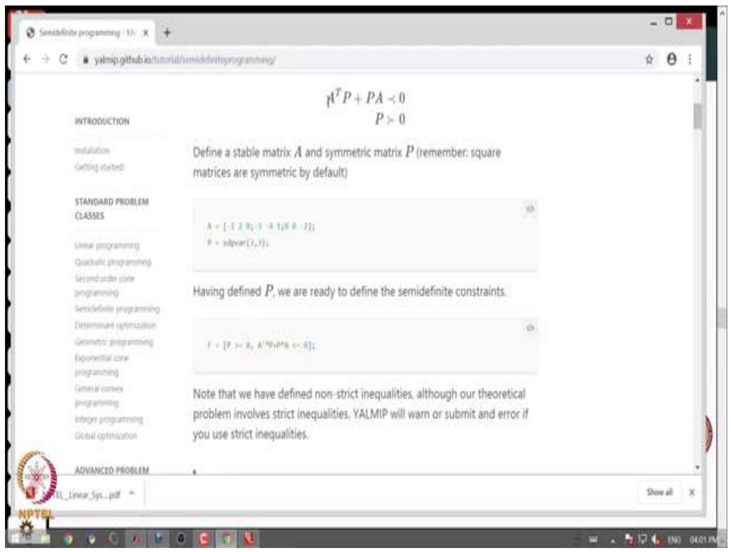

And the Lyapunov condition says that I need to solve for $A^T P$+ PA < 0 with P is a symmetric matrix which is greater than 0 and so on. So, how do you define? So, A you can just write in the standard MATLAB way as defining a matrix, P is called my it is called an sdp variable and solving for P. So, that will be called a sdp var of; so, its be our dimension 3 x 3.

The LMI that I am really trying to solve is F, so I will just write down those these two constraints in the code P > 0 and then $A^T P$+ PA ≤ 0, ok. So, there is a little warning which YALMIP tells us that, we are not really solving for strict inequalities, although theoretically we are looking for strict inequalities, right that P > 0, $A^T P$+ PA <0.

So, to do this what we could do is just add a little some kind of a threshold to P that this do not check for P > 0, but instead say check for P being say greater than point, so the condition which YALMIP checks do by default is this one, but you could also check for P being greater than or equal to 0.001, right just to add act with this strictness here.

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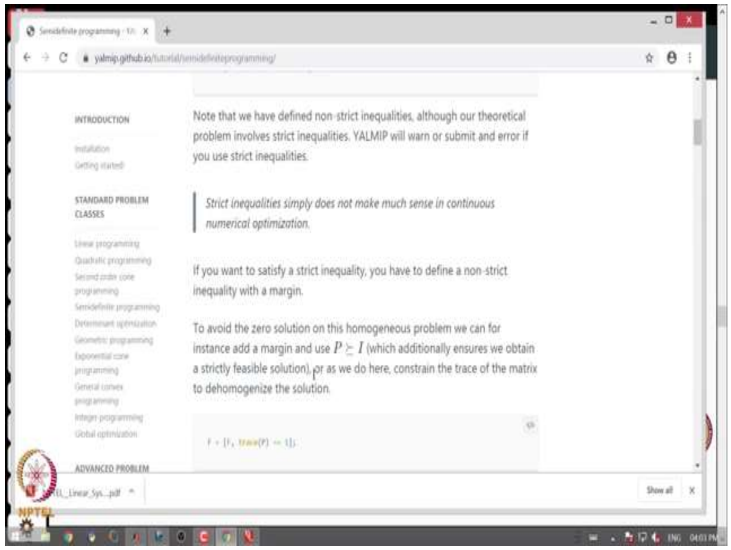

There also other ways which these guys tell us, you can have the trace of P to be 1 and so on, ok.

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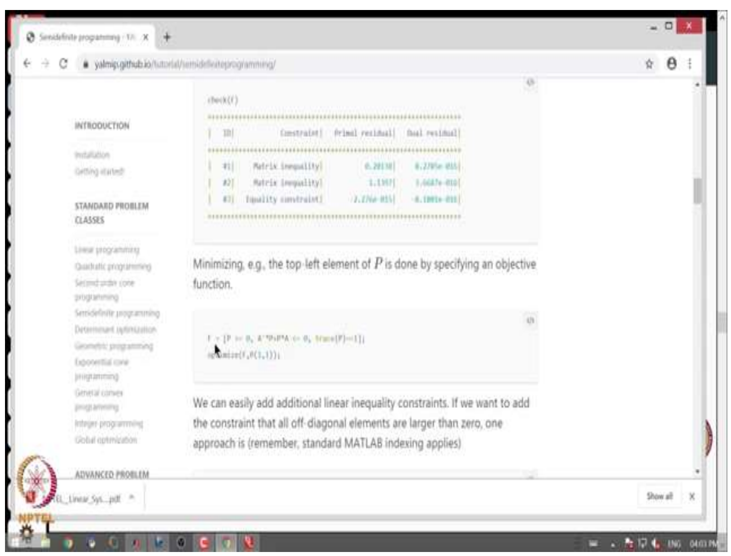

So, once you write this constraint , F is such that P is greater than 0.

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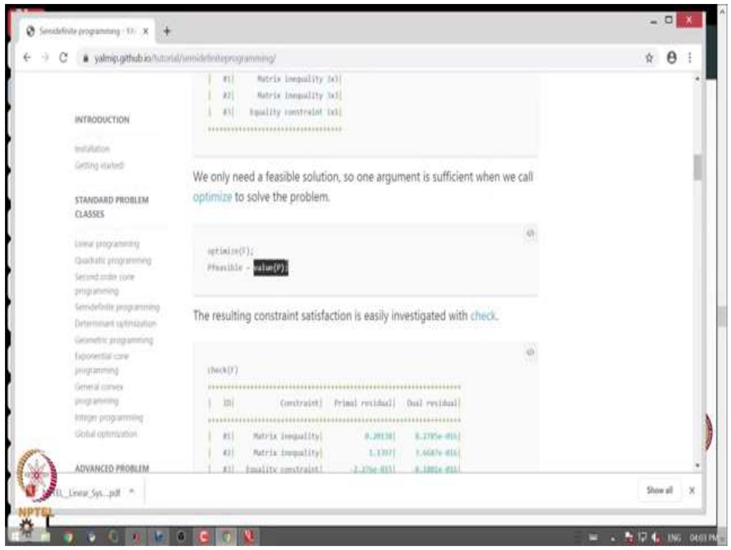

Then the way to solve is you just say optimize F and then the feasible value of P would be the value, so you have to write value of P, right because P is just this variable here and P feasible value you just type down value of P it will give you the appropriate value of P, ok.

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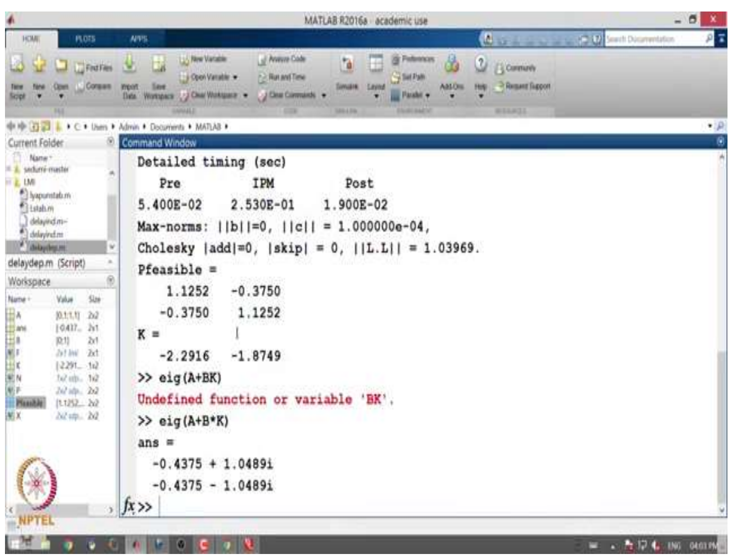

So, let us do some very very basic code here.

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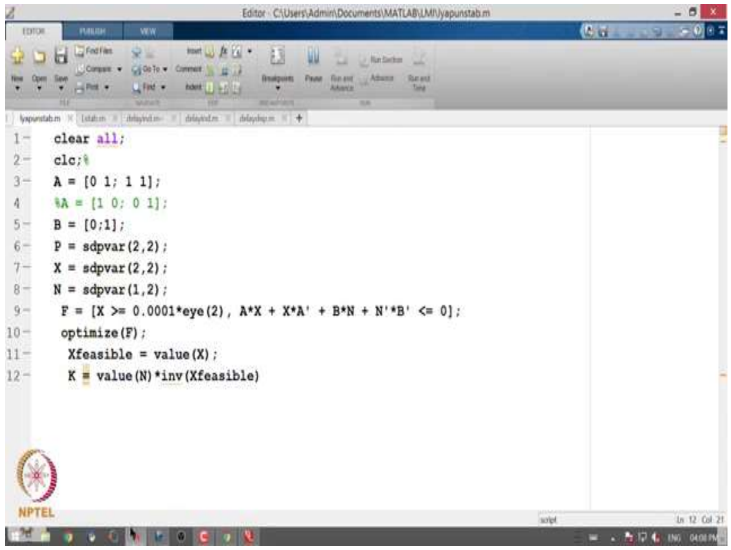

So, I just go to the very standard problem where A is given as say something very very simple, right.

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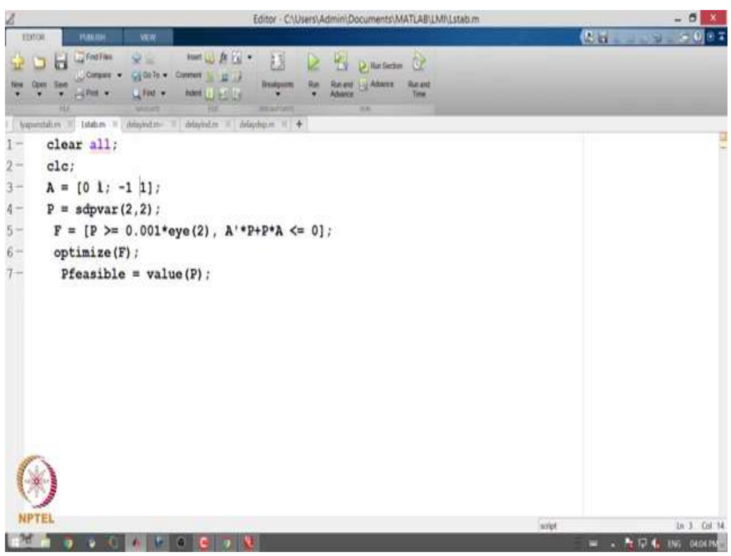

So, A, let me start with this A as $\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$ and of course, it is easy to really verify that this is this stable. But I just run this code for you and check what MATLABs actually gives you, right.

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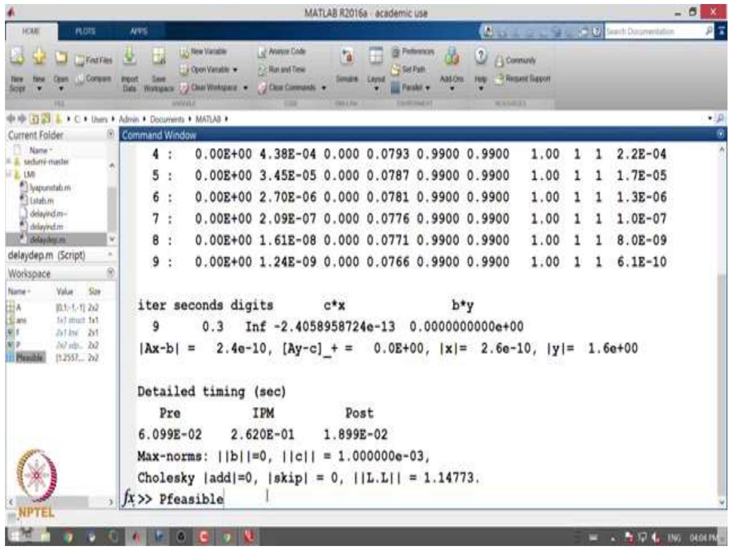

So, it actually tells you that this problem is solvable and you could also check for P feasible.

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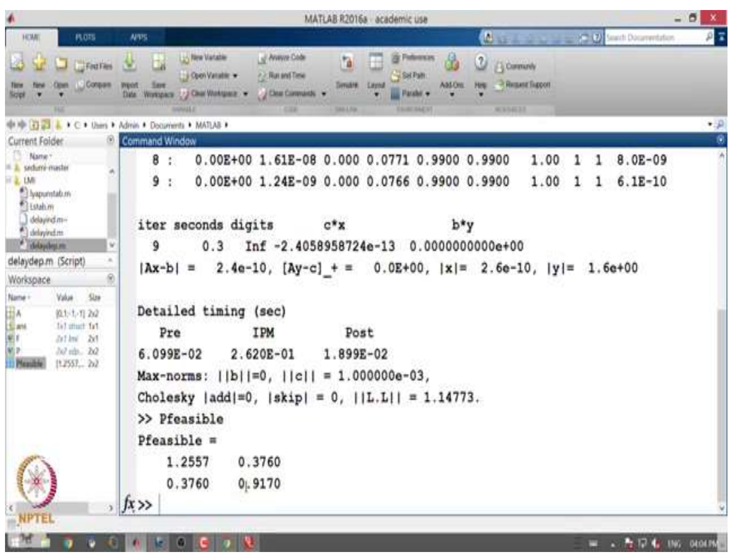

So, the matrix P that solves this problem will be given by this. So, you can check that this matrix is a symmetric and it is positive definite, ok. So, and let me just do something else, right. Let me just start some, ok. So, this is this unstable, right. So, A is now instead of, so I just take another A which is $\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}$ and check how this LMI solves (Refer Time: 19:17) the instability across for me, right, ok.

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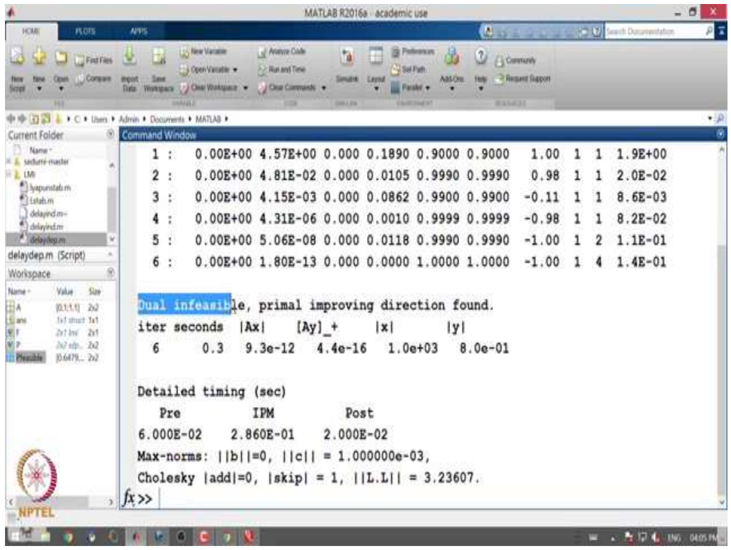

So, I go through this process again and actually it tells me that it gives me an infeasible infeasibility thing, right, that the problem is actually not solvable and this because it is an it is an unstable system that no there is no P that will solve the linear matrix inequality, ok. And just to add a threshold I just checked put as P as greater than or entries of P > 0.001, right and this is how just write the little code, ok. Now, let us go one step further. So, if we, ok; let us me just open slides of lecture 1.

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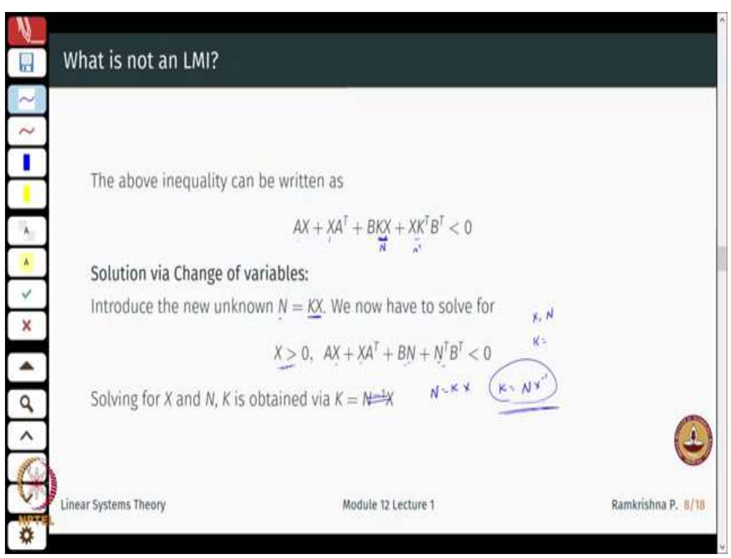

In lecture 1, we had this stabilization problem, right; so, how do I check if A + BKis a stability matrix.

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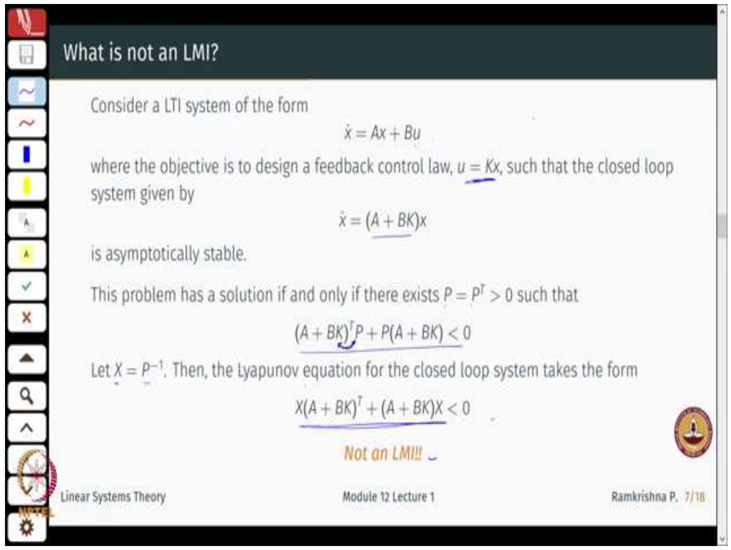

So, we kind of found that found out at the standard $A^T P$+ PA does not work because of these two unknowns k and x, ok. So, and then we had an equivalent LMI formulation for that, right. So, AX + XA^T+ BN +$N^T B^T$, where K was eventually given by this one, ok. So, we will try to solve a problem like this using MATLAB.

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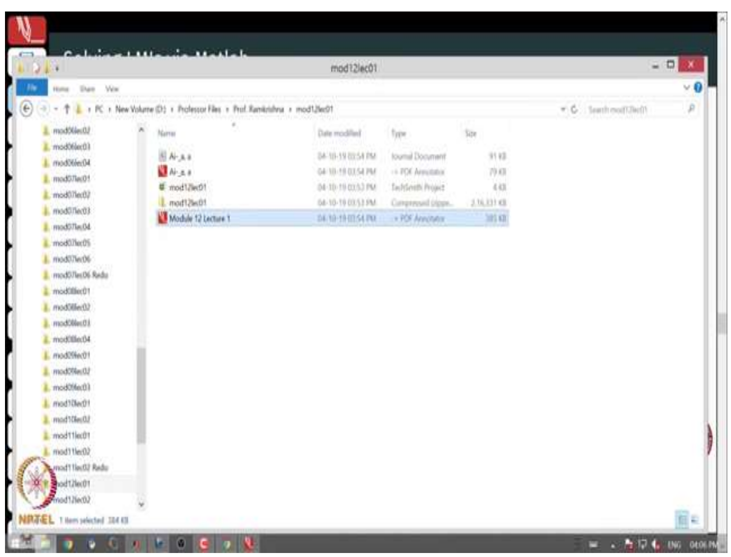

So, I just take this unstable A here $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, which we had which had here and say let B is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and I need to search for a K or find whether or not the K exists such that this system is stabilizable, ok. So, my P is usually of, the P I do not need any more. So, my X is again 2 x 2 matrix, this is my sdp variable, yet again going back here, right. So, X is what I am solving for and P is just X^{-1} that I can easily find out. N is of dimension 1 x 2, right this is also an sdp variable.

So, there is no real constraint on A, right, so on N, right I am not really fixing N to be all entry should be greater than 0 or whatever; only thing that is to be fixed is for x and then y. So, this LMI I just this LMI just plug into this equation and I say optimize F and then I have this P feasible or what knocking down the feasible X is this maybe I should call this X feasible instead of P, ok. And then I find this value of K, right I just run this little code and it gives me, ok. Let us check if, ok.

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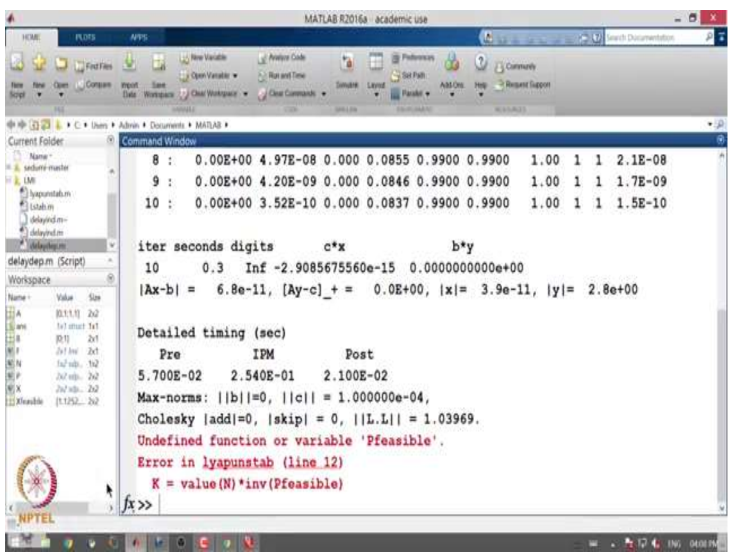

Something went wrong, ok. I should call this Xfeasible instead of P feasible, ok. Let me run this code again.

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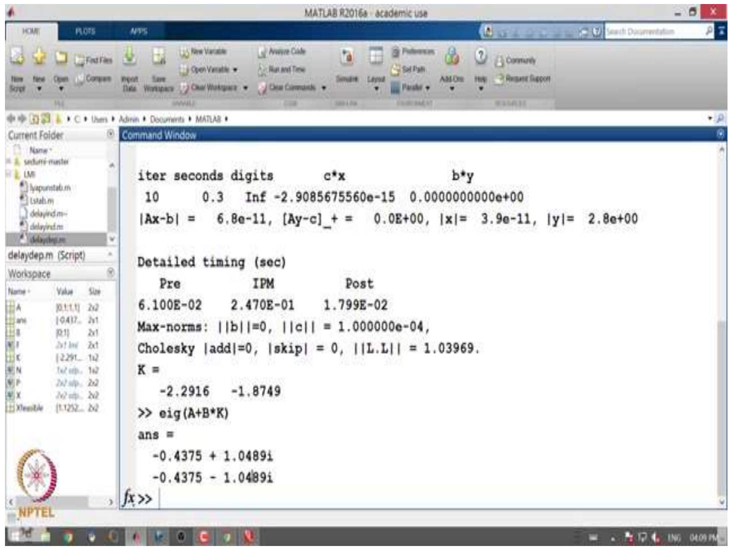

So, it gives me this values of K, right. And I can easily check what are the eigenvalues of A + BK and you can see that the closed loop system is actually stable, right, ok. So, this is some very very basic examples that you can try out discrete time version of this you can try out Schur complements and so on, but I will just leave that to you. It is just a matter of typing down; typing down the code, ok. A few more examples I would like you to quick, to quickly do here.

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So, first is the delay LMI. And what is this delay LMI tool?

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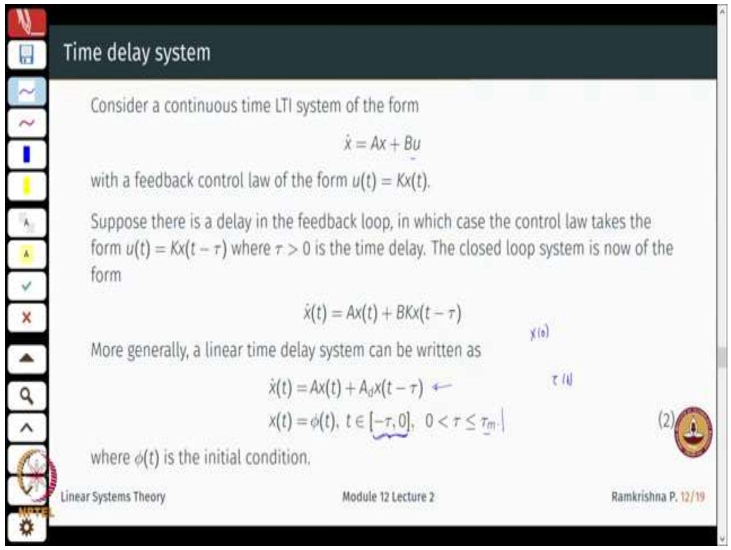

So, let us take a small problem or how do we motivate ourselves to do this problem of why are delays important.

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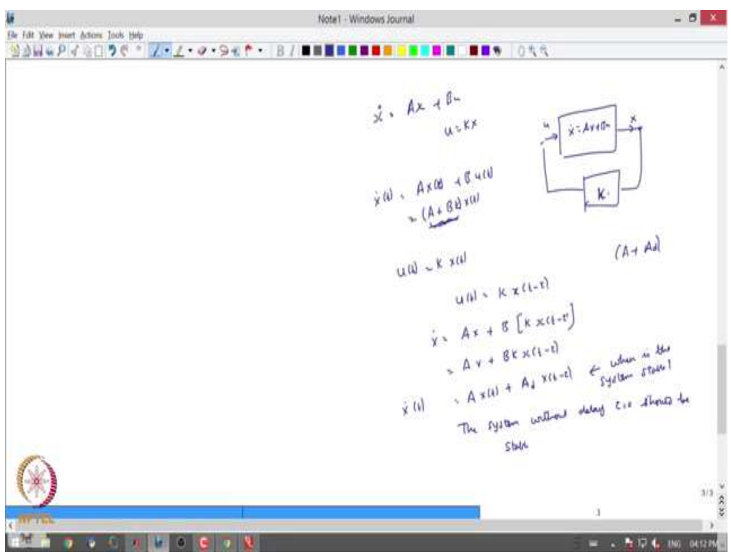

Let us say I have a plant $\dot{x}$ = Ax + Bu. So, this is my plant. So, this is $\dot{x}$ = Ax + Bu. Let us say I measures all the states this is my input and typically I will have a k here, which will just measure, this will be measured instantaneously the control law will be computed and then fed back all will happen instantaneously. But you know in practice things may be a little different, right that you may there may be some computation time.

If you are communicating via network channel that will cause further delays and so on and therefore, your control closed loop control law may not be instantaneous. So, if I say I am looking at u = -kx, ok; let me put this piece here $\dot{x}(t)$ = Ax(t) + Bu(t), ok. If in the standard case u(t) is k, x (t)and my system looks like this A + Bkx(t). And these piece problems I know how to solve a A + Bk is stable then my closed loop system is stable, that is ok.

Now, usually u may not arrive instantaneously and it may arrive with some delay let me call this τ, ok. Under this situation how does the closed loop system look like? I have $\dot{x}$ = Ax + Bx(t-τ). So, this will be Ax + Bkx(t – τ), ok; so, longer in this form, right. So, the eigenvalues of A + Bk would not tell me much about the closed loop system.

So, in general we can write that a delay system of the form $\dot{x}$(t)= Ax(t) plus let me call this some A_dx(t-τ), ok. So, when is this system stable? (Refer Time: 25:30) is this is the question, ok. So, first, the necessary condition is that the system without delay should be stable, right. So, the system without delay; without delay which means τ = 0 is stable or it should be stable and that is A + Bk or A + A_d first must be a stability matrix, ok.

Then we can ask our self a few questions, right is the system stable independent of delay that, ok, no matter whatever the τ is my closed loop system will still be stable. We will see if that could be true or derive, what under what conditions these systems are actually stable independent of the delay; means whatever even for large delays my system would still be stable. There could be cases where my system can or I can guarantee stability say if the delay is like say 2 seconds or say 4 seconds, but anything larger than 4 seconds I might go to the verge of instability, ok.

So, let us see how we can actually quantify those results. Again, as I warned you that this is not really an introduction or even its not even introduction you know forget about a comprehensive review of delay systems, but it just like this is just teaching you a bit of how to formulate problems as LMI problems, ok. So, what is the result that we will be interested in?. So, I start with $\dot{x}$= Ax + Bu and I write a delay system which is of this form. The initial conditions now need to be defined on all this time interval from -τ to 0.

So, just to solve a continuous differential equation, I need just x of 0 whereas, to solve a differential equation I need the set of all initial conditions which happen in this range or in this time interval -τ to 0. Of course, I will not go into the details of that and also assume that the delay is bounded from above.

One more assumption we make is the delay is constant with time, but they are also the results where you can say that my delay changes with time and then you will say how fast it changes how slow which changes and so on, but will restrict ourselves to a constant delays. So, this system, I will call this system 2, is asymptotically stable, if there x is symmetric matrices P and S such that P is greater than 0 and an LMI like this holds, ok.

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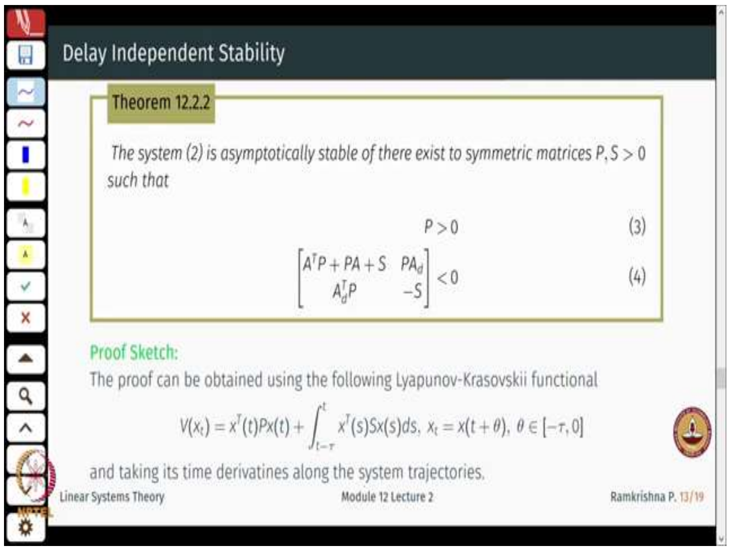

The stability certificates are still are again obtained by using my making use of Lyapunov like function. So, this was a standard Lyapunov function for the system without delay. So, now we have another term which takes care of how to handle the delay terms, ok. So, let us see if we can really quickly do a proof on how we arrive at this delay LMI.

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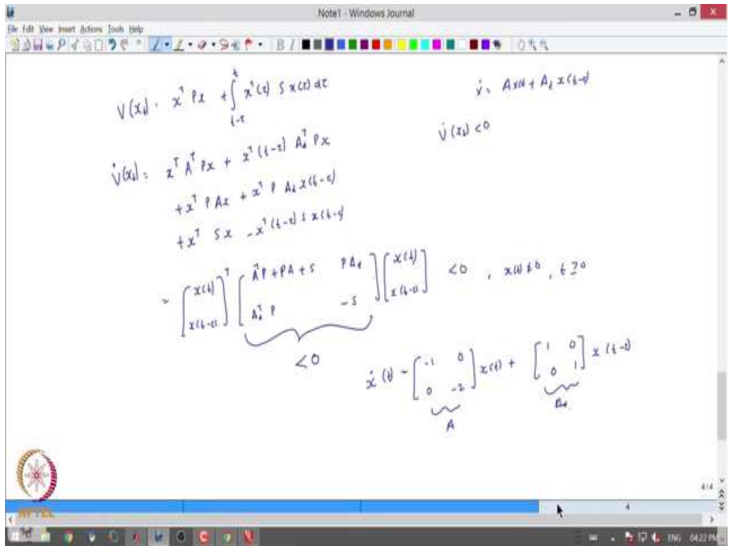

So, I have V(x(t)) = x^T Px + $\int_{t-\tau}^{t} x^T(\tau)\, Sx(\tau) d\tau$ ok. Then I do the differentiation $V(\dot{x(t)})$. Again, along the system trajectories what are the system trajectories system trajectories are $\dot{x(t)}$ = Ax + A_dx(t-τ), ok.

So, I just do the standard differentiation I get this is $x^T A^T$Px + x^T(t-τ)*,* A_d^TPx + x^TPA x +x^TP A_dx(t-τ) + x^TS x $x^T(t-\tau)$Sx(t – τ). Now, rearrange terms in the following way, right.

So, I have x(t) here, I have x(t – τ) at the transpose. What I am left here is this $\begin{bmatrix} A^T P + PA + S & PA_d \\ A_d^T P & -S \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}$ < 0, right. So, x of t this with this constraint and t ≥ 0.

So, $\dot{V(t)}$ < 0 translates to this matrix being less than 0, ok. Now, this is this is an LMI, right. So, look at this. So, this is what I am interested in falling P > 0, again matrices S which are also positive and symmetric such that this LMI holds, ok. So, I will just quickly show you some examples again, ok. So, let me go here. So, let me, ok; comment this out, ok.

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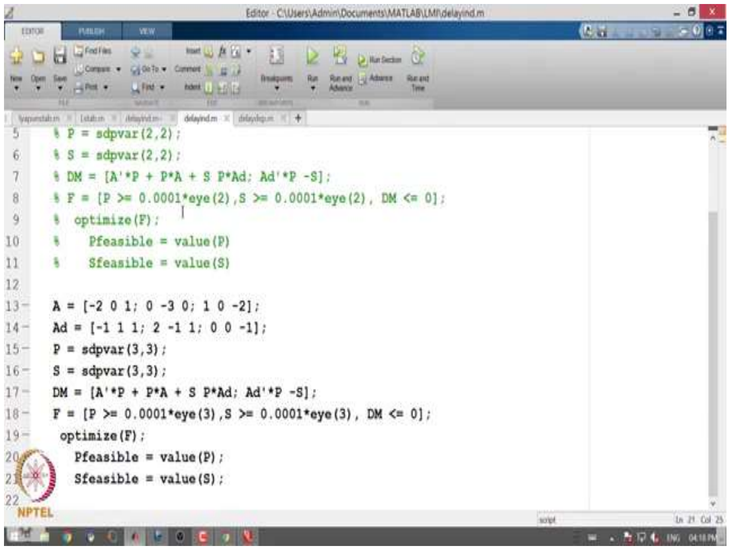

So, what I will be interested in is to solve is to check for stability of this particular delay system $\dot{x}(t) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$x(t) + $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$x(t-$\tau$), ok. And if I just run this code here, so I just plug in. So, my unknowns are P and S of they are 2 x 2. So, what I need to check is I just plug in the LMI condition here that is P>0, A^TP + PA + S and all this entire matrix, I just plug it in here.

So, P>0, S > 0 and this DM which is the matrix inequality which we had, so this also, this should be less than or equal to 0. I just run it for optimizing F and this is what I get, ok. I get an error.

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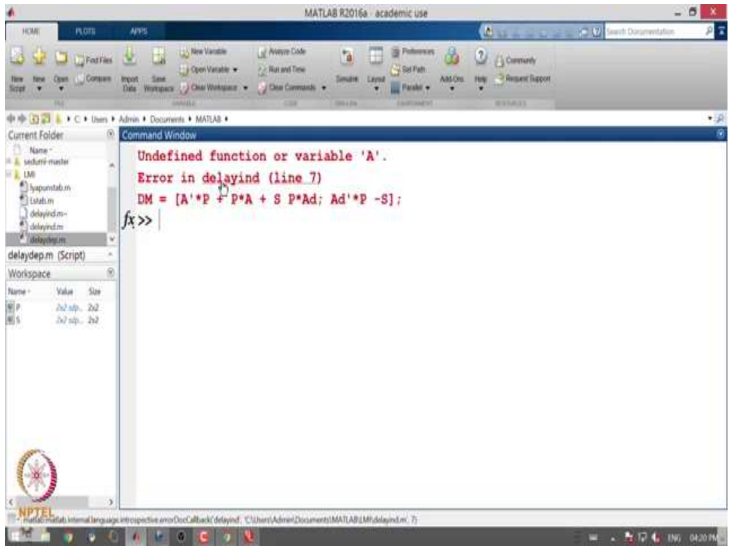

I forgot to define the A.

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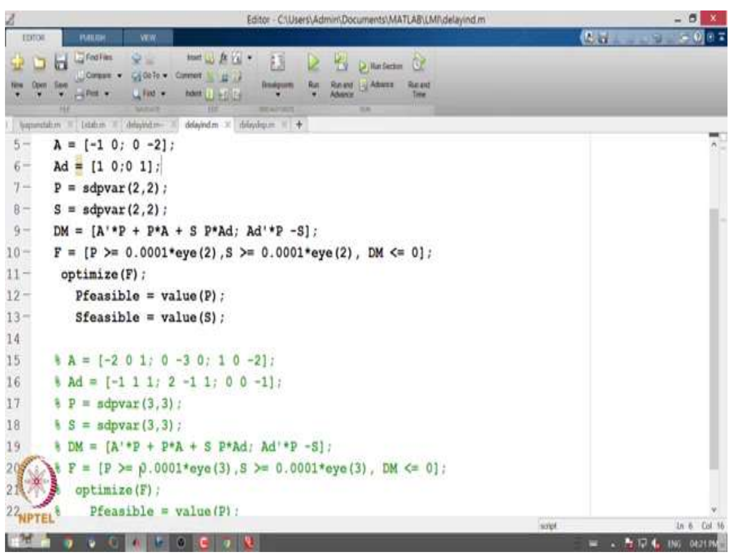

So, my A was this one was [-1,0;0,-2] the A_d was just the identity that is [1,0;0,1], ok. And let us see if this works.

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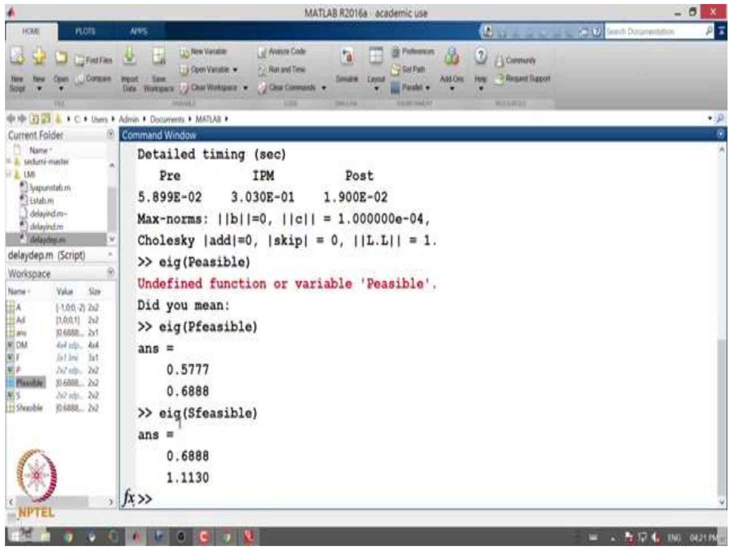

So, I get I can just check the eigenvalues now of Pfeasible, ok. So, P is positive similarly I can also check for the Sfeasible, ok. So, this means that this the system governed by this dynamics, with this particular A and A_d is stable independent of delays, ok. Now, let me do something else. This comment this out and go to some other example; where, ok; let me write down the problem properly.

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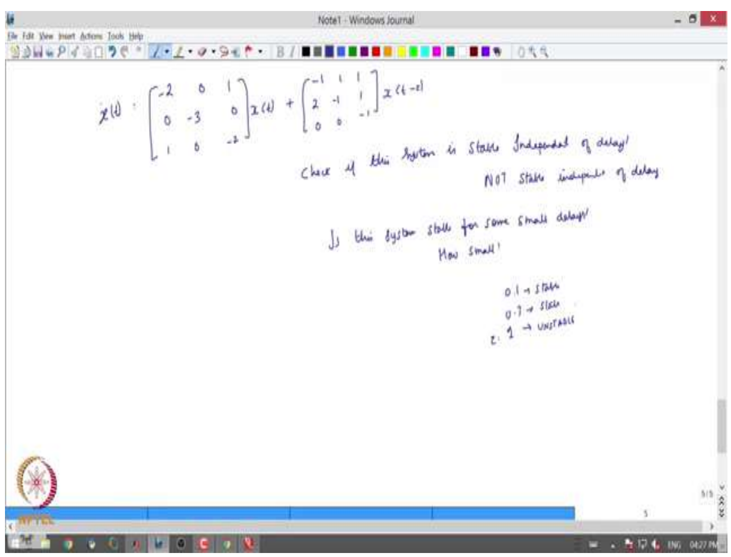

So, in this example, so, $\dot{x} = \begin{bmatrix} -2 & 0 & 1 \\ 0 & -3 & 0 \\ 1 & 0 & -2 \end{bmatrix}$x(t) + $\begin{bmatrix} -1 & 1 & 1 \\ 2 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$x(t-τ) and check if this system is stable, independent of delay, ok. So, let us see. So, I just plug in the A and A_d, so my matrices are now unknown matrices are of dimension 3 x 3 and I just run this code, ok. See that actually gives me an infeasible condition here, right, that this system is not independent of delay.

So, the answer is the system is not stable independent of delay, ok. Whereas, the this system actually was, this actually was stable independent of delay, ok. Now, the next question that you will ask is this stable say maybe for some small delays. And then we ask the question how small, ok. So, let us see if there are there are results which tell us that, right.

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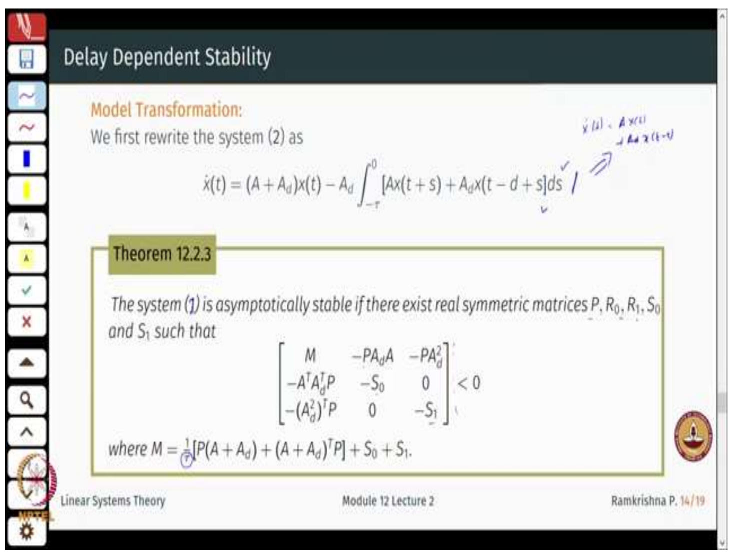

So, the delay dependent stability, ok. I do some model transformation, I will skip these steps. So, this system is; so I do a system transformation here such that when this system is stable the original system is also stable $\dot{x}$= Ax(t) + A_dx(t-τ). So, if this is stable this is also stable. The converse may not be true and therefore, this transformation adds a bit to the conservatism of the results, but ok, we will not at the moment worry about that.

So, this system is asymptotically stable, ok. So, should be 2 here. If there exists matrices of this form such that I have an equivalent matrix LMI of which looks like this, right. So, unknowns here are P, S_0 and S_1, where M has this term τ now. You see there is some something some dependency of the LMI on the delay term tau, ok. This can be you can just h solve for this $\dot{V}$, ok. I will not go to too much of the details, but, ok.

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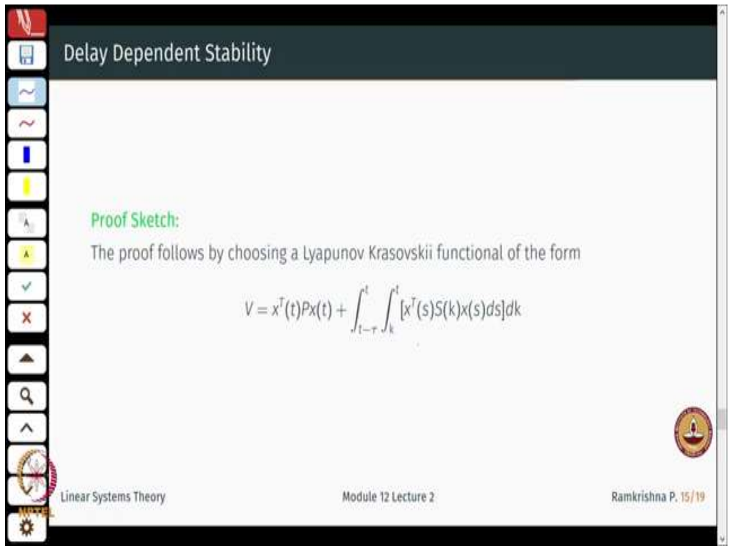

So, the this the delay term explicitly appears in the LMI in form of this M, whereas, in the previous case there was no delay term, right. So, all these terms are independent of delay, right.

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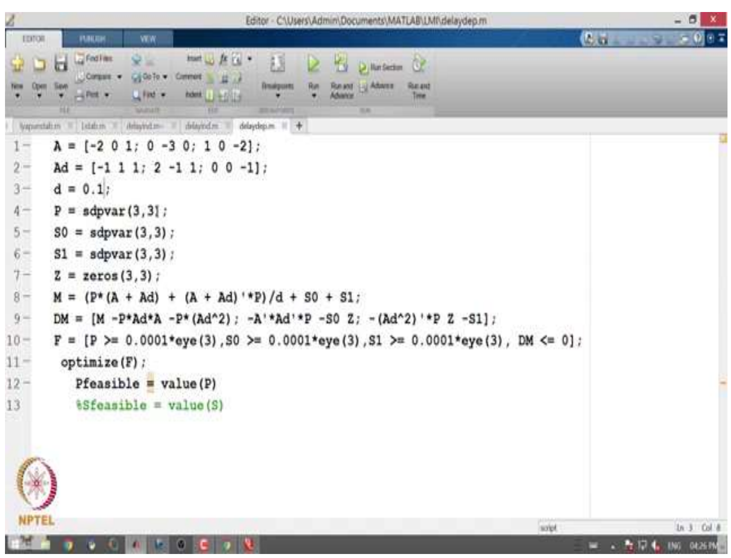

So, now let us check if this same system, this was stable independent of delay now let us just check what happens for say small delay. I just put a delay of say 0.1 and I just plug in the code, ok. So, let us see what happens, ok. So, this looks good, so there is no in feasibility condition.

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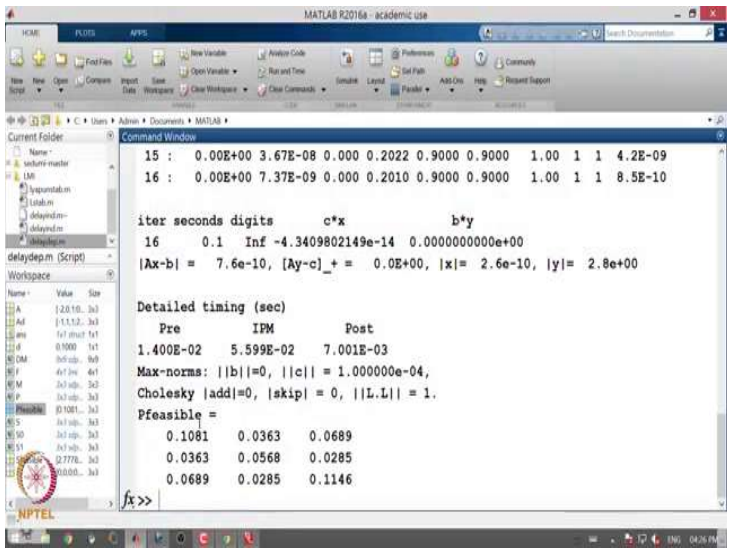

So, for 0.1 the system is stable. Let us say now I go to say 0.3, ok. That still seem to be good, right.

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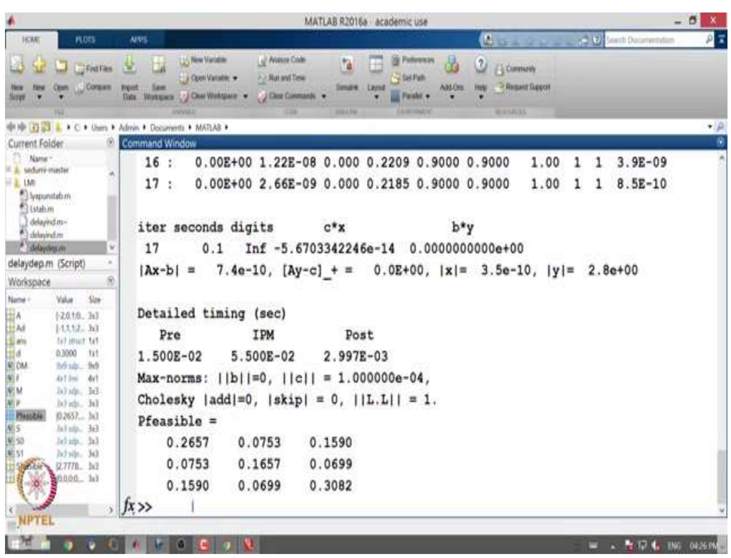

So, there is no in feasibility condition here. Now, let me say, say d, d is say 1.

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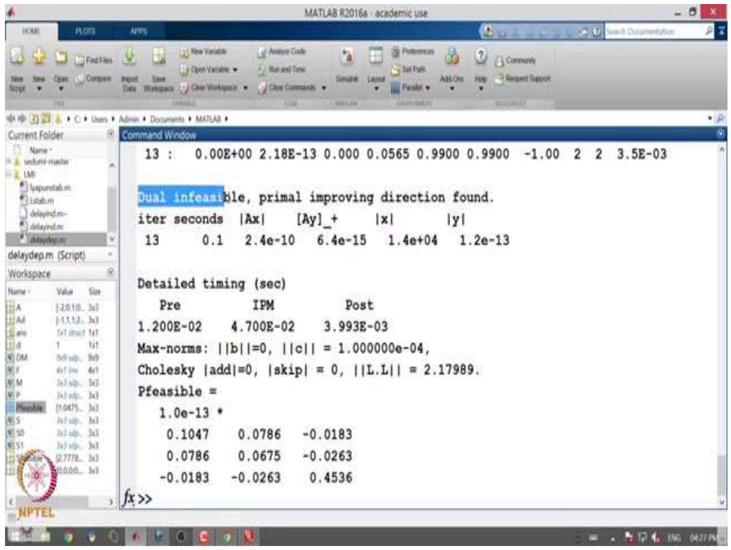

No, I some I have some feasibility condition, right. So, two things that I found out was for 0.1 it is stable, 0.3 is stable and for 1 or for τ equal to τ or d whatever, $\tau = 1$ the system turned unstable, ok. Then maybe you can just write a little loop to check for what is the maximum value of tau or d that the system is stable, so it will be some 0.4 or things like that.

So, we can just play around with the quantum of delay and check whether or not the system is stable. So, what is observation here? Even though there are some systems which are stable independent of delay some systems are stable well depending on for some small values small values of delay are delays are acceptable say in this case 0.4 seconds, but for larger values my system goes to the verge of instability. So, this is a little better result than then what we had previously, right.

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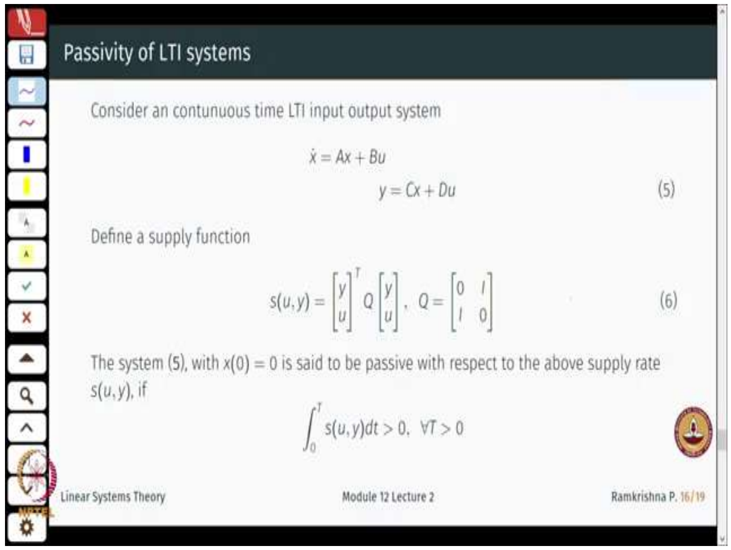

The last thing is about passivity, ok. So, this is something which is value as we kind of do very very often, but we do not really think that this is or we do not work within the realm of passivity, this is in the concept of passivity. So, I start with an input output system, some type of $\dot{x}$= Ax + Bu; y = Cx + Du. I just define a supply function which is quadratic and say Q is of this form, ok.

So, as this system is with say some 0 initial condition or maybe no not necessarily, but so for ease of representation. So, this system is said to be passive with respect to the supply rate if and only if something like this, some relation like this holds, ok. So, what does this mean from the physics point of view?. So, let us do the things in you know in a way that we can understand.

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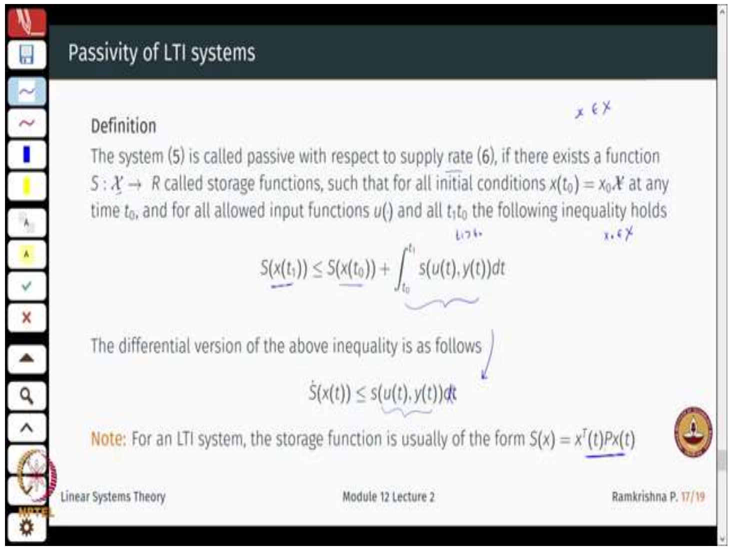

The system is called passive with respect to some supply rate. It is actually the physical supply say some power supply you can call supply rate. If there exists a function called the storage function where is the x is the state space, right say all x are coming from x, ok. So, some type o here. Of course, storage function, so that for all initial conditions x(t) with $x(t_0)$ =x_0, x_0 belonging to x, right at any time t_0 and for all allowable input functions, right and for all t_1, ok.

So, this should be t_1>t_0. The following inequality holds that, the value of the storage function at time t_1 is less than or equal to the value of the storage function at time t_0 plus some supply rate, ok. Now, I can just draw it a slightly better different better looking differential version of this one, right. So, that $\dot{S}$ dot is less than or equal to as less than or equal to the supply rate, ok. This will not be here anymore.

And for the LTI systems, this is simply quadratic this the storage function is just of the form x^TPx, very similar to the Lyapunov function and actually has very direct consequence also to the Lyapunov thing, ok.

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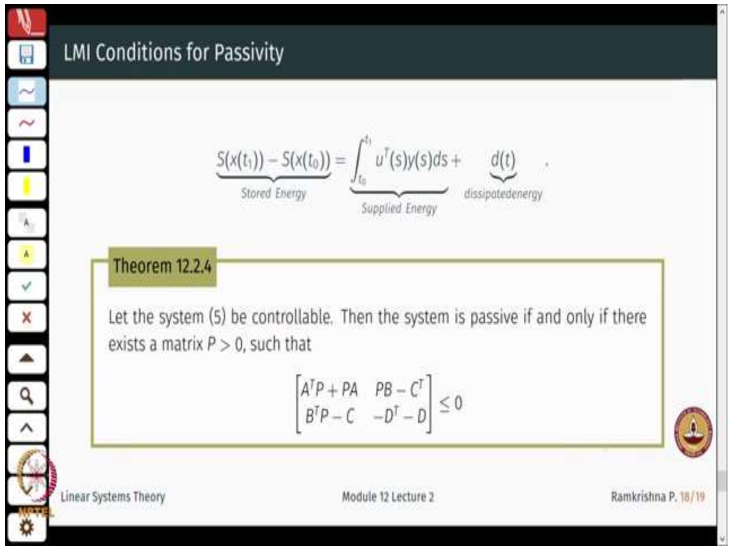

So, what is the interpretation of this? That S(x(t_1)), S($x(t_1)$) – S(x(t_0)) is the stored energy or the rate of change of energy, is equal to this supplied energy plus the dissipated energy, that is what any physical system would obey, right. It is just the simple law of conservation of energy, ok. So, I will do another very small example of a passive system, ok.

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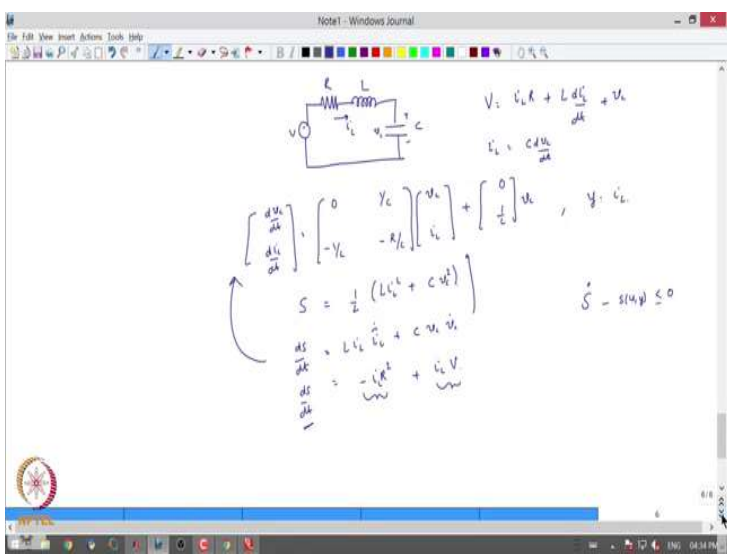

So, just take a R with some voltage source V, ok. So, this R, L; L and C are all passive elements we can extract only a finite amount of energy from these systems whereas, this voltage source is not a passive element it is an active element, ok. So, let us see what are we looking at here when I talk of the cvt, ok.

So, the dynamics of this would be V is ok, let me call this i_l, the voltage across this as v_c, ok. So, V = i_lR+$L\frac{di_l}{dt}$ + v_c. And what is this v_c? v_c is simply $C\frac{dv_c}{dt}$, ok. So, I can write an equivalent state space form of in the following. So, $\begin{bmatrix} \frac{dv_c}{dt} \\ \frac{di_l}{dt} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} \\ \frac{-1}{L} & \frac{-R}{L} \end{bmatrix} \begin{bmatrix} v_c \\ i_l \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} v_i$. Now, let us say that the output is just i_l, ok.

Now, what is the total energy of the circuit? The total energy is $\frac{1}{2}$(Li^2+ Cv^2). And if I take what is $\frac{dS}{dt}$ along this dynamics, ok. So, this will be L$i_l\dot{i_l}$ + C$v_c\dot{v_c}$ and I substitute for $\dot{i_L}$ and $\dot{v_c}$ from here and what I get is the following. So, this is -i^2R + $i_l v_i$, ok.

So, this is exactly the passivity relation, right or the arrange energy balance equation that we are here, that the stored energy is supplied energy plus the dissipated energy or the rate of increase of stored energy, so this is my power dissipated and this is just my input power, ok.

Now so, if the system is controllable then passivity is equivalent to solving the following LMI, ok. So, look at this little carefully. So, how do we derive this condition? So, just starting from here. So, what I am looking at is $\dot{S}$ – S(u, y) ≤ 0, ok. So, let us start from this.

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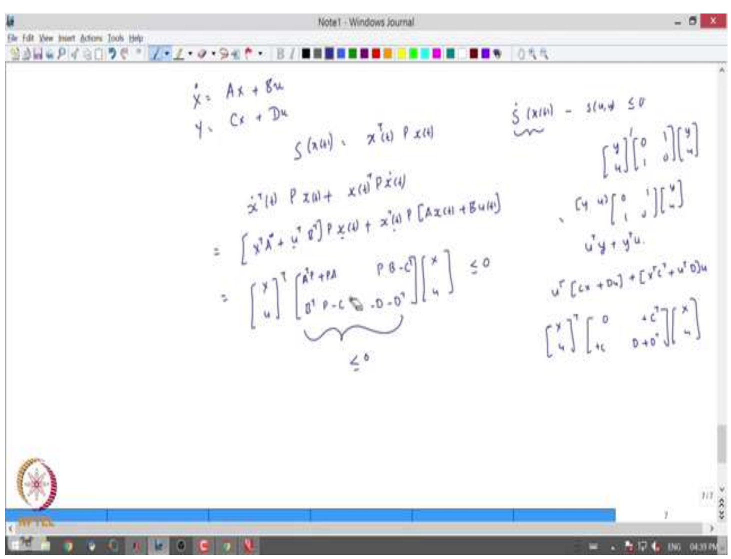

So, I have $\dot{x}$= Ax + Bu, y = Cx + Du, ok. My storage function S(x(t)) is say x^TPx, ok. Let me leave out the half for the moment, ok. Now, what I should find out is how does $\dot{S}(x(t))$-S(u,y) ≤ 0, ok. So, first let us look at this term. $\dot{S}$ is, so I get $\dot{x}^T$Px + xP$\dot{x}$(t). This is, so this is $(x^T A^T + u^T B^T)$Px(t) + $x^T(t)$P(Ax(t)+Bu(t)) So, this I can a club it in the following way. So, I have x u transpose x u.

So, what do I have here? I have A^TP + PA and the term involving x^T and u would be P B and the term involving u^T and x is B^TP and a 0 here, ok. And what is my supply rate? So, supply rate the way I define is in the following way. This is [y u]$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} y \\ u \end{bmatrix}$, ok. So, let us do this.

So, y what is y? y = Cx, ok. So, let us write it down this first. So, this will expand this is transpose. So, [y u]$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} y \\ u \end{bmatrix}$, ok. Let us just make it a little scalar system, right, so that notations would be easier. So, I have this u^Ty plus in general this will be y^Tu, ok. Now, u^Ty is or Cx + Du, again y^T this will be x^T transpose, transpose, transpose, $x^T C^T$ + u^TD u, ok.

So, I can write this. So, just plug this I, so, I will have again terms relating to x u with the transpose and x u, ok. So, I have terms relating to u^TC. So, I will have a -C-C^T there will be a 0 here, here will be D +D^T, ok. So, I just subtract it here. So, what I get is now the following that relation which looks something like this, ok.

(Refer Time: 51:38) be a C. PB - C, -C^T here this will be D, because of this minus here and this must be less than or equal to 0 and therefore, this is equivalent to saying that this is less than or equal to 0. That is essentially what I have here, ok. So, some things x^TC will have C transpose and a C here, ok. So, this will be C transpose this will be C, ok.

So, this is how I derived that particular LMI. So, to check whether or not a system is passive I just plug in this inequality, right. And of course, there is a bunch of literature on what can we do with passivity, how the passivity relate to stability and so on and how also we can do things related to things called passivity based control and a lot of it.

Again, the idea is not to really go deep into concepts of delay or passivity or even D R stabilization and other stuff, but it is just to give you some idea of what you can do with LMI, why do we need a LMIs, starting from the Lyapunov equation to the stabilizability conditions and then there is a bunch of stuff that you could do and we will just play around with some of these this equations or these systems, ok.

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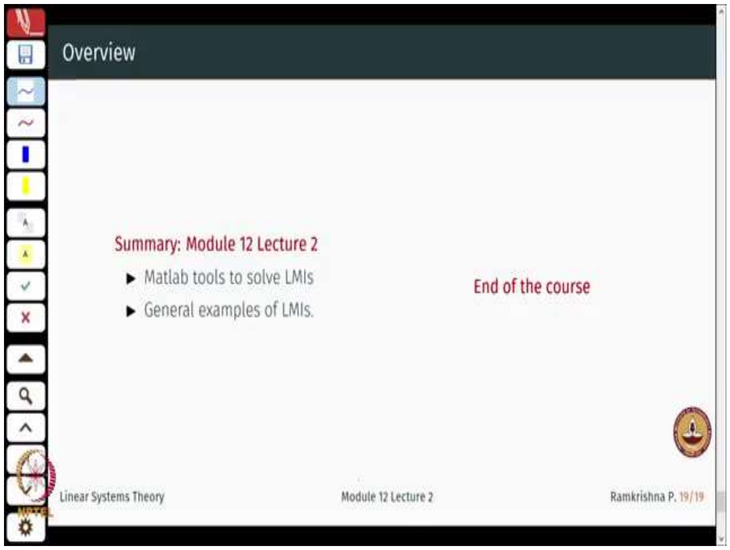

So, that kind of concludes what we what I had to say in this course of 12 weeks. I hope the course was useful to you. That was just to give you a little introduction to some higher or advanced level courses in control theory. This might give you a way to maybe pursue courses in optimal control or non-linear control and a bunch of other things that will open up for you. I hope you enjoyed the course.

Thanks a lot.