Linear Systems Theory
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Module - 12
Lecture – 1
Linear Matrix Inequalities

Hi everyone, welcome to this last week's lectures on the course on Linear Systems Theory. So, far we have done a bunch of things starting from basics of linear algebra to look at solutions to systems of equations both time invariant time varying discrete time continuous times and so on. We had a bunch of tools to analyze the stability of linear time invariant systems or also and also linear time varying systems.

One of the most powerful tool we used was that of Lyapunov stability. We did a bunch of things for analysis of controllability, what to do if the system is only partially controllable or only few states are controllable, what if all the states are not measurable then we had the notion of observability. Then we had design problems where we had problems relating to designing controllers to designing observers simultaneous design of controllers and observers. We also had looked upon of reduce order observers and towards the end also looked at some problems relating to optimal control when we have constraints on the control energy or the control input we also have constraints on the time and so on.

So, what we will do now is to just revisit those things and look at things a little more from a computational point of view.

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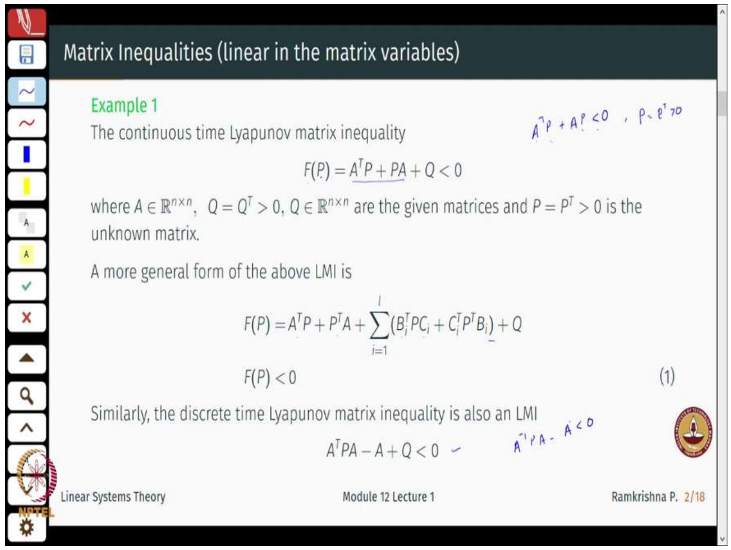

So, let us start with the one of the basic equations that we had looked upon while we were interested in instability of systems. So, that is the traditional Lyapunov equation A^TP + PA. So, here A is given to me, P is the unknown and I can say if you if you look at the last condition for Lyapunov stability we had a condition where something like this was equal to asymptotic and also exponential stability.

A it was given to me P was the unknown and if there exists a solution P which is symmetric and positive definite then the system was asymptotically stable and if you look at it closely this equation is linear in P. So, we are solving for the matrix P or the elements of the matrix P and there is an inequality and that is somehow motivates the name linear matrix inequality ok.

So, in general I can write that. So, I am solving for this equation for P in general my P could be of the form A^TP + PA again some B's and C's of this form and some Q where my objective is to find a P such that this entire expression f(P) < 0 ok. Similarly, I have the discrete time Lyapunov matrix or the Lyapunov matrix inequality this was also a linear matrix inequality. You can check that this is also like this. So, this inequality A^TPA – P< 0 was also linear in P ok.

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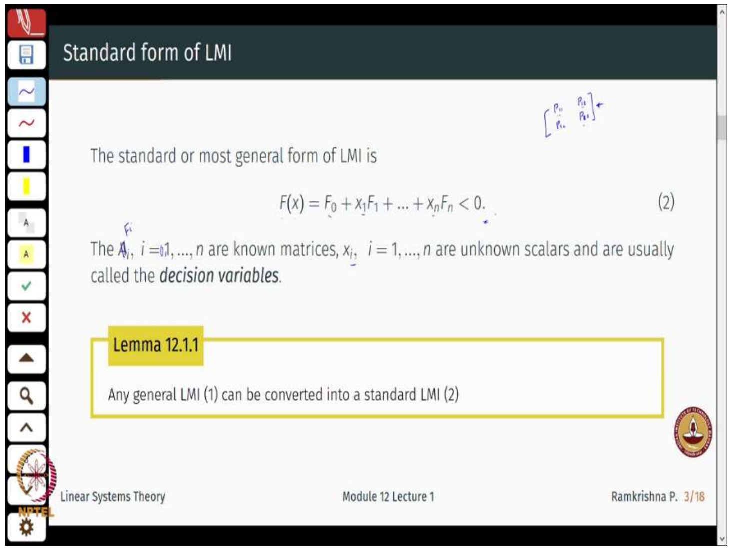

So, the in general so, if I look at say a two dimensional system, so, I was essentially solving for these three unknowns p_{11}, p_{12} and p_{22} when I was looking at a solution to this equation right. So, in general I can write LMI in its most standard form in terms of these unknowns p_{11}, p_{12} and p_{22}. So, here I call this xi s are my unknowns as so, in general form I can write F(x) = F_0 + x_1F_1 +….x_nF_n < 0 where ok.

So, this should be F_i, well all this, F_i's, i going from 0 till n are known matrices this x_i's are unknown scalars like all this P s in this in this entries in this matrix are scalars and they are usually referred to as the decision variables. And there is a little result which I will not prove in general, but I will give you a little example of that any general LMI can be converted. So, an LMI of this form here can be converted into an LMI in the standard form ok.

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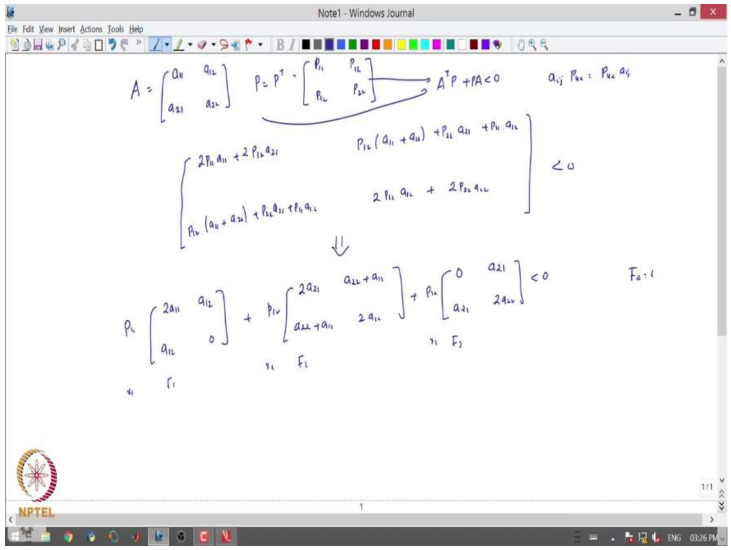

So, let us start with a matrix A which looks like this: $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, P is symmetric and has entries of the form $\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$ ok. So, I am just looking at how does this equality look like it or this inequality A^TP + PA < 0 I just substitute into things. So, , just plug in the values of A and P here and what I end up is something like this I have twice $p_{12}a_{11}$ + $2p_{12}a_{21}$ and then because these are scalars I can always write a ij P with k l would be equal to this 1 ok.

So, and therefore, we will have this 2 s here. The second entry here would be $p_{12}(a_{11} + a_{12})$+ $p_{22}a_{21}$ + $p_{11}a_{12}$ then you have $p_{12}(a_{11} + a_{22})$ + $p_{22}a_{21}$ + $p_{11}a_{12}$ this is $2p_{12}a_{12} + 2p_{22}a_{22}$ is less than 0 can also be currently written in this form. So, I have $p_{11}\begin{bmatrix} 2a_{11} & a_{12} \\ a_{12} & 0 \end{bmatrix}$ + $p_{12}\begin{bmatrix} 2a_{21} & a_{22} + a_{11} \\ a_{22} + a_{11} & 2a_{12} \end{bmatrix}$ + $p_{22}\begin{bmatrix} 0 & a_{21} \\ a_{21} & 2a_{22} \end{bmatrix}$< 0 ok.

So, this I converted this LMI into what I called as a standard form of the in this unknown race x_1F_1, x_2F_2 and, x_3F_3. So, this is x_1F_1, x_2F_2, x_3F_3 and then the F_0 = 0 ok. So, I will not do a general proof, but the general proof will follow some similar arguments. So, I am just keeping things a little simple for the moment ok.

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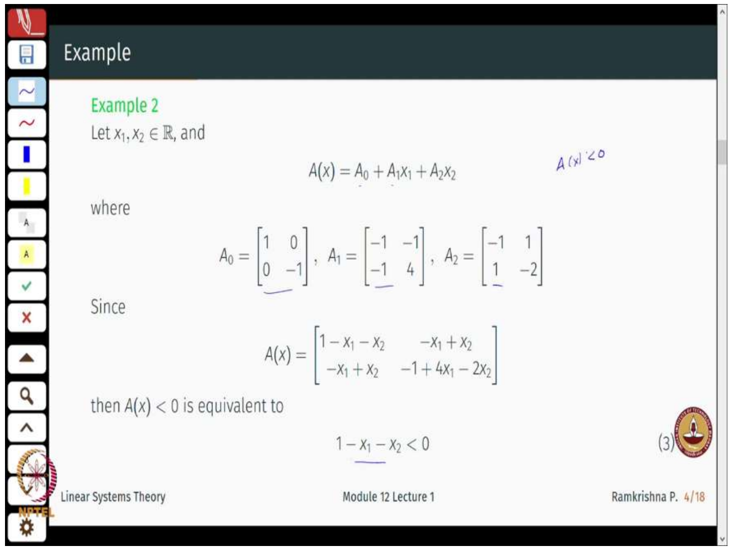

So, what does it mean geometrically right when I write down an inequality of this form and what am I actually searching for? So, let us do a little example and say F and LMI of this form $A_0 + A_1x_1$ with A_0 of this form A_1 and A_2 and searching for A with for A x being less than or less than 0 would be equal to these 2 conditions first 1 - x_1-x_2 < 0 and second I am looking at the determinant right.

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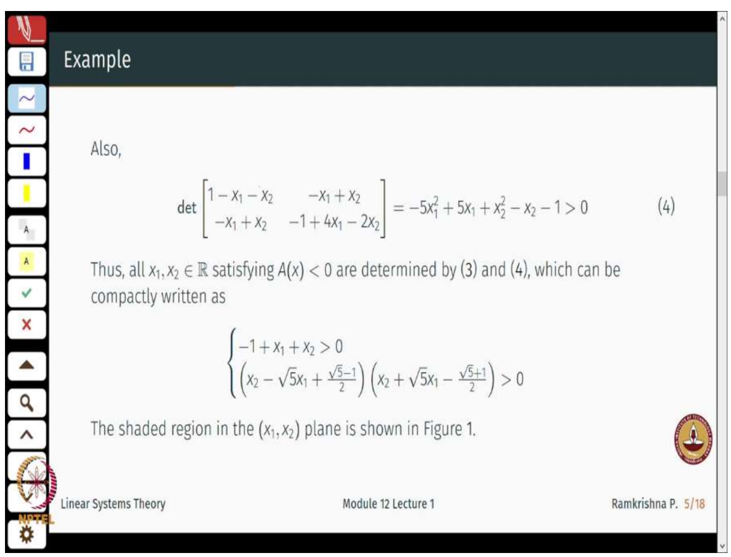

So, I just compute the determinant and it will be something like this ok. So, I am just looking at now the x_1 and x_2 which satisfies this relation plus this relation which can together be written as a as a relation of this form.

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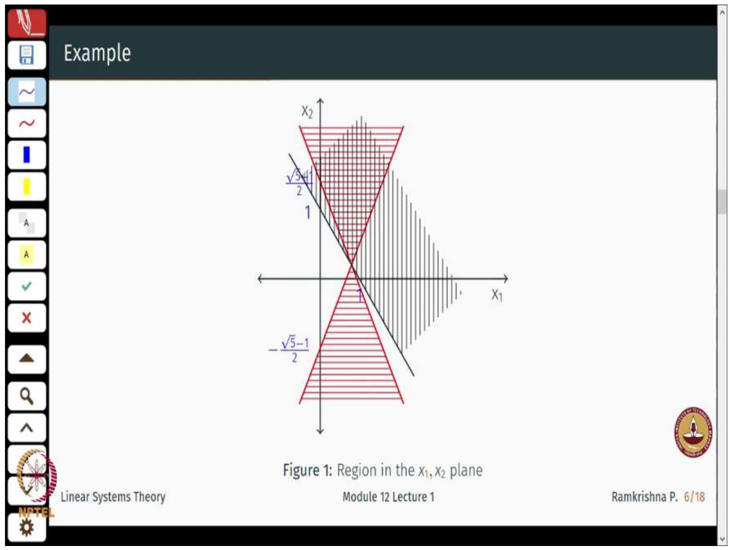

So, if I plot these two regions individually this could be. So, this is this double shaded region here would give me the feasible set of all points x_1 and x_2 that satisfy this inequality of a x being less than 0 ok. So, this is easy to plot and check and therefore, ok. So, will not spend much time on the solutions of this, I just leave this to you because they are fairly looking equations here. So, but what we will what the aim of this is to show that just to give an illustration of what I am actually looking for.

So, I am just looking for solutions of this which just look as this as a region all given by this one ok.

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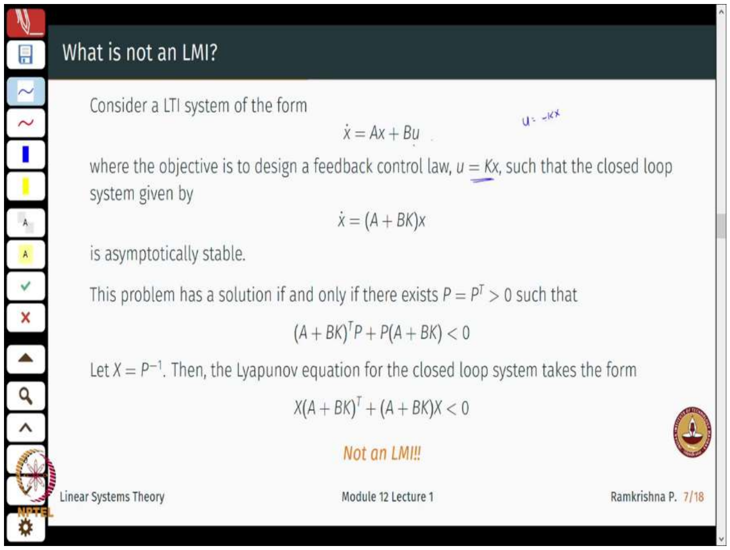

So, what if things are not LMIs what is not an LMI right? So, I know I gave you a general form of LMI what is not an LMI. So, we will start with the standard LTI system with a control input where the objective is to design a feedback control or u = Kx ok. I am just doing a little abuse of notation here whereas, traditionally we you would use u = -Kx, but just for some ease of notation I am just using u = Kx, but now nothing really changes in this right.

So, not that all right to talk to you earlier was different than now. I am just omitting that minus for some convenience of notation here ok. So, my objective here is to design a feedback control law such as the closed loop system given by $\dot{x}$= (A + BK)x; (A +Bk) x is asymptotically stable ok. What does Lyapunov theory tell me that this problem has a solution if and only if there exists a positive definite P and of course k such that this entire expression or this inequality holds that; A plus Bk $(A + BK)^T P$ + P$(A + BK)$< 0 ok. So, if you see that I have two unknowns here P and k.

So, this is not an LMI because I have two unknowns here and they are not linear in P and k because this is this cross term here ok. So, what do I do with equations like this ok? First step that I will do is let me just take introduce another variable X as P^{-1} and just substituting here and doing some manipulations. I get an expression like this is also not an LMI ok. So, what do we do under in situations like this ok?

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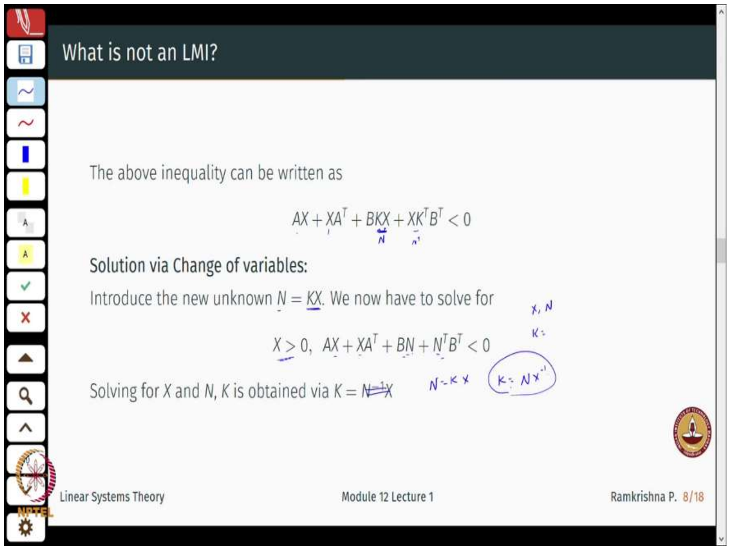

So, I can expand that this inequality to write as the following AX + XA^T+ BKX +X$K^T B^T$. So, the unknowns again are my X and the unknown is K and because of this cross terms the linearity is lost ok. I can do a little trick here right. So, I will just do a little change of variables and I introduce a new unknown X simply as KX ok.

So, once I do this K times X becomes n this becomes N^T and I have a solution. So, what I have to solve for now? I have to find an X which is greater than 0 because P is greater than 0 P^{-1} will also be greater than 0 and therefore, X will also be greater than 0 such that AX +XA^T+ B N + $N^T B^T$< 0. Now I can so, this is an LMI right. So, this is linear in X and N.

So, if I find X from this if I find N for this to me to find K is easy right from this. So, K will simply be ok. So, K will simply be this is type 2 here since N =KX, K will simply be NX^{-1} ok. So, that is that is a nice trick here right, we just introduce a new variable N as KX I solve for N I solve for X and I can easily realize what my what my K is ok.

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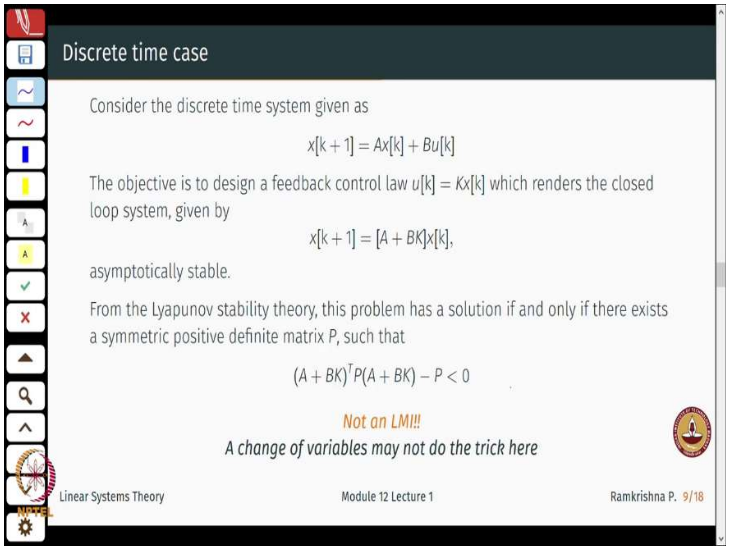

So, now let us see what happens in the discrete time case. So, if I go to the discrete time case I have a system that x(k+1) = Ax(k) + Bu(k) where the objective again is to design a control law u(k) = Kx(k) such that the closed loop system is asymptotically stable again I just dropped the minus here for some obvious reasons again without loss of any generality ok.

Now from the Lyapunov stability theory what I know that this problem has a solution if and only if there x is a symmetric and positive definite P such that this inequality holds and again this is not again an LMI because I have this cross terms here ok. So, not only that if I just and it should be easy to verify that I if I just do a change of variables that trick may not work here directly. So, I may have to look at look at some other tools that will help me solve problems of this kind ok.

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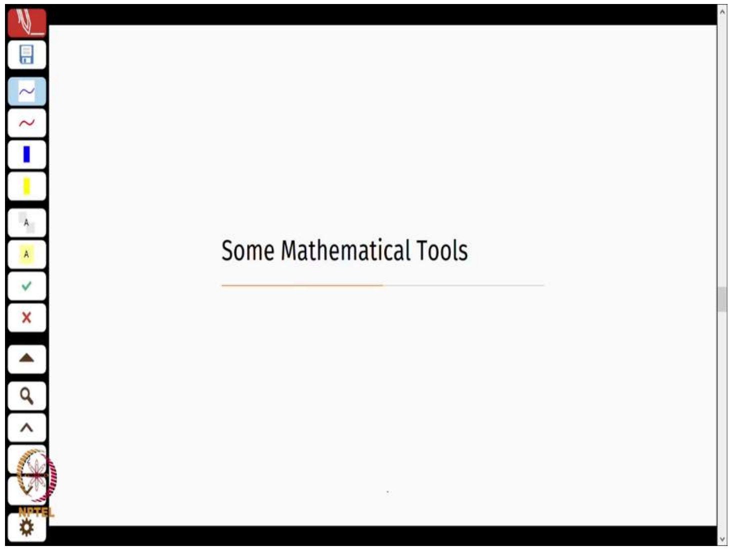

So, let us take a little break from LMIs let us go back to matrix theory and see if there are some tools that matrix theory teaches us in such that we can arrive at solutions to this or at least formulate them as LMIs ok.

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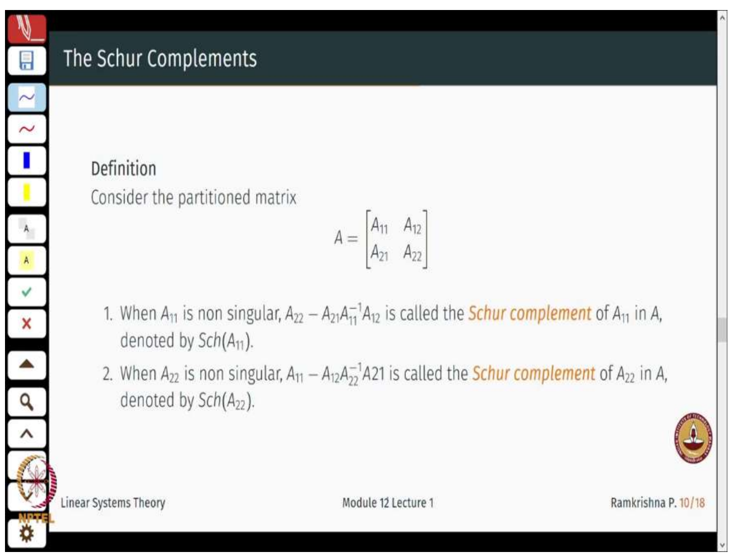

So, one of the very powerful result in matrix theory is the Schur complement. So, let us say I start with A matrix A and I partition it in this way $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ and when assume that A_{11} is non singular then A_{22} - A_{21} $A_{11}^{-1}A_{12}$.

So, this entire expression is called the Schur complement of A_{11} in A and denoted by this Schur notation. Similarly when A_{22} is non singular then A_{11} -A_{12} $A_{22}^{-1}A_{21}$ (Refer Time: 15:10) A 2 1 is called the Schur complement of A_{22} in A and is denoted in the compact way as the Schur of A_{22} ok.

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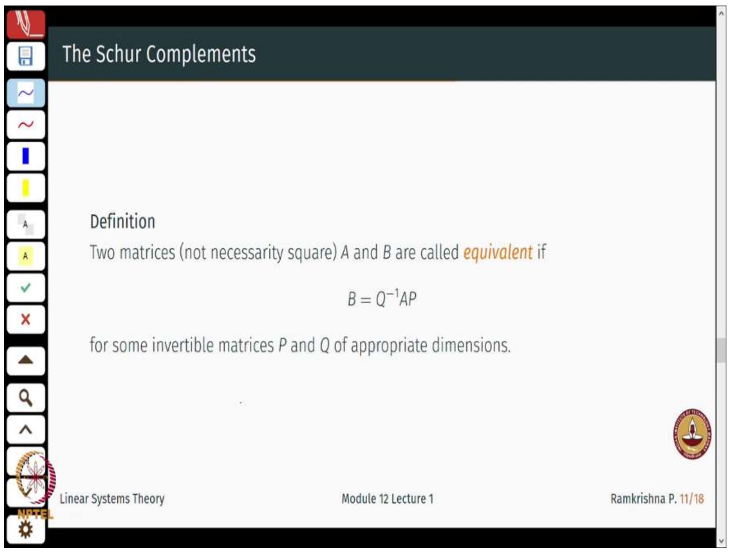

So, what does this do to us ok? So, some deserves before we do this or some definitions before we go further 2 matrices I am not really dealing with square matrices. So, A and B are called equivalent if I can write B as some Q^{-1}AP for some inevertible matrices P and Q of appropriate dimensions. Slightly different than what I do in the similarity transformations because they are the underlying assumption is that the matrix is A square matrix.

So, P^{-1}AP will give me an $\bar{A}$ and so, on. Here its A little more general way of looking at things ok.

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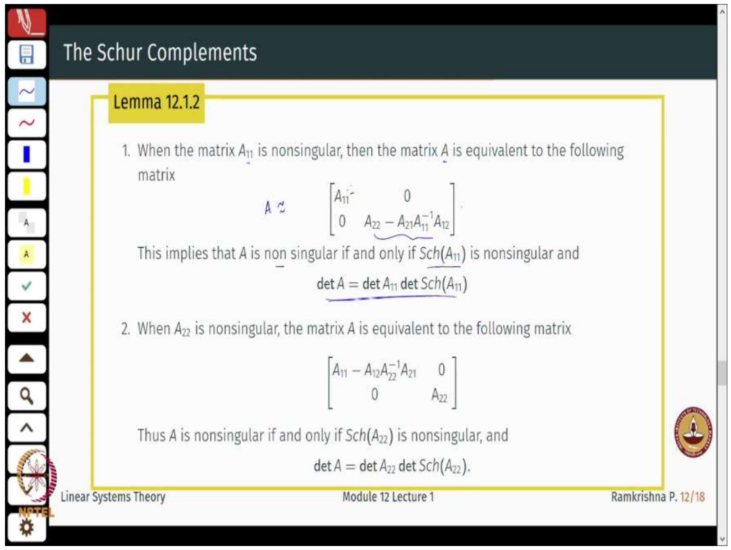

So, this is good ok. So, what happens in this case? So, when A_{11} is non-singular then the matrix A is equivalent to the following matrix and this implies that A is non-singular if and only if the Schur of A_{11} is non singular and the determinant of A is just the determinant of this guy plus the determinant of this guy ok. So, I have A, I say that this is this A is in a way equivalent to this matrix ok.

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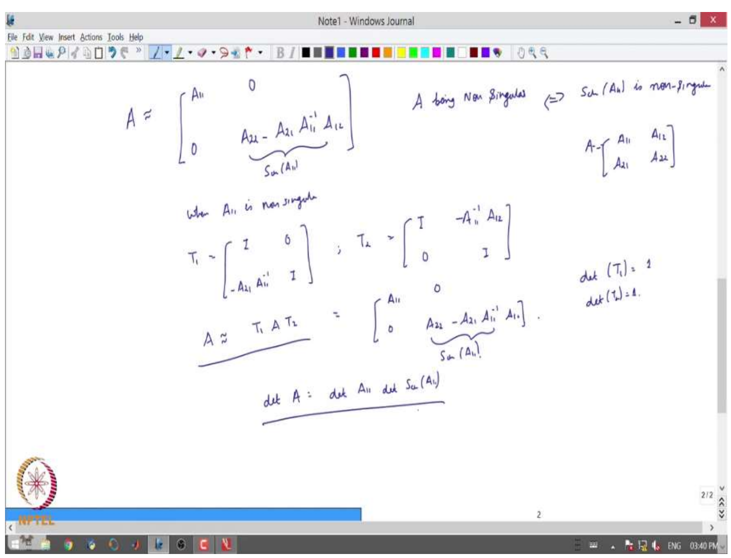

So, I have $\begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} - A_{21} A_{11}^{-1} A_{12} \end{bmatrix}$ this is also called as the Schur of A_{11} and what the result says is that A being non singular it is equivalent to saying. So, this is possible if and only if the Schur of A_{11} is non singular ok. Now, let us do it. So, when A_{11} is non singular now that is the condition right, so, this is necessary for this Schur to exist. When A_{11} is non singular let us define these two matrices T_1.

So, its $\begin{bmatrix} I & 0 \\ -A_{21} A_{11}^{-1} & I \end{bmatrix}$ similarly T_2 as the $\begin{bmatrix} I & -A_{11}^{-1} A_{12} \\ 0 & I \end{bmatrix}$ here and fine and say that A can be obtained as T_1 A right where A is no partition like this right. So, that was the condition A is partitioned as $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, $T_1 \mathrm{A} T_2$. So, I just substitute for all of this and I just get the following.

A_{11} 0 I will I will skip the steps A_{22} - $A_{21}A_{11}^{-1}$ A_{12} this is the Sch(A_{11}). So, what does the results say or what does the theorem statements say that A is non singular if and only if the Sch(A_{11}) is non singular and additionally determinant of A is just given by the $|A_{11}|$.|Sch(A_{11})| ok.

So, the second so, this shows the equivalence of 11 sorry of A being equivalent to this matrix which means that A is singular if and only if Sch(A_{11}). So, A sorry A is non singular if and only if the Sch(A_{11}) is non singular right. So, A_{11} being non singular is anyways A necessary condition right ok. And now look at what do I do with the determinants ok. So, what we had to prove was A determinate of A was the determinant of A 11 times the determinant of the Schur of A 11 ok.

So, if I look at this expression here what do I have is easy to check that the determinant of T_1 is 1 and also the determinant of T_2 is 1 ok, therefore, this is this is now kind of trivial to check good.

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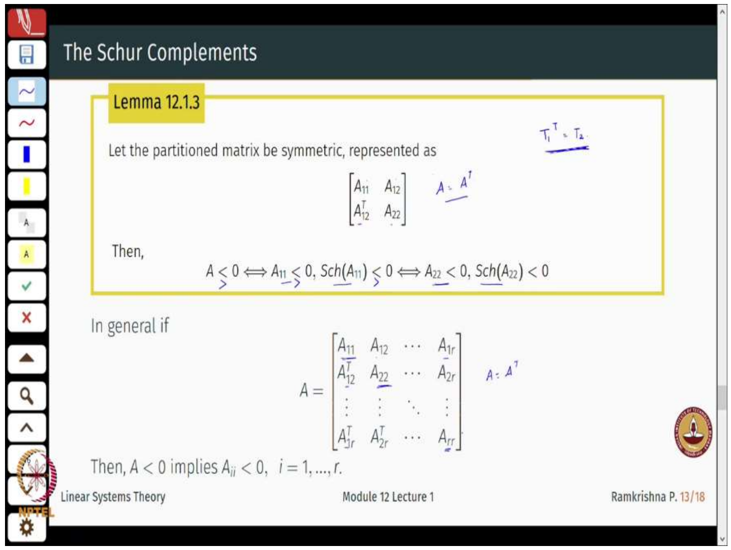

Now what happens when the when the matrices are symmetric? So, we will, so, through the course we were interested in symmetric Ps and so on ok. So, what happens when the matrices is symmetric? So, let me say that the matrix is partitioned in the following way of $\begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix}$ here which is that that A = A^T ok. Then the result says that if A < 0 this implies that A_{11}< 0 and the Sch(A_{11})< 0 and similarly A_{22} and this also implies that A_{22}< 0 and the Sch(A_{22}) < 0 ok. I am I can write exactly the same results by replacing this with this and so on and similarly here ok.

So, I think the proof should be should be a kind of easy because when the matrix A is symmetric then the previous transformations are such that $T_1^T = T_2$ and once we establish this then the results are like easy to check ok. And in general now this is also A straightforward consequence is its the following that I skip the proofs, but its easy its important to know these results that if A is as blocks which are of this way. So, A_{11}, A_{12}, A_{12}^T …A_{1r}^T …and so on such that again A = A^T then A < 0 implies that each of these entries are 0, A_{11} A_{22} all the way till A_{rr} are less than 0 ok.

So, I will do maybe a very short proof of the sentence skip the longer one.

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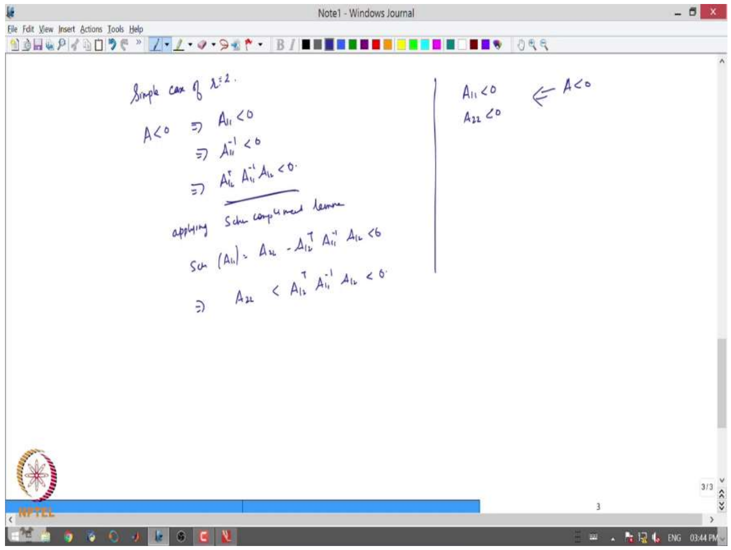

So, let us take a simple case of r = 2 ok. So, well then A < 0 its obvious to say that A_{11} < 0 which also means that A_{11}^{-1} < 0 and $A_{12}^{T}A_{11}^{-1}A_{12}$< 0 ok. So, applying Schur compliment applying Schur complement lemma which was essentially this one at the what we had here where A < 0 implied A_{11} < 0 and the Sch(A_{11}) < 0, similarly A_{22}< 0 and this ok.

The Sch(A_{11}) which is A_{22}- $A_{12}^{T}A_{11}^{-1}A_{12}$< 0 this implies that A_{22} < $A_{12}^{T}A_{11}^{-1}A_{12}$ and what do I know from the from the Schur complement lemma are. So, I just then I just have for this one. So, from this condition I just also have that A_{22}< 0 right. So, these two are true; A_{11}< 0 and $A_{22} < 0$ right. So, A less than 0 implies these two conditions ok.

For r its just maybe there will be A couple of more steps to for greater than 2 will be A couple of more steps, but I will just skip those things, but this is A good for A little understanding of what the result actually is trying to say right.

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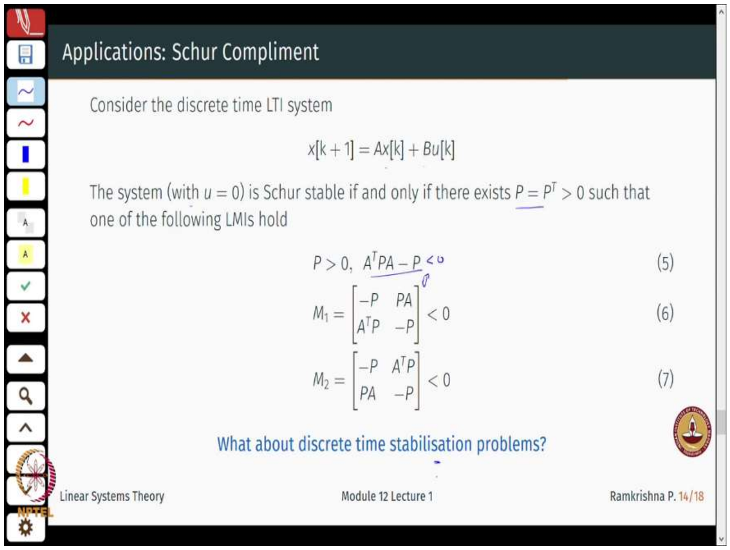

So, what do we do this with this Schur complement ok? So, I start with again the discrete time system x(k+1) = Ax(k) + Bu(k). System with u = 0 Schur stable if and only if there exists again P = P^T > 0 such that any one of the following LMIs all right. So, P > 0, A^TP + PA this should be less than 0 and these two inequalities ok.

So, this is again its very straightforward consequence of applying Schur compliment and then I will skip the steps, but we will go to the more interesting ones. Interesting ones where what about in the case of discrete time stabilization problems where we said I cannot use a change of variables, I can also to it was not an LMI in the continuous time case we could use a change of variables to arrive at a nice looking LMI which was solvable whereas, in the discrete time case that was not possible. So, what does what happens to the case of discrete time systems here ok?

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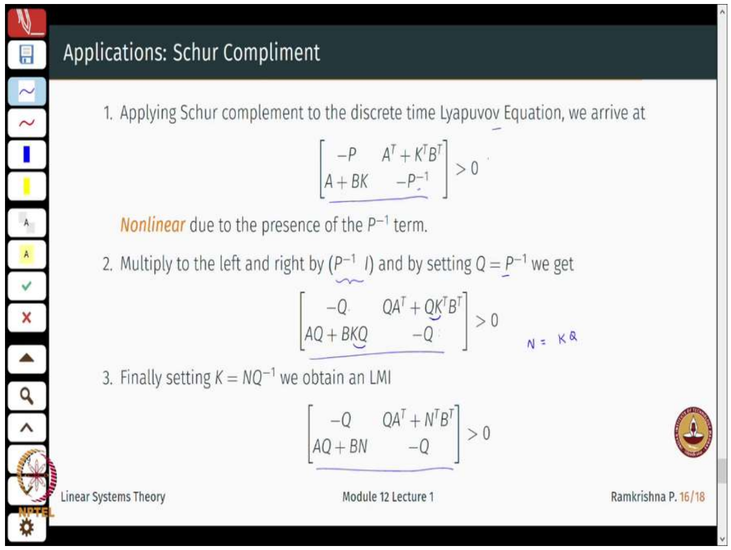

So, this is not an LMI and I will say what does the Schur complement does to me ok. So, first is applying the Schur compliment to the discrete time the Lyapunov equation we first arrived at this expression and then there is still non-linear because I have a P^{-1} ok. Now next what I do is I add I multiply to the left and right by this matrix P^{-1} I and set Q as P^{-1}, I obtained something like this I have qs here I have QA^T and some this Q is also Q and k both are unknowns I have some cross terms here ok.

But, I can do some other trick I can just do I can set another I can call N as K Q and then rewrite all this terms here like this ok. Now this is this is can be easily verified to be to be an LMI. So, by making use of the Schur compliment I could convert the discrete time stabilization problem with unknowns k and P to a nice looking LMI again I can just solve solving for Q and, I can find out what is also the k in this case ok.

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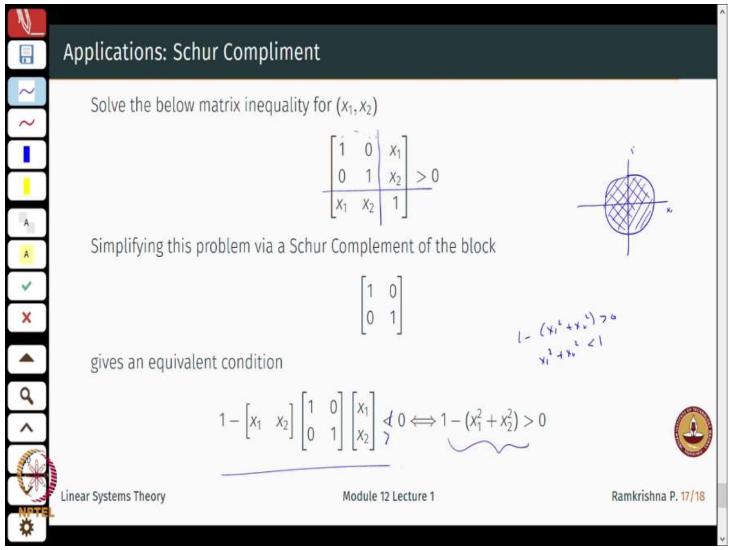

So, one more example it is not necessarily for stabilization problems, we could also have we will some complex looking LMIs which can be simplified via Schur compliment.

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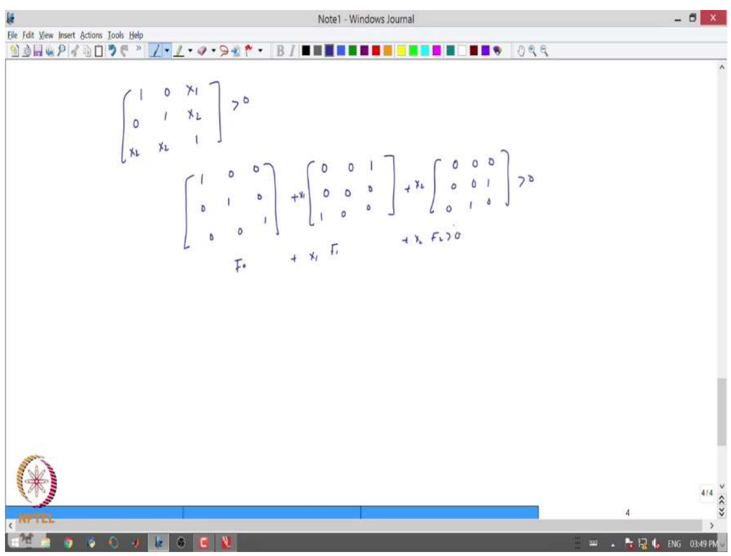

So, if I give you this thing to solve ok, so, I have this one right. So, $\begin{bmatrix} 1 & 0 & x_1 \\ 0 & 1 & x_2 \\ x_2 & x_2 & 1 \end{bmatrix}$ and I have to solve this for being greater than 0 I can write it in the standard LMI of the form.

So, I have $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ plus and I write it in terms of x_1. So, it will be $x_1 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + x_2$

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ > 0. So, its left F naught plus x_1F_1+x_2F_2.. > 0 and then we are go about computing the feasible set geometrically or even analytically.

But what we see here is if I just apply the Schur compliment to this block here I just partition a matrix this way and if I apply; so, this is invertible right. So, if I apply the Schur complement to this block here what I get is something very nice looking here right. So, an equivalent condition for this LMI is something like this ok. So, this is equivalently written in the following form.

So, I have a solution 1 -x_1^2 - + x_2^2 > 0 it should be a greater than 0 here, And therefore, we have a condition that x_1^2+ x_2^2 < 1 and then if I just plot that region I just get that the feasible region of this LMI is all this circle of all the points within the within the unit circle in the x_1 and x_2 plane ok.

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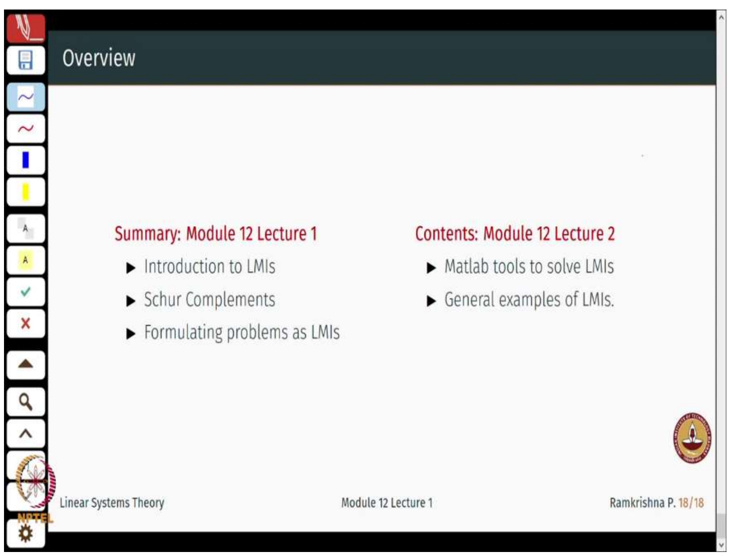

So, that brings us to the conclusion of part 1 of this lecture where we had introduced LMIs or the Lyapunov solution problems as some LMI formulation including the stabilization of continuous and discrete time systems. We saw how via Schur compliments we can translate difficult looking LMIs or even non-linear matrix inequalities in LMIs. So, next time we will just I will just teach you some MATLAB tools to solve this LMIs and in general some examples of formulation of LMIs of systems that we are more likely to encounter in real life.

So, that just coming up in the next two lectures thanks for listening.