Linear Systems Theory
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Module - 11
Lecture - 03
Tutorial for Module 11

Hi everyone. So, today we will spend some time solving some problems related to LQR and also look at problems from different methods that we had devised or derived earlier and see what they mean in the context of our solution right. So, I will just solve some 3 or 4 problems and I just suppose to be a bunch of them online just for your own benefit. So, let us start with the first one that could be that ok. Let us start with what happens in case of a scalar system.

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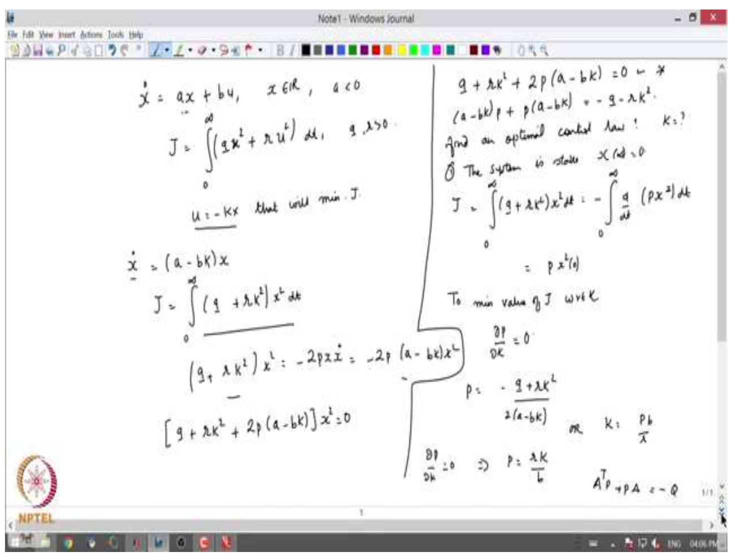

So, I have $\dot{x}$ = ax + bu, x is in R and so is a b and u. Let me assume that the system matrix is stable by itself. And I have a performance index of the form $\int_0^\infty (qx^2 + ru^2)dt$. And as usual q and r are numbers which are greater than 0 ok. So, we need to find an optimal control law that will minimize this j, next some control law of the form u = -kx, that will minimize the j ok.

Now let us do a first step by step and then relate to the general theory that we have derived earlier. So, with the application of this control law $\dot{x} = (a - b k)x$ ok. and then if I substitute this u = -kx into my cost function this will be a $\int_0^\infty (q + rk^2)x^2 dt$ ok.

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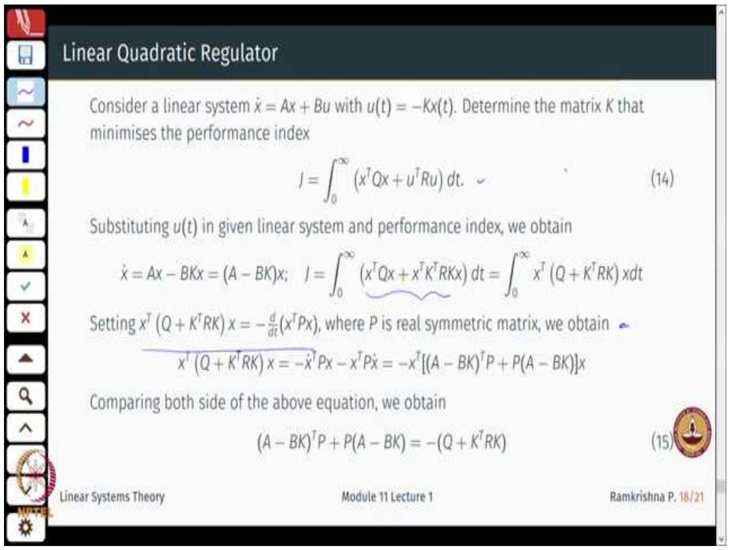

Now, let us just said check what we did here right. So, we had this cos function which was more general and what did we do is set something like this right. Set the guy inside the integral to $-\frac{d}{dt}(x^T \text{P x})$ and then we had a couple of justifications and why the why we were doing this kind of a choice coming also from the from the invariance the Hamiltonians and so on right.

So, we will stick to this and then see what this means in our context ok. So, I have so, once I do this, so, what does it mean that I am doing a (q + rk^2) x^2 is -2px$\dot{x}$. So, this will be -2px$\dot{x}$ is a - b k and the 2 x x will be x^2 ok.

Now, I can just rearrange all these 2 terms and I can write this in the following way; (q + r k^2+ 2p(a - b k)) x^2 = 0 ok. Now for this to hold for all non zero x what we require is the following; this (q + r k^2 +2p(a - b k)) = 0, right. This is just the scalar form of the of the Riccati equation that we derived this one.

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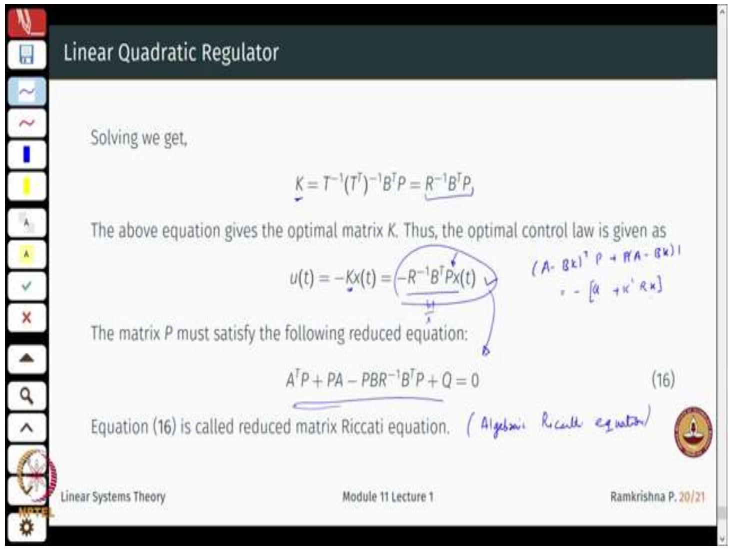

Let us say a^Tp + pa will be twice p and so on right ok. So, what is the objective now? So, one more step is if I look at this equation I can also write the equation equivalently as a Lyapunov equation for a closed loop system right.

So, the when we say closed loop system stable? So, if we just look at the standard Lyapunov equation for stability that will be that if there is any q which is symmetric and positive definite, for which I can find a symmetric positive definite solution to this Lyapunov equation then my system is asymptotically stable ok. So, that is the equivalent of that here in this optimality case would be something like this and in the scalar case I can just write it in the following way; $(a - bk)$p + p(a – bk)= -q - rk^2 ok. Now what do we need to do is find an optimal control law right.

So, I am not just interested in finding a stabilizing control the system is stable by itself I am also not interested in this placing the eigenvalues at certain location it could just be solving a linear equation for K. So, I am finding an optimal control law here that will minimize my cost function ok. So, let us find. So, essentially I am finding what is the value of K ok.

So, 1 first thing is when the system is stable the system is stable then x at infinity will be 0 ok, so now j which is $\int_0^\infty (q + rk^2)x^2 dt$. So, I am just taking it from here. This will be equal to the following minus $\int_0^\infty \frac{d}{dt}(px^2)$dt ok.

So, this will be what? This will be simply $px(0)^2$ because x at infinity is say 0. Now to minimize the value of J given some x(0) with respect to k ok. So, what are we looking at? We are looking at a solution to the equation this one; $\frac{dp}{dk}$= 0. So, P from this equation can be written as $\frac{-q+rk^2}{2(a-bk)}$ and then ok $\frac{dp}{dk}$= 0. I will do all the math here and that will turn out that this is P = $\frac{rk}{b}$ or in other words. The K, the optimal control law K = $\frac{pb}{r}$. Similar to what we had here right, so, what was the optimal control law here; this R. So, in the scalar case this will be $\frac{pb}{r}$ that is what we just derived here in a very simplistic way right ok.

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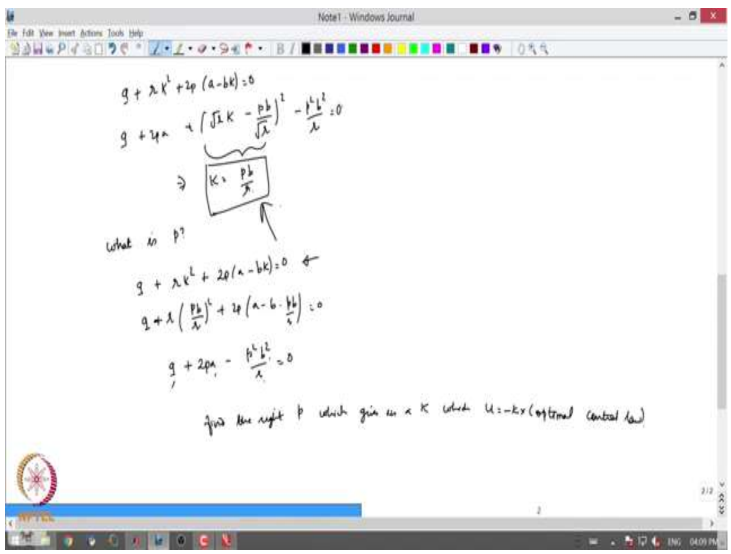

So, we can also look at it in a slightly different way. So, if I take this equation q + rk^2+ 2p(a – bk) = 0. I can write this as q + 2pa + $(\sqrt{r}k - \frac{pb}{\sqrt{r}})^2$- $\frac{p^2b^2}{r}$=0. So, this will have a minimum when this guy over here is 0 so, the essentially means that this is k = $\frac{pb}{r}$ ok. Now this is what I derived right. Now what is the value of p? To find a K I need to find what is what is p?

Now, p can be found out by the. So, I substitute in this equation plus rk^2 + twice p(a – bk)= 0; I substitute for K and what I end up with is what is k? This is q + r$(\frac{pb}{r})^2$+ 2p(a-b$\frac{pb}{r}$) = 0. And I end up with solving a quadratic equation of the form also q + 2pa - $\frac{p^2b^2}{r}$ = 0 right. So, in this q is known to me, a is known, b is known, r is known and I can find an appropriate p, find the right p, which gives me a K which is the which can be used to compute the optimal control law u = -kx this is my optimal control law ok.

In a way I in other (Refer Time: 11:32) also maybe just quickly differentiate this with respect to K and then will get this you get the same relation right. It is just a very simple simplistic way of looking at it from a scalar system point of view ok.

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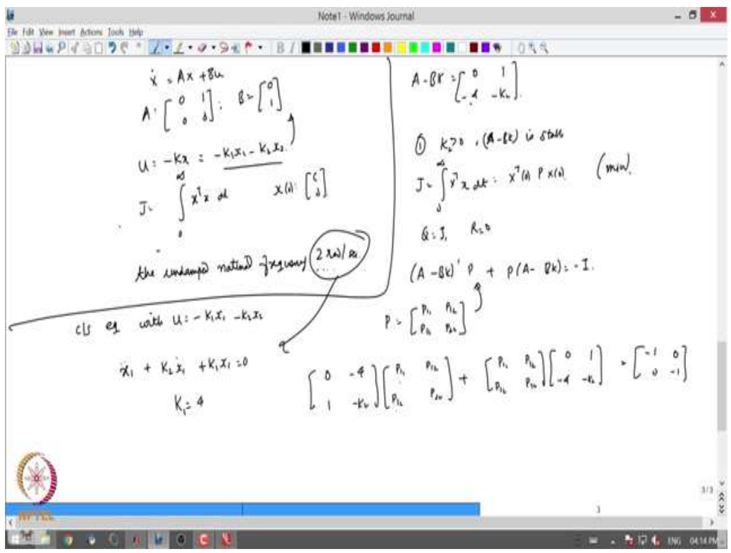

So, let us move it forward and say well I have $\dot{x}$ = Ax + Bu with A of the form $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ because both of these problems I am just taking from the standard textbook of got all basic control. So, assume u = -K x is of the form -$k_1 x_1$ - $k_2 x_2$ ok.

And let us say well I have a little easier looking cost function to be minimized that is $\int_0^\infty x^T x dt$ ok. And so, for some initial condition x(0) and just say $\begin{bmatrix} C \\ 0 \end{bmatrix}$ ok. Now I want to minimize this cost function with the entire natural frequency. Under natural frequency to be 2 rad/s right ok.

So, this is my problem statement right. So, what is an appropriate u that will minimize this function and also place the closed loop poles such that the un damped natural frequency is 2 rad/s. So, substituting the control law u = -$k_1 x_1$ into here, I find out first the characteristic equation with u = -$k_1 x_1$ -$k_2 x_2$. So, that will be. So, I have $\ddot{x_1}$+ $k_2 \dot{x_1}$+ $k_1 x_1$ = 0 next. So, this substituted here would mean that k_1 = 4 ok.

So, essentially I am now looking at A - B K of the form $\begin{bmatrix} 0 & -4 \\ 1 & -k_2 \end{bmatrix}$ ok. Now first is well when is A- BK stable whenever $k_2 > 0$, A - B K is stable ok. Now again I will do the same exercise of computing what is the optimal control law.

$\int_0^\infty x^T x dt$ t= $x^T(0)$Px(0) or so. So, I have to find when is this optimal when is this cost function minimized. This cost function is minimized if I put for Q = I, R = 0 into my Riccati equation over here or even say over here I get the following expression. I get $(A - BK)^T$P + P(A – BK) = -I ok.

Now, let me assume that P is of the form $\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$ is symmetric so, this will also be p_{12}. This will be p_{22} ok. So, I just write it in this equation right I what is my A - B K of the form is. So, $(A - BK)^T$ would be $\begin{bmatrix} 0 & -4 \\ 1 & k_2 \end{bmatrix}$, I have $\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -4 & k_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ ok.

So, I get now P in. So, I just have to solve for $\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$ in form of in terms of k_2.

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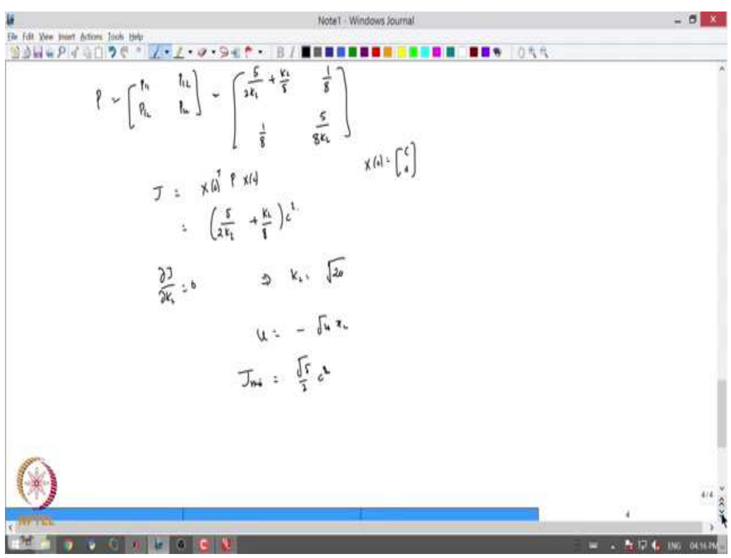

So, what do I get is I rearrange terms and go through some computations; $\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$ = $\begin{bmatrix} \frac{5}{2k_2} + \frac{k_2}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{5}{8k_2} \end{bmatrix}$ 1 over 8 5 over 8 K 2 ok. The performance index is about J = $x(0)^T$Px(0) right. So, I just substitute for all these things ever x 0 was of the form c and 0. So, what do I get here is $(\frac{5}{2k_2} + \frac{k_2}{8})\, c^2$ ok. So, to minimize J I just do the following $\frac{dJ}{dk_2}$ = 0 resulting in $k_2 = \sqrt{20}$ ok.

And therefore, u = -$\sqrt{20}\, x_2$ and then the minimum value of J therefore, is because I know P now is all this, ok. So, this is a minimum value of J ok. So, some straightforward steps of how to compute the optimal control law given a certain problem ok. So, the last problem for today we will do is the following.

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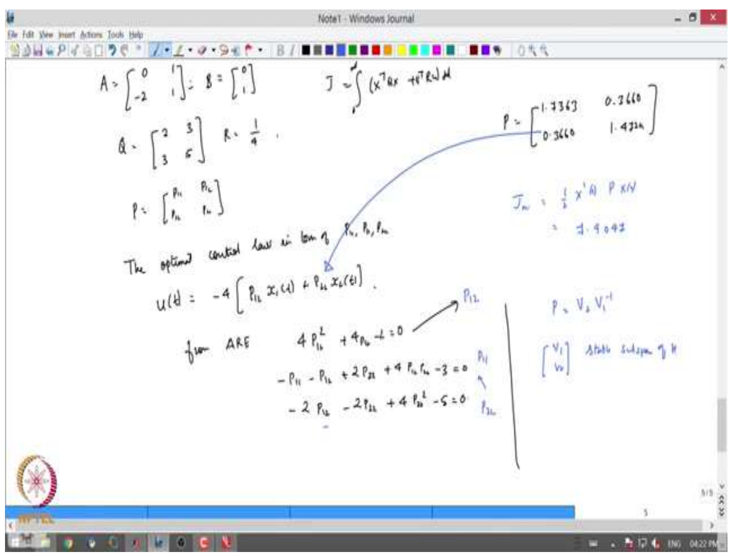

So, I will again do standard second order system with A = $\begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}$, B = $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, Q = $\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$, R = $\frac{1}{4}$ ok. And then of course, it is a infinite horizon problem as usual. So, I just again fix p as $\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$ and I will here I can use the standard cost function that we have here right.

So, J =$\int_0^{\infty} (x^T Q x + u^T R u) dt$ ok; so, the optimal control law in terms of elements of P that is $\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$. How do I derive this that optimal control law? That is simply over here right. I know what is R, I know what is B, I know how the terms of P look like. So, that will simply be I just I will just skip the computations.

So, u would be -4$p_{12}x_1(t)$ + $p_{22}x_2(t)$ ok. And then now I again go back to the Riccati equation which is this one, I look at what are the solutions I am given Q and I am given R and from that so, from the Riccati equation ok. I will skip all those computations, but what substituting what P will substituting for P or substituting for A P B R and so on, will give me the following things.

So, I have 4 $p_{12}{}^2$ + 4p_{12} - 2 = 0. I have a -p_{11} - p_{12} + 2p_{22} + 4$p_{12}p_{22}$ -3 = 0, and lastly I have -2p_{12} - 2p_{22} + 4 $p_{22}{}^2$ - 5 = 0 ok. So, I have 3 equations here solving this equation will give me this will give me p_{12}.

I can use that p_{12} over here to get p_{22} and a substitute here to get whatever is remain that is p_{11} ok. I will just say this is this is very standard process. So, and the P that we get will look in the following way 1.7363 0.3660 0.3660 1.4729 ok.

Then I can plug this P into my optimal control law right. So, that is and then of course, all the all the steps will follow that I just substitute for P here and the j minimum will just be as usual $\frac{1}{2}x(0)^T Px(0)$ and that will be 7.9047 ok. So, that is that is again I am just taking you through the process and see what that what that things that we learnt yesterday be ok.

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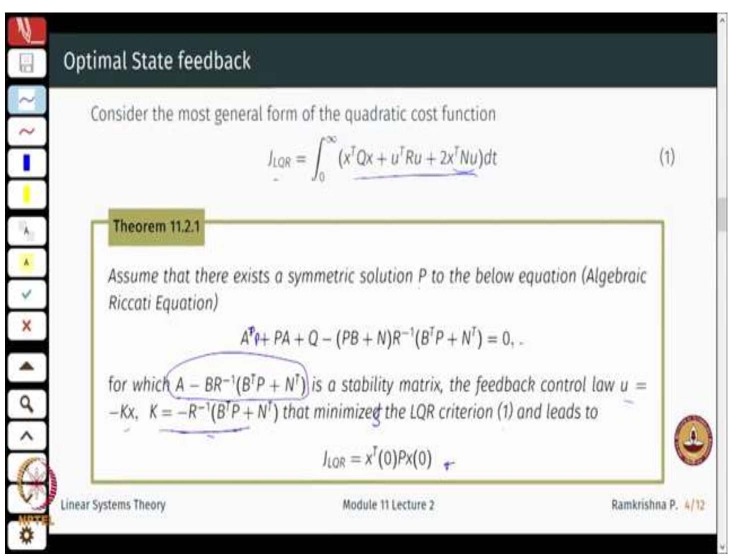

So, what we also learned yesterday was a very nice looking thing right.

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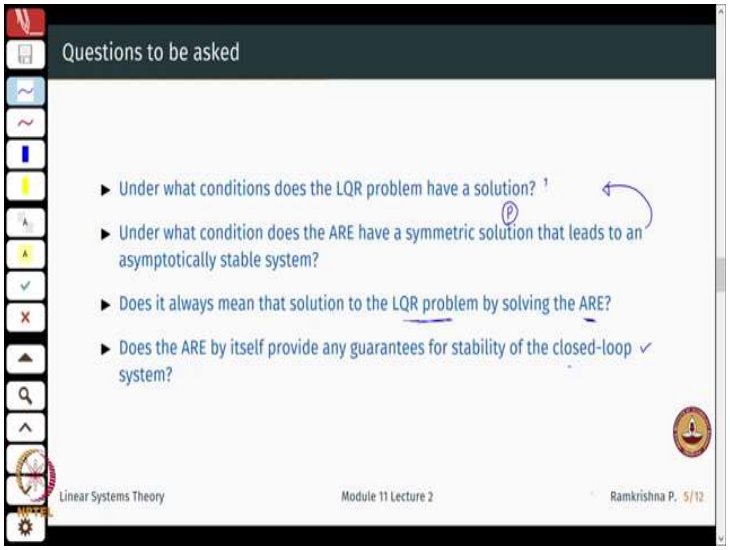

Of we are also has questions when can we solve this equation.

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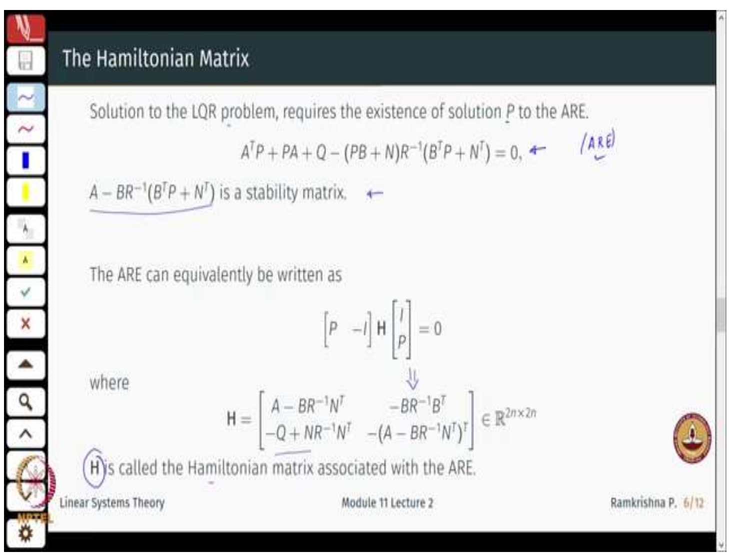

Then we constructed something called a Hamiltonian matrix.

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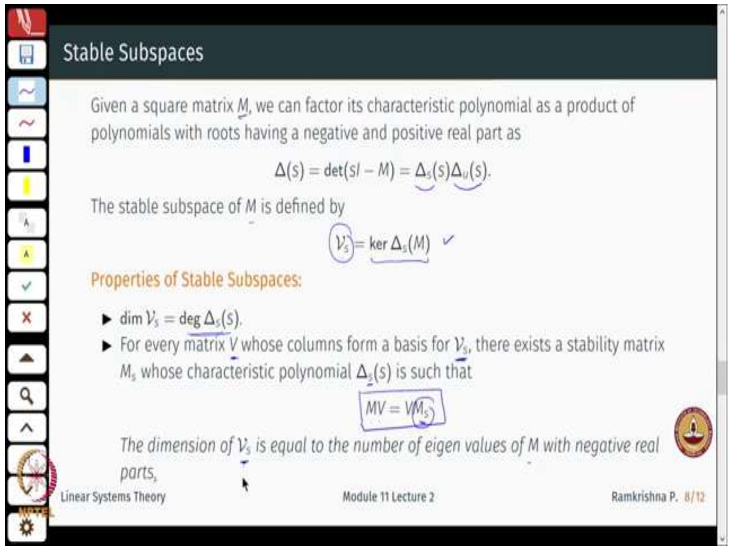

And then a bunch of results which give me conditions based on this expression, which I said that the Hamiltonian H is said to be in the Riccati operator, if a condition like this holds; H_s was he was the closed loop stability matrix and so on.

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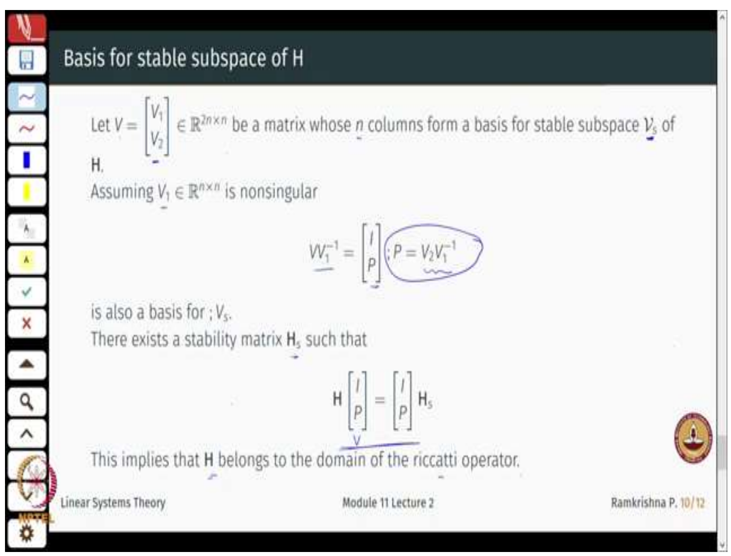

So, if I just quickly go through this and then what I find is something nice here right where P = V_2 and V_1^{-1} ok, so let us just write on it. So, P was $V_2V_1^{-1}$. Now where does V_2V_1 come? V_2 V_1 is the stable subspace; so, V this vector $\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$ was from this table subspace of H.

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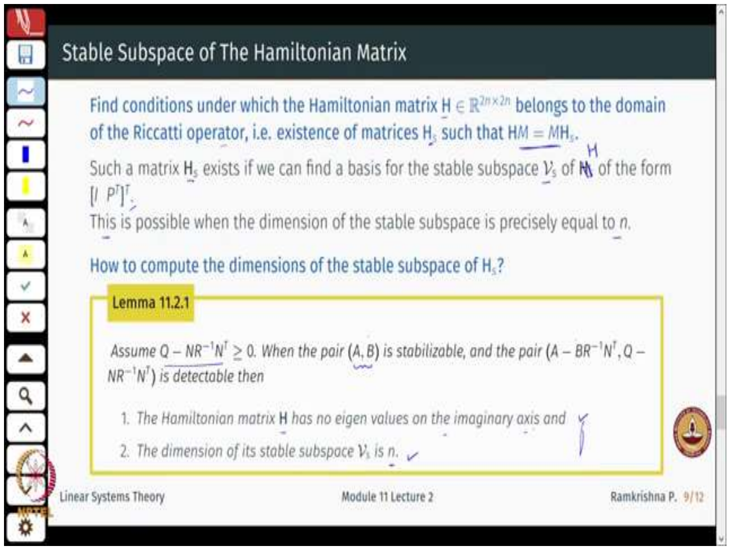

What do I know about H? That when a solution exists then the dimension of a stables of space is N or that there is no eigenvalues on the imaginary axis right. So, how does H look like? So, let us do this quickly right. So, I will just quickly run show the matlab code.

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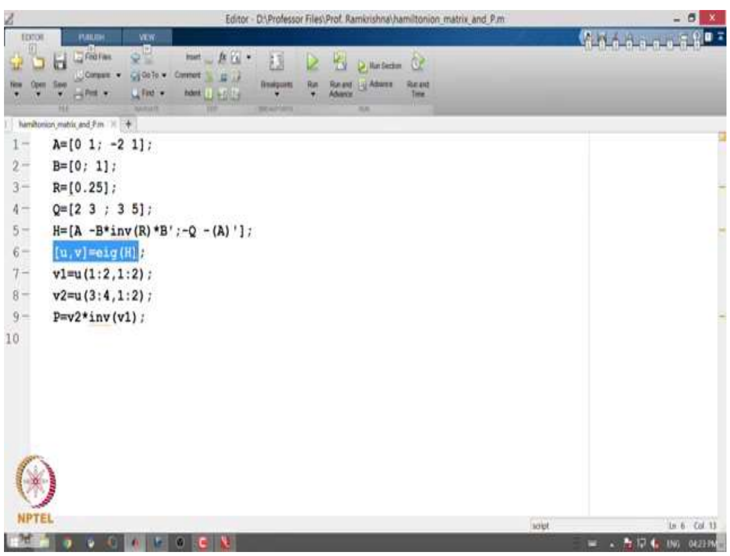

So, I have A from the problem statement which looks like this, B of this form, R is simply this, Q and so on. So, H I just write all the equivalent expressions for H and so on ok.

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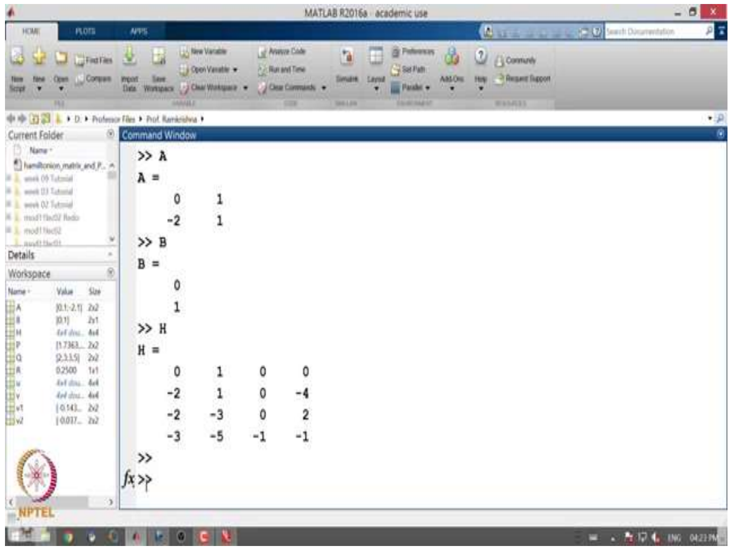

So, let us let us quickly do this. So, I have A from the problem statement like this. I have B and similarly define my Q and R as the problem statement. I construct H in the following way right. So, this is how my H looks like. First let us just see what this thing here means. So, if I run this, copy this and let me paste it 0 ok.

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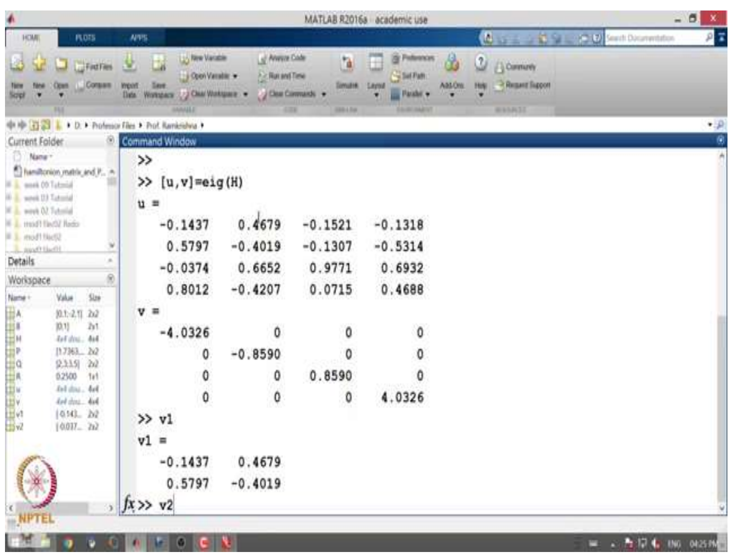

So, what I get is H as in the theorem statement has 2 eigenvalues which are positive and 2 eigenvalues which are stable ok. Now this thing here this with so, these two columns here, starting from the column sorry the column of this and column of this will be the basis for my stable subspace ok, and then the first two rows here this and this will be my will be my eigen sorry will be the basis of the stable subspace right this these 2 correspond to the stable eigenvalue say this 0.403 and minus 0.403 and minus 0.85 ok.

So, these two the first two columns of u will now be my basis for the stable subspace. What is that from the slides? I am just constructing these things. The first 2 columns of dimension R^{2nxn}. In that case it will be in our case it will be R^{4x2}. So, the first 2 columns and then I split those first two columns as V_1 which is 2 x 2 V_2 which is 2 x 2. So, V_1 will be the first four elements of u. So, V_1 right so, this is V_1 which is just there is the first 4 elements of here.

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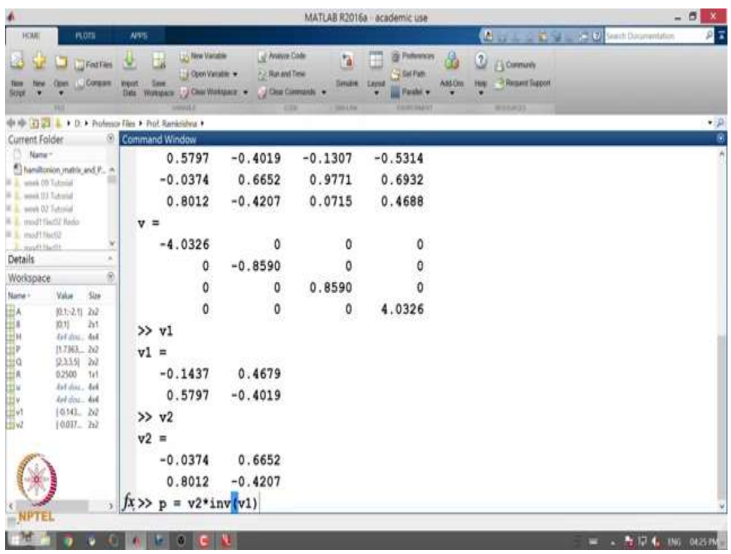

Similarly, I have V_2 which is like these elements 0.03 which is like the last four elements of the first 2 columns ok.

Now, what was P? P from the formula was $V_2V_1^{-1}$ right. So, let us just compute; so, p is $V_2V_1^{-1}$ ok.

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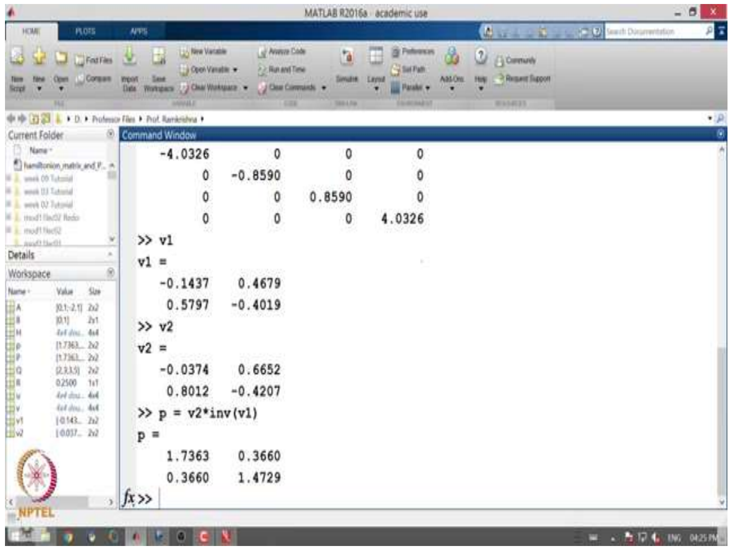

So, what we have? This is good right. So, we have exactly the same P that we computed over here 1.7 0.3660 and so on so, exactly the same P right. So, this is another very nice elegant method of computing the P right which we started off with the question when can I solve this and under what conditions does that actually exists a solution to the algebraic equation or under what conditions can actually find the optimal control law.

So, I have these 4 problems or 3 problems and the third problem with 2 different methods will actually give you good insights on how to solve problems for the assignments and any future problems if any. If you come across any interesting problems or something which you are stuck at you can just post them online and I will be able to help you out.

Thanks for watching.