Linear Systems Theory
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Module - 10
Lecture - 02
Output Feedback

Hi everyone. So, welcome to this lecture number 2 of week 10 on the course on Linear Systems Theory. So, just to have a little recap of what happen or in the last lecture.

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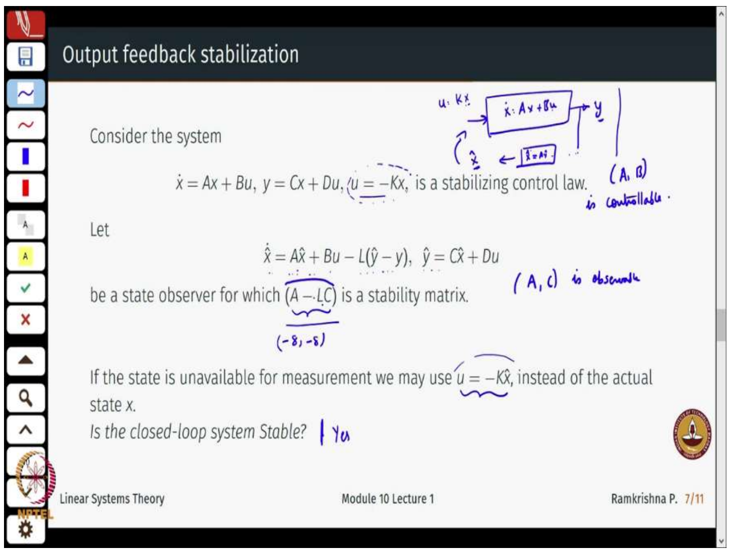

So, we were looking at simultaneous controller and observer design.

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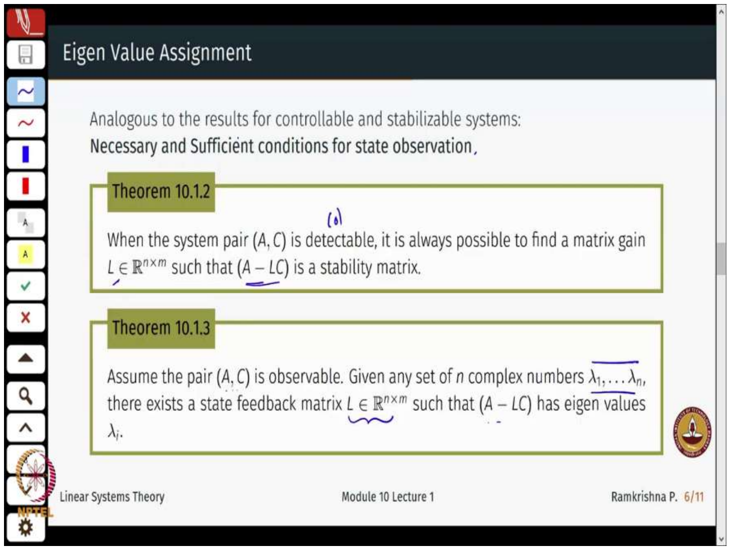

And we first derived what were the necessary and sufficient conditions for state observation. And then we proceeded on to the design procedure of how to actually construct the state observer. And it turned out that we were looking at situations; where we had to design an L such that this A -L C is a stability matrix.

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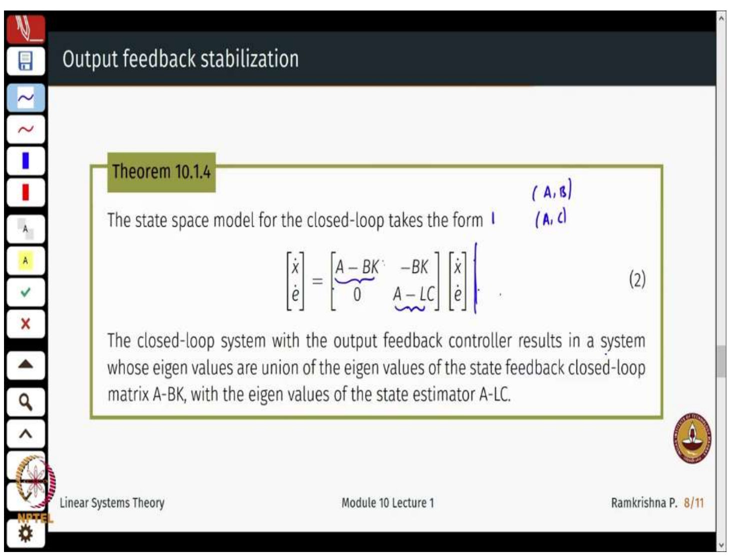

Where ok, then you have the state space model for the closed loop system taking this particular form. And here you have the controller design part here you have the observer design part.

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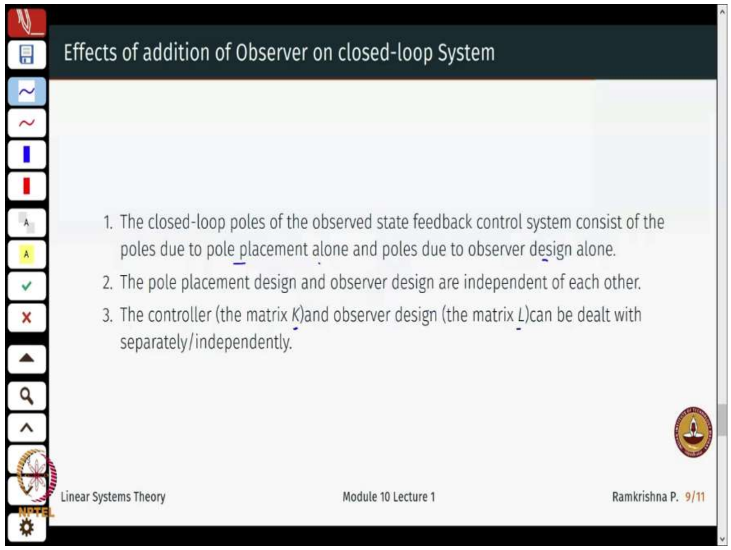

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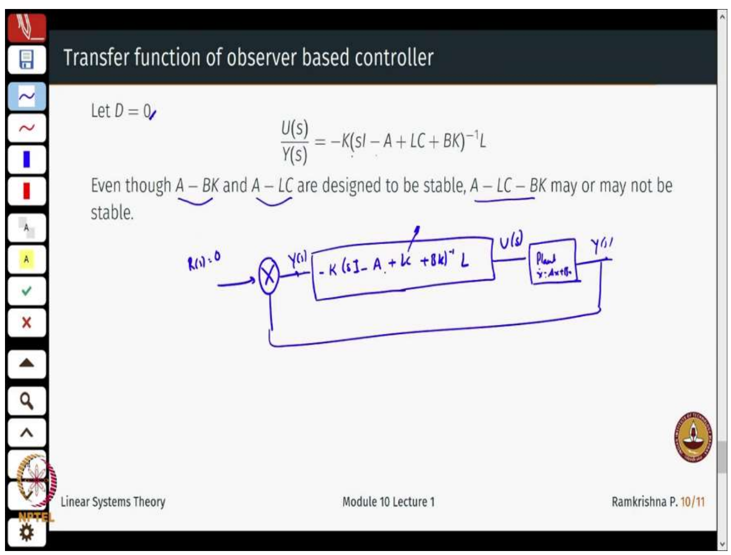

And what you also what we also saw at the end was that the design of the controller and observer. Do not actually interfere with each other and this is how the how the closed loop in terms of a of a transfer function looks like right.

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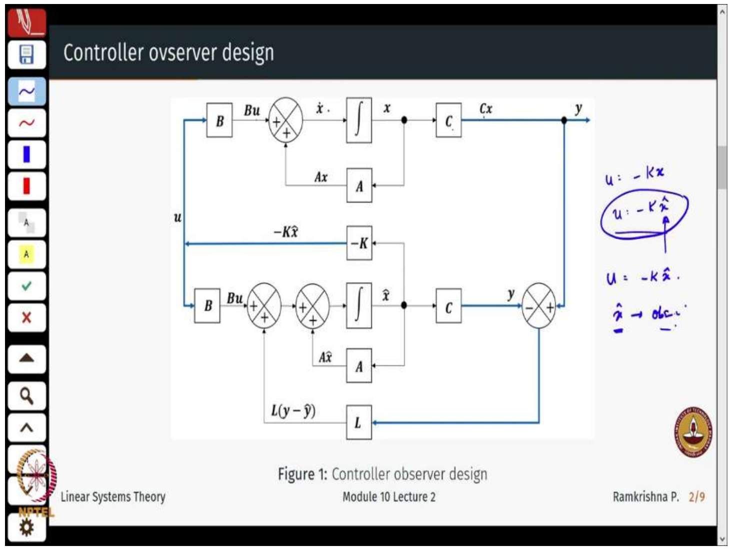

So, in summary if I go to just draw ah block diagram representation of it, it would just look looks something like this right. So, here I have my state $\dot{x}$ = Ax + B u, y = C x I want u of the form -k x right. So, this is the u I do not have x for measurement. So, I will have the estimated state as u = -k$\hat{x}$ hat and this $\hat{x}$ comes as a result of the of the of the observer design right.

So, where the observer part looked something like this yeah. So, this plus this gave us the observer design. So, like $\dot{\hat{x}}$ = A$\hat{x}$ + Bu and if you just plug in all this you get a block diagram representation, which looks something like this right ok. So, we today we begin by looking at some small very small design problems; I also use the help of MATLAB a little later. But let us first do a very very basic design procedure by hand ok. Let us say I we change the colour right ok.

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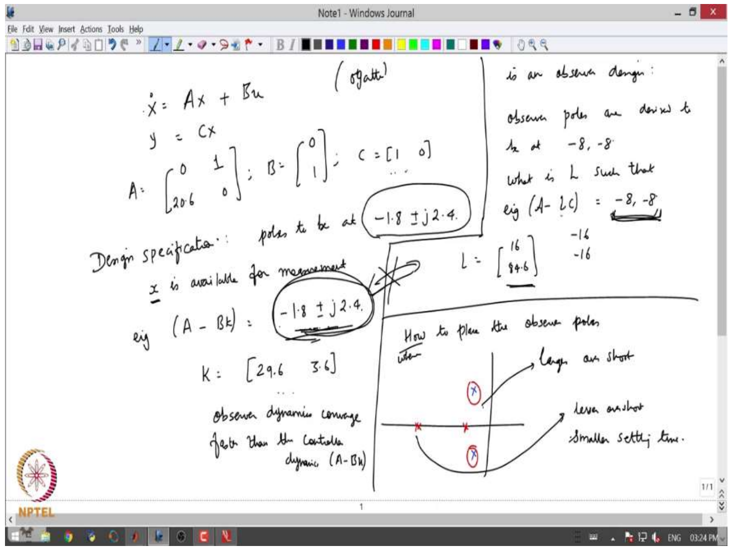

So, let us start with the system $\dot{x}$ = Ax + Bu, y = Cx; where A = $\begin{bmatrix} 0 & 1 \\ 20.6 & 0 \end{bmatrix}$ B = $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and C = [1 0] ok. This example I just directly take from ogatta. So, just in case if there is some confusion you can always go back to ogatta and refer ok. So, what are the design specifications ok? So, there is a design specifications are such that we want the closed loop poles, to be at -1.8 ±j 2.4 ok.

So, assuming that x is available for measurement I will use the standard pole placement techniques to find a value of k such that A - B k has its eigenvalues at minus 1.8 ± j 2.4 ok. I will not go into the details of that steps ah, but that it turns out that the k here would be 29.6 and then 3.6 ok. So, you can just use any of those formulas that we had last time starting from just matching.

The two characteristic equations or a the ackermans formula or to the controllable canonical form and so on ok. So, that is that is not important right so, but what happens here is that x is not directly available for measurement. So, an additional step we need to do is that often observer design ok. So, in most in most problems we will only be given what is the desired plan performance.

So, this could be in terms of could be in terms of a certain overshoot or settling time and so on. And what we remember from basic control course is that this will translate to some kind of locations of the poles of the closed loop system. Or what we called what there as the as the dominant pole analysis. So how do we go about doing the observer design ok?

Let us say that there is a specification such that the observer poles are desired to be at -8 and -8 ok. Like we design a observer for second order here ok; because we have to observe that two states ok. So, I can similarly use the techniques I had. So, what the observer design problem translates to is, to find the locations or assign -8; -8 in this example to a to this matrix A - L C right. So, this is or over here.

So, A - B k we had poles at the minus 1.8 ± j 2.4 and A - L C should have poles at say -8 ok. So, I just do the exact same procedure I can use bunch of methods as stated earlier. To compute what is the k or L in the in this case or what is L such that eigen values of A -LC are at exactly -8 and -8.

I can again compare the characteristic equations of what is the desired and what are the unknowns in terms of L. So, it turns out that L will be something like this. So, this is a little design procedure to show to tell how I place the controller poles and how I place the observer poles. And what we also know is that these two do not interfere with each other right.

So, even if I place this at -16, -16 nothing here would change and vice versa. So now, just to plot how the closed loop response looks like, I can just compute the transfer function right; so which we had derived over here last time right. And I can do a bunch of things to check this step response and so on ok. So, what is important here is to is a location of observer poles right.

So, one question that we will answer shortly is how to place how were actually where to place the observer poles ok? And if you look at look at the closed lope system ah, what we want is that the state $\hat{x}$ is the estimated state ah. So, there is there is two process right. So, one is u should be computed as k$\hat{x}$ and this $\hat{x}$ is computed as result of an observer design.

So, first so the observer should give its $\hat{x}$ the estimated state to the controller and so on right. So, first is which dynamics should be should be faster right. So, we will first just check numerically right just try to play around a bit with the poles of the observer and check how do I appropriately choose the poles of the of the observer ok.

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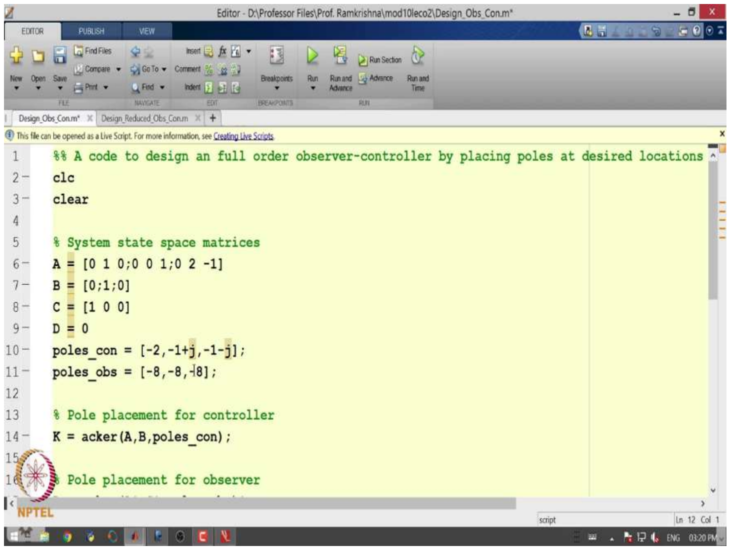

So, I will do a little example here ok. So, I just directly jump into the system here which has A matrix, now which looks like this certain B and the certain I will I will post this code online so, that you could check it for yourself. So, what is desired is that, the closed lope poles are at -2 and 1 + j and 1 - j right.

So, two complex conjugate poles and one pole on the on the real access. And let me say well I can I want my desired observer poles to be at these three locations -8,-8, -8 so, these many things ok.

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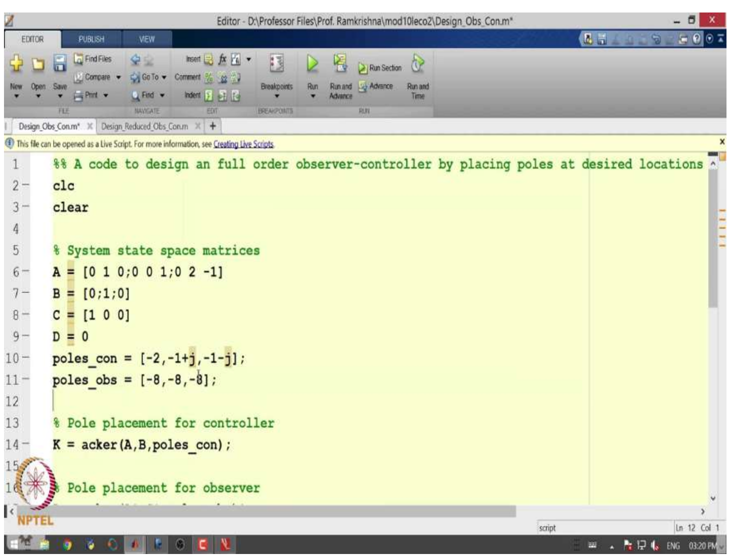

Now, ok, so, MATLAB gives you bunch of tools to do the pole placements. So, I will just use the command for ackerman's formula. So, I have A; which is my system matrix which is defined by this thing over here. So, I have B which is the input matrix I know what are the desired poles? So, once you just give this three inputs. So, this command here it will it will plot it will show you what are the what is the controller gain similarly for the observer gain.

So, I have A; I plug it in here I have C and then I have also the location of the observer poles -8, -8, -8. And I do I and I do the designs simultaneously and at the end I just plot how my closed lope step response for example, looks like. So, let us just run this code ok.

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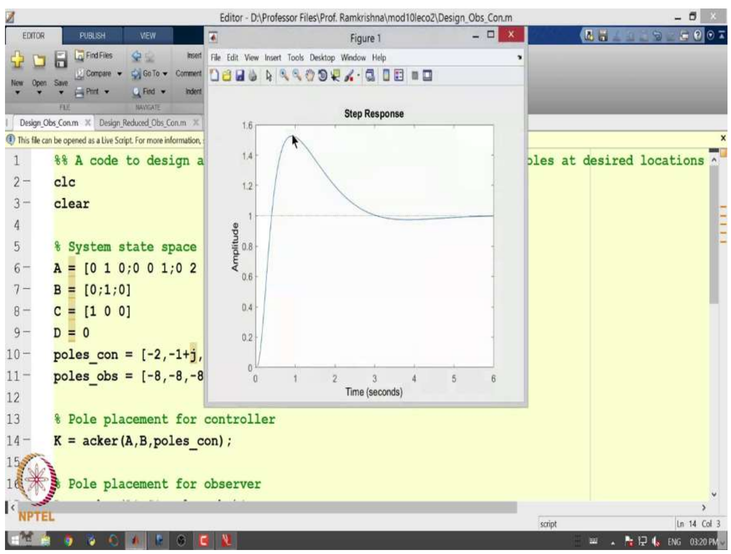

So, what I see here is well I have nice looking response even though the overshoot is say about 50 percent in this case, settling time about 5 seconds and so on ok.

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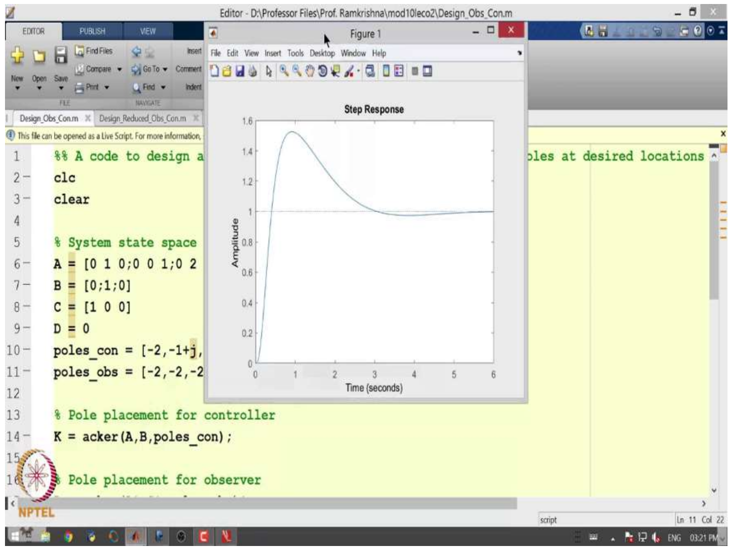

So, let me do something else. So, let me say I place the poles at -2, -2 and -2 and I run the code again so, ok. So, let us compare this two ok.

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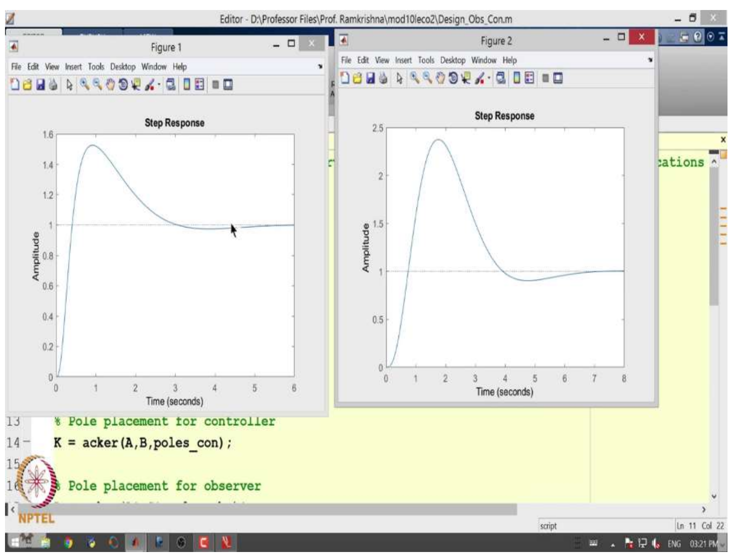

So, this was; the figure 1 was where I had the observer poles at -8, -8, -8. So, this had an overshoot about 50% this is response looks horrible right. So, you have an overshoot which exceeds 100%, the settling time is much larger than what it was here. And you if you see if you that you place it at -1, -1, it will get much much worse.

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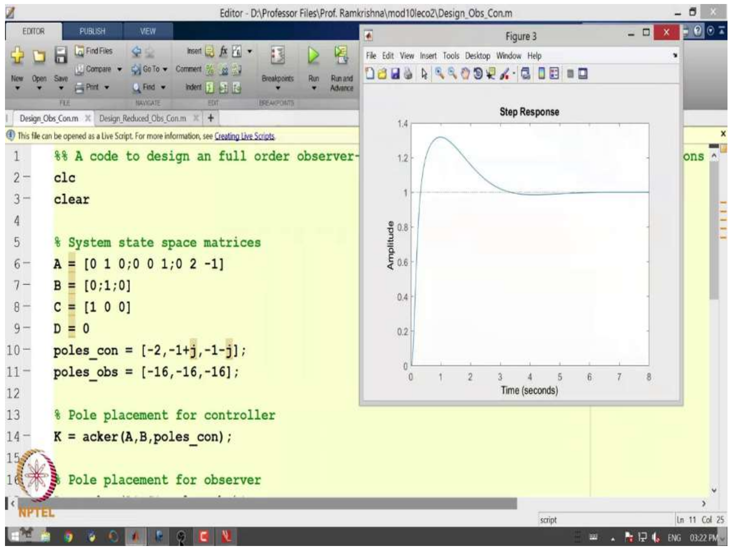

Let us do another trial here let us say - 16, -16 and -16 ok. And I run the code this looks. So, let us compare figure 1 and 3.

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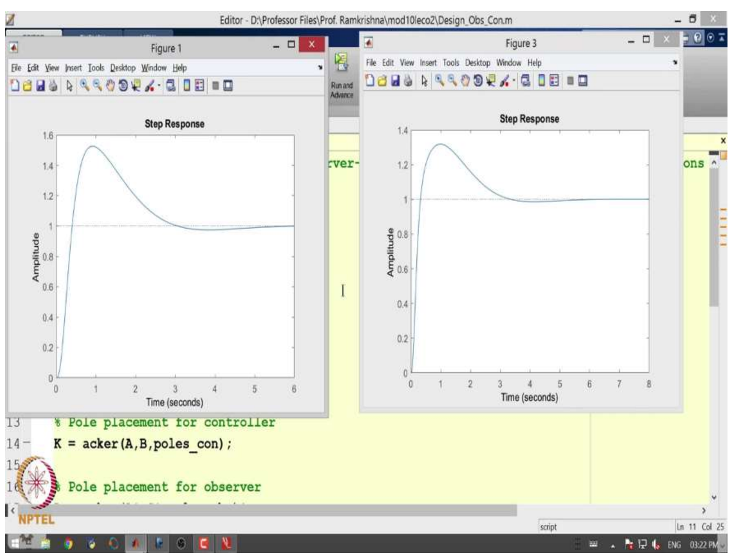

Now, ok. So, well here I have a reasonably good settling time of about say 5 seconds or even less. My overshoot has drastically decreased ok. And of course, these are this is nice things that that I have a lesser overshoot and a faster settling time is always desired in any control design ok. So, one observation from here is the following right.

So, whenever the poles so, when whenever so, in that example we had three poles. But say whenever the poles were close to the open loop poles then we had like a much larger overshoot ok. So, let me just draw a little diagram here right let us say well I just denote ok. These are my control pole poles with the blue line and say these are my observer poles let us say they are somewhere here right or as both of them were real. So, let us say that they are somewhere here ok.

In this case we will have a larger overshoot ok. And on the other hand when the poles were here this is do not change right; because this come from the requirements of or this come from the design specifications. So, in this case you have a lesser overshoot and also ah smaller settling time ok.

So, well this is also kind of kind of obvious right. That your observer dynamics should converge faster than the controller dynamics or then the then the controller dynamics. So, here when I say controller dynamics I essentially say the dynamics of A - B k ok. So, this kind of because what the controller or what the observer based controller does is it just first computes the estimate state and then it has to feed it back to the plant.

And therefore, we would expect the observer to have a faster response than the controller itself right. And then therefore, so one thing is to choose the observers poles to be much further away from the desired poles of the closed loop plant ok. So, that is that is about well little example illustration of observer design we could do a bunch of example.

So, you can actually construct your own example. So, lots of people over the forum post messages that please do more examples I think we can construct examples. So, there are lots of examples which you which can be found online, you can just work those out for yourself and they are they are pretty forward right you know. So, just you can just check for different values and just check for performance of a controller and the observer simultaneously.

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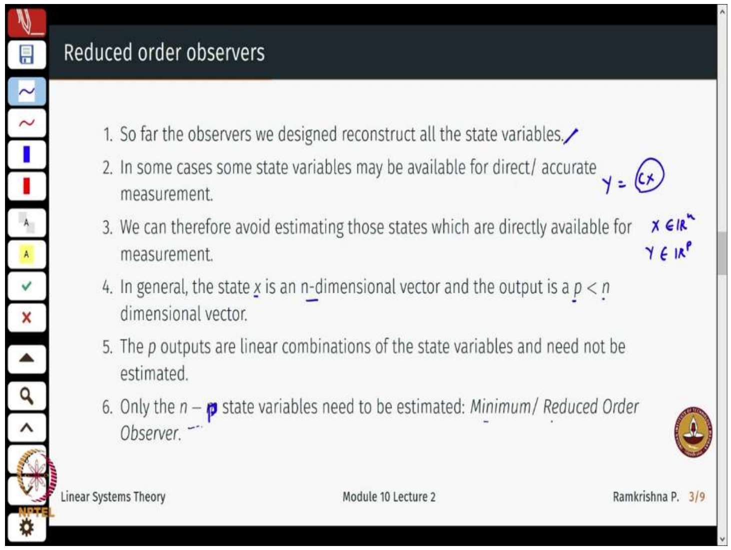

In the observer design so far what we saw was that, we reconstructed all the state variables ok. So, in some cases some state variables might be available for direct or even accurate measurement. And therefore, one can avoid estimating those states right. So, whatever is available for measurement, we can just directly measure those and feedback and whatever are not available for measurement we can possibly construct an observer.

So, in general your state is an n dimensional vector and output is say some p; which is typically less than n dimensional vector. So and when I say y = C x and say x is in R^n y is in R^p. Then these p outputs are usually linear combination of state variables and this need not be computed. So, what we need to estimate is only the remaining n - p state variables ok. So, when we need to estimate only the remaining n - p variables this is called a minimum or a reduced order observer.

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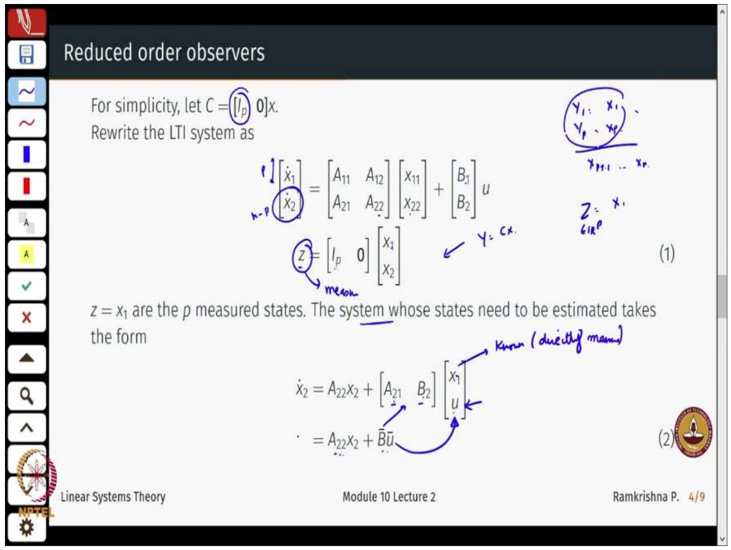

So, let us see how this works ok. So, let us for simplicity say that p variables are of the form or the p outputs or of the form $y_1 = x_1$ till $y_p = x_p$ ok. And the remaining once are the once which I which I need to estimate I do not know what is how does x_{p+1} one look like till x_n right. But I know how x_1 till x_p looks like right I just making a very nice assumption here that this holds.

In general what we said that there might be linear combinations of state variables. But here I just make a much more stronger assumption just for ease of computation ok. So, now I can I can split my variables or my state equations into the following form. So, I have $\dot{x_1}$ and $\dot{x_2}$ in this way. So, these are the first p variables these are n - p and I split my a matrices the B matrices accordingly.

And my output now looks something like this I just call this z has the p dimensional identity matrix times x_1 and 0 and just rewriting y = C x in a in a different way here ok. Now, these are all measured right. Because z is simply x_1 yes both are of dimension p ok. Now, let us look at how the system whose states need to be measured look like. So, these are the states set which need to be measured.

So, have $\dot{x_2} = A_{22}x_2 + A_{21}x_1 + B_2$ u and ok. Now, what is there this is I know x_1 right; because I can directly measure this I know u of course, I know A_{21} and I know B_2.

So, that I just call so, this a system $\dot{x_2}$ = $A_{22}x_2$,+ $\bar{B}\bar{u}$ where this is $\bar{B}$ and this is there the $\bar{u}$ right. So, this is a system whose states need to be need to be estimated right.

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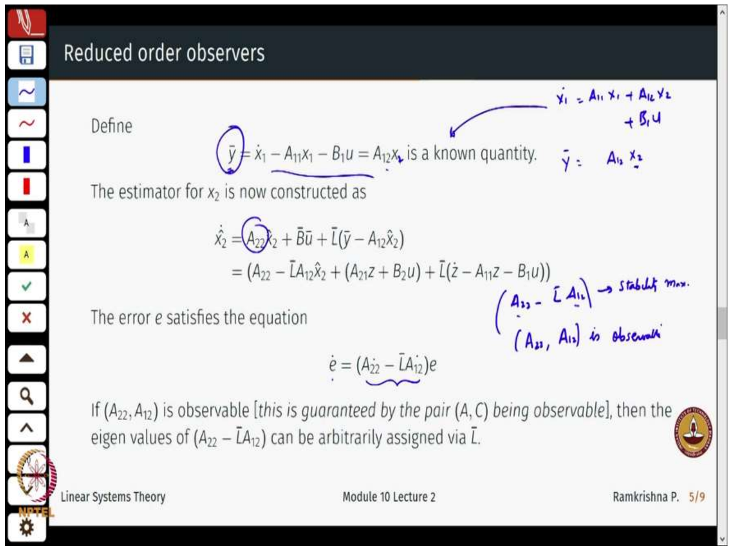

Now similarly, I will define a new output in the following form that $\bar{y}$ is from the first equation how this first equation look like? First equation is $\dot{x_1}$ = $A_{11}x_1$ + $A_{12}x_2$ +B_1u. So, from this I can I can write something like this. So, where the new output $\bar{y}$ is a new C matrix which is $A_{12}x_2$. So, these are the states that need to be to be measured ok.

And this $A_{12}x_2$ from this expression takes this form ok. Now we construct for a for an estimator for this system, which whose states are x_2 and output is this $\bar{y}$ ok. And I used the standard formula which I used over here to construct right. How did I construct the state observer here is just with this equation ok. I will use exactly the same thing here to construct a state to construct an observer for those un estimated states in the following way right ok.

I will just skip those computations they are like fairly easy to check the error e takes the following form in terms of A_{22} and A_{12} that is exactly what I wanted to estimate right. So, so given over here in this system right, this $\dot{x_2}$ = $A_{22}x_2$ + $\bar{B}\bar{u}$ is the system that needs to needed to be estimated ok.

Now, what do I know is that when the error should converge to 0. This A_{22} - $\bar{L}A_{12}$ should be a stability matrix; with some eigen values or with eigen values at the desired locations right. So, if the stability matrix then I know that the error converges to 0. So, when can I do this? When can I place poles of A_{22} - $\bar{L}A_{12}$ at desired locations? This I can do if and only if the pair A_{22}, A_{12} is observable ok.

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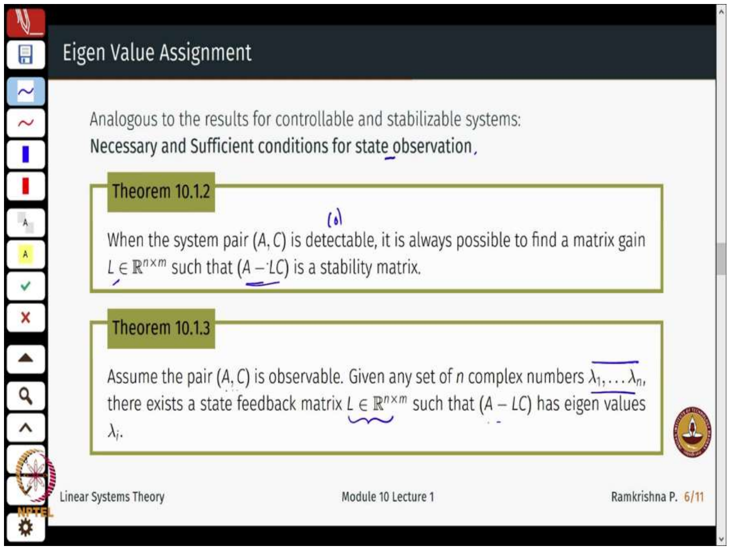

So, very similar to what were the conditions here right what was the necessary conditions for A, A - L C to be here. Stability matrix was that the this pair A and C must be detectable. Or observable in our case right or when the pair A, C is observable then I can place all the eigen values of A - L C at the this desired locations right.

Similarly if I were to place the poles of A_{22} - $\bar{L}A_{12}$ this is observable ok. Now who guarantees this? Well, we know that the pair A, C is observable and this guarantees that this pair A_{22} and A_{12} is observable ok. You may just want to write down the proof of it as quickly for yourself right I mean I will skip that these are.

So, we have done lot of lots of proofs and how to compute controllability or observability via the duality the eigenvector test and so on. So, you can just make use of one of those tests to show that the pair A, C being controllable is enough for me to guarantee that the pair A_{22}, A_{12} is observable ok. So, A, C being observable is equivalent to saying A_{22} to A_{12} is also observable ok. So, and therefore, I can assign eigen values to A_{22} - $\bar{L}A_{12}$ right ok.

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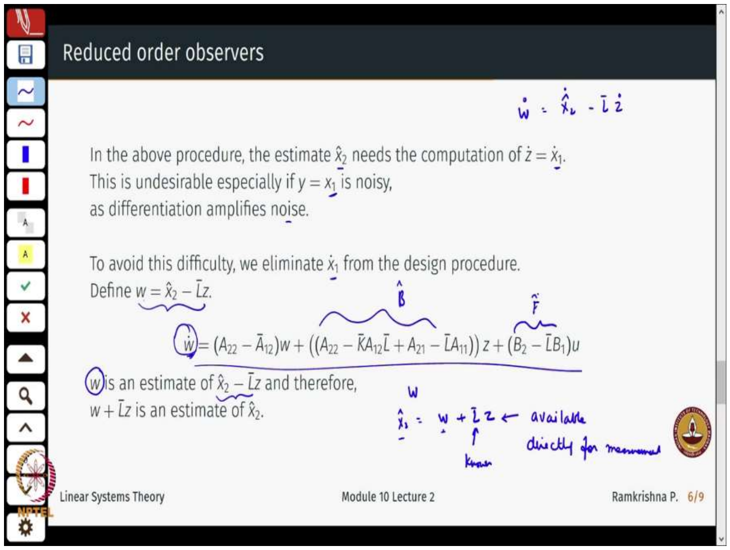

So, one drawback here is that I was estimating $\dot{x}_2$ and this $\dot{x}_2$ had an expression $\bar{y}$ and $\bar{y}$ contained derivative of x_1 right. So, x_1 was. So, what was the x_1? x_1 came from here right. So, so z was equal to x_1 and it. So, the $\dot{x}_2$ needed computation of x 1 dot ok. Now, this is un desirable especially if x_1 is noisy.

So, if even though I can measure even though the states are available for measurement, but there could be lots of sensor noise for example right. And derivative of noise is not a desirable thing that differentiation of that signal actually amplifies the noise ok. So, how do we get rid of that ok? To avoid that we eliminate $\dot{x}_1$ from the design procedure.

And if I knew variable w as $\widehat{x_2}$ - $\bar{L}$z ok. So, this kind of works out pretty neat that I can write down my expressions or in terms of not in terms of $\dot{x}_2$, but in terms of $\dot{w}$ ok. So, I can so, quickly compute what is $\dot{w} = \dot{\widehat{x_2}}$ - $\bar{L}\dot{z}$ ok. Now, I know what is $\dot{x}_2$ from here, I know what is $\dot{z}$ from here and I can simplify this equation to look something like this ok.

Again this is just not even a laborious process, but just couple of steps you can you can write this down and you can arrive at this particular expression ok. So, w is an estimate of$\widehat{x_2}$ - $\bar{L}$z. And therefore, so, I know w. So, what is x_2? Well, $\widehat{x_2}$ is now simply w +$\bar{L}$z. So, this is what I estimated now I know this right.

So, this is known, this no this comes from the computation of the of the observer poles this also know this comes some from the measurements. So, this z are available directly for measurement ok. So, that is kind of kind of pretty need; because we do not have to now go through the procedure of computing what is $\dot{x}$ ok.

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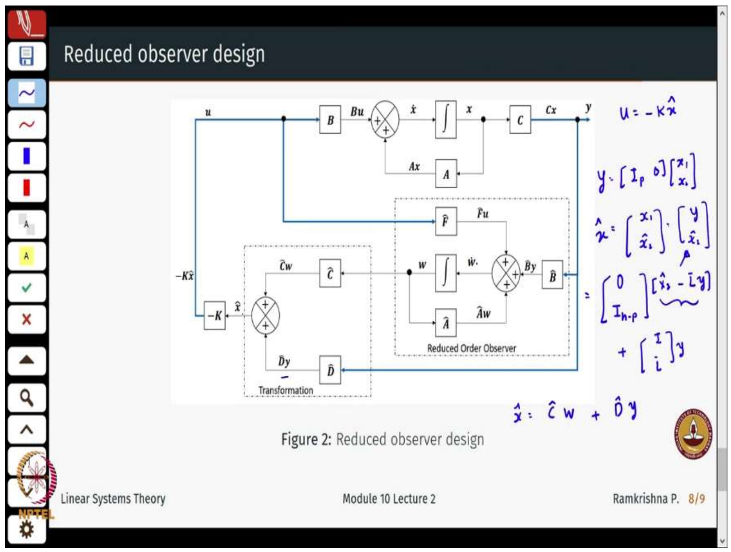

So, again coming back to the block diagram of it; so, again I have $\dot{x}$ = Ax + Bu ok. So, let me write down the few steps here. So, that it is easy for us to understand how the how the block diagram actually looks like ok. So, again so, u comes as a result of -k$\hat{x}$ ok. So, what is $\hat{x}$? So, first is I estimate w and then once I estimate w, $\widehat{x_2}$ is constructed as w+$\bar{L}$z.

So, here right, so $\hat{x}$ comes as a result of this signals. So, my y was initially of the form some p dimensional identity and x_1 and x_2 ok. Now, the estimated state is has two components this is already measured directly and I have $\widehat{x_2}$ ok. What was x_1? x_1 was simply y and this $\widehat{x_2}$ was to be estimated ok.

o, this I can write equivalently as $\begin{bmatrix} 0 \\ I_{n-p} \end{bmatrix}$[$\widehat{x_2}$ - $\bar{L}$y] this $\bar{L}$ I know how to compute right from the from the observer design plus I have a identity here $\bar{L}$ and a y ok. So, you just rewrite this and you will realize get this back ok. So, this $\hat{x}$ now can be written as so, I will call this as some $\hat{C}$ times. So, what was the signal right y was it was also equal to z from this expression right.

So, this was my original y = C x ,I call that as z. So, this turns out to be $\hat{C}$ times w plus let me call this some $\widehat{D}$y. So, that is exactly what is happening here $\hat{C}$w + $\widehat{D}$ times is y gives me x hat and this via minus k goes back to my to my controller and like I equivalently derive the transfer function and so on. And similarly we will also verify the notation for the expression for w right. So, $\dot{w}$ = A_{22} - A_{12}w and so on ok.

So, you can just right check for this also. And this $\hat{B}$ and $\hat{F}$ are just these two terms here. So, the term associated with y I call this entire thing as $\hat{B}$ and this as $\hat{F}$. And you can have the nice looking block diagram realization like this ok.

So, this is just as an nice pictorial interpretation of what how designing the reduce order observer ok. So, what was the assumption here was that C was a kind of had a beautiful expression like this. C was the p transfer identity and 0 ok; what if C is not in this form then we know bunch of tricks that. We know how to transform C into a form which looks like that.

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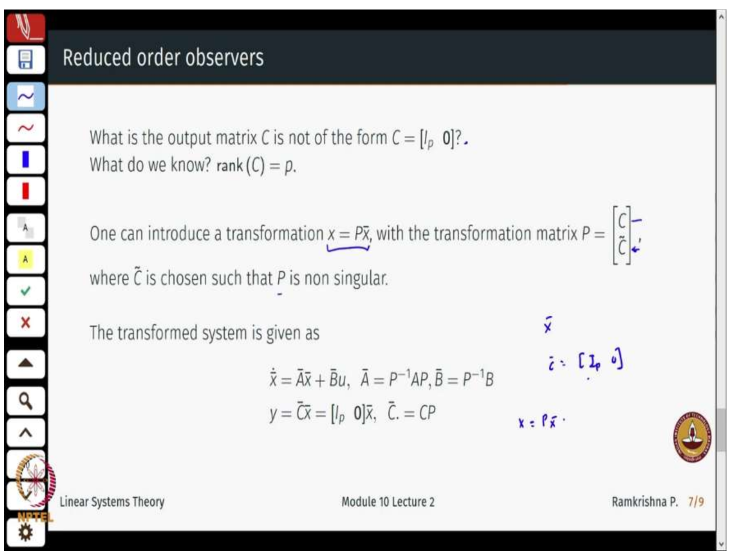

So, what if the output matrix is not of the form $[I_p \ \ 0]$ well, we know what is the rank of C right rank of C is p. And therefore, we use a standard trick of coordinate transformation that let x = P$\bar{x}$ be a transformation with P given by C. And then $\tilde{C}$ and then $\tilde{C}$ is chosen such that P is full rank that as in the same way as we did for the observer decomposition or the even in the dual way the controllable decomposition.

Now given the system I can write it in to a system in a transformed form where in the new coordinates $\bar{x}$ my $\bar{C}$ takes the this form $[I_p \;\; 0]$ ok. Now, I do the design here because I know nice looking formulas here. And I just use the reverse coordinate transformation that x = P$\bar{x}$ to get see how the observer looks like in the original coordinates right.

Like what we do even for the for the controller design that you just get it into the controllable canonical form you design the k. And then use the transformation P to get back to the to the original system ok.

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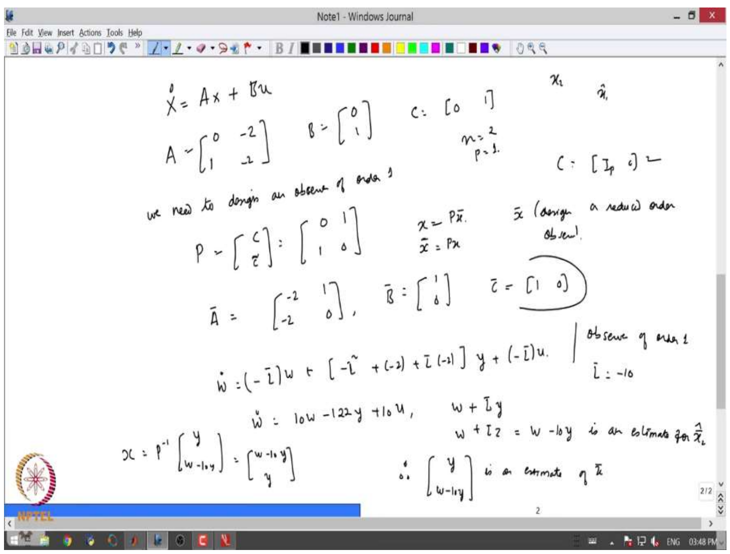

Now can I just do a little example ok? So, again I start with $\dot{x}$ = Ax + B u with A = $\begin{bmatrix} 0 & -2 \\ 1 & -2 \end{bmatrix}$ B is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$; C is [0 1] which means I can actually measured x_2 directly and I need to find what is the estimate of $\widehat{x_1}$ ok. So, what is the reduce order observer here. So, we need to design an observer of order 1 that because here n = 2, P = 1 ok.

Now, how do we do this? First is C in the form $[I_p \;\; 0\,]$well the answer is no, but I therefore, I would use a transformation P C C bar that was the notation we had use here right C $\tilde{C}$ such that ok. What is C? C is [0 1] and $\tilde{C}$ is [1 0] such that P has now as a as full right ok. Now, I use the standard transformation $\bar{x}$ sorry $\bar{x}$ = Px and design a an reduce order observer for the system in $\bar{x}$.

So in this so, my new A will be of the form ok, here I use x = P$\bar{x}$ simplicity; x = P$\bar{x}$. So, if I use the transformation it will just simply change to you can also use $\bar{x}$ = P x right nothing really changes just that instead of P AP^{-1} it will be P^{-1} A P. So, we just the transformation I will have my new a matrix $\bar{A}$ as $\begin{bmatrix} -2 & 1 \\ -2 & 0 \end{bmatrix}$; $\bar{B}$ is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and not surprisingly $\bar{C}$ will now have the form which I want write 1 and 0 like this ok.

Now once I have this I just use the expression to design the observer which is this one ok. I just substitute all the values and what I get is $\dot{w}$ as - $\bar{L}$w plus etc, will be plus you just substituting value right. So, this is now we have an observer of order 1 ok. I am just rushing through this steps but you could just substitute each of these values. And check should be pretty straightforward right ok.

Now, let us say I just arbitrarily choose this to be -10 and therefore, I have $\dot{w}$ = 10w - 122 y +10 u and w plus w (Refer Time: 35:05) also a w , + $\bar{L}$ z, now what is this? This is simply w - 10 y ok, this is an estimate for $\widehat{x_2}$. And therefore, y, w - 10 y is an estimate of $\bar{x}$ ok. So, this all I am in the new coordinate right.

This should be $\overline{x_2}$ hat and therefore, the original estimate x will simply be $P^{-1}\begin{bmatrix} y \\ w - 10y \end{bmatrix}$this will simply turn out to be $\begin{bmatrix} w - 10y \\ y \end{bmatrix}$ ok. So, and nothing much happening here the first step I do is to convert C the system into a form where the C matrix looks like this. The next step would be to follow these steps here ah.

And then what happens here is that this w + L z is an estimate of $\overline{x_2}$; because w is an estimate of this thing ok. So that is pretty straightforward right and then I can go back to the original transformation to see what how it looks like in the original coordinates ok. So, just as we saw I will re run the previous example with reduce order observer.

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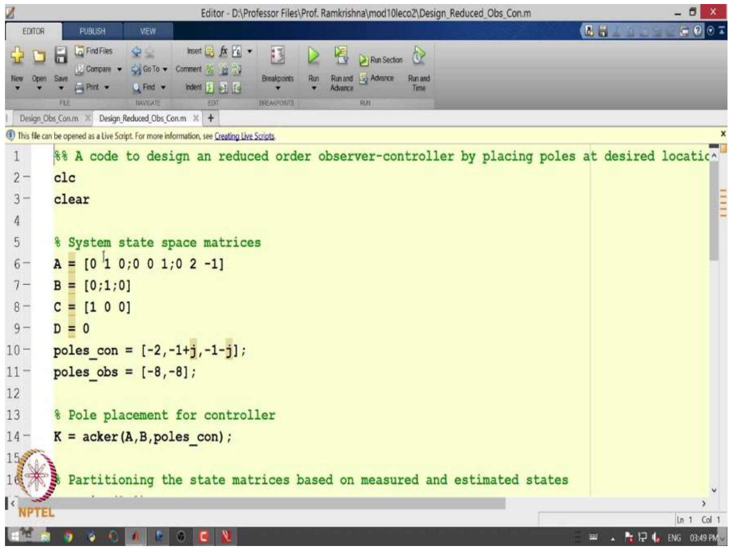

So, I have a third order system again A = $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$ C is [1 0 0] well I just for simplicity. I just choose it to be in a way that that suits me ok. And then now I have the desired poles of the closed loop system similarly as what I had earlier. Just that I am now designing a reduce order observer. So, the observer here will be of order 2 which was of three earlier. So, it will be just be -8, -8 instead of -8, -8, -8.

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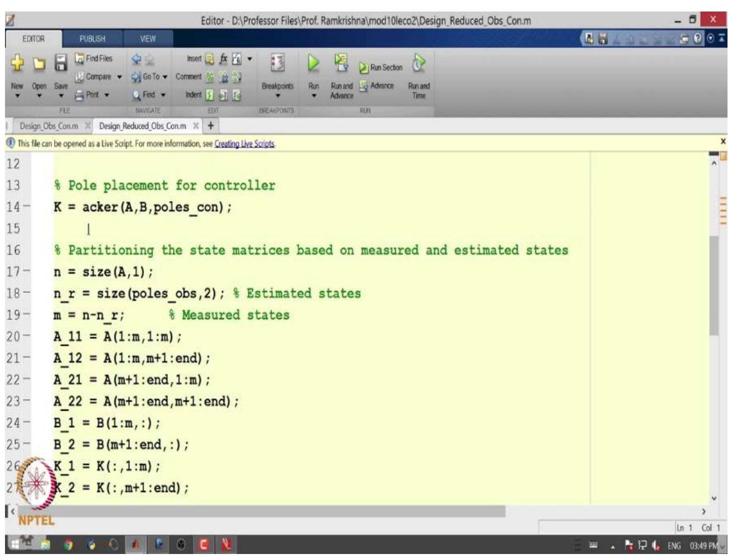

Then I just do the sa[me]- same thing for k to find the controller gain that will be the same ah. And then now I design a reduce observer based on the states which look like this right. So, I have to now design sorry design an observer for. Or in other words I can also say that I want to design a full state observer for a system which looks like this in x_2 right and a certain output here right.

So, a designing a reduce order observer for this system turns out to be designing a full order observer for this system over here ok. And then I just write down the equations in that form right. So, I just partition A into its appropriate matrices and so on.

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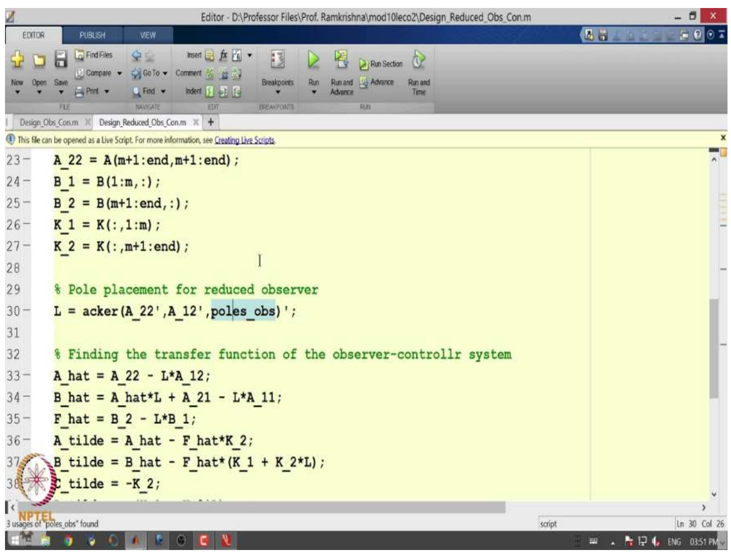

Design an observer for this A_{22} minus this pair A_{22}, A_{12} and this we know was observable right. And I just run the code for, so the observer design for this pair A_{22} and A_{12} with the poles being at -8, -8. And the rest of the process remains the same.

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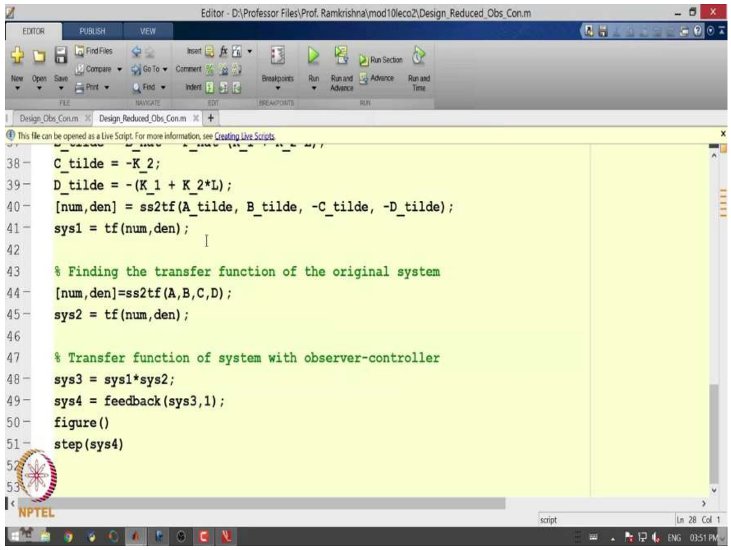

I will upload the code and you can just play around with this. And again the closed transfer function would look would look something like this. You can also look at changing the poles for example, to say.

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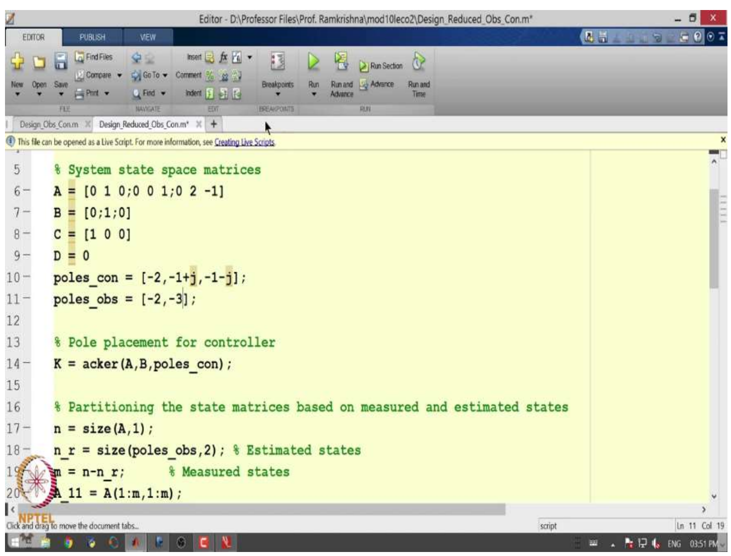

-2 and say -3 and check how the performance changes.

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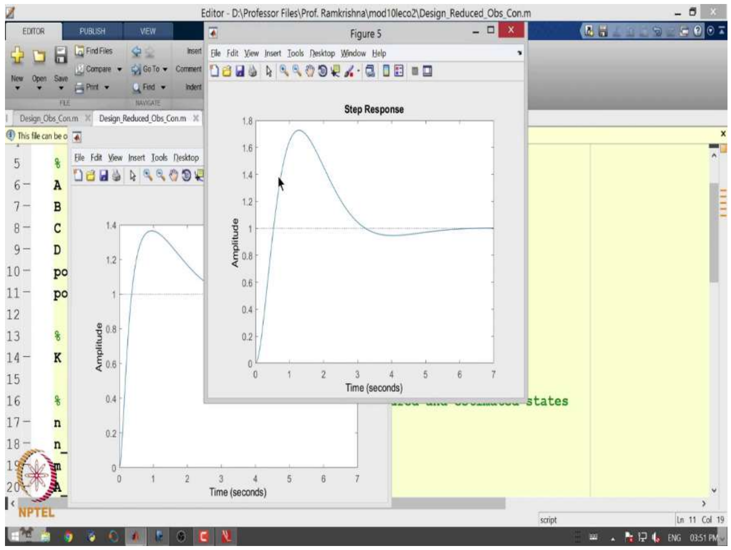

You can see how different locations of the observer poles affect the closed loop transfer function or the or the closed loop step response of the system ok. So, I will just leave to you to run a little more codes and check for yourself ok. So, this kind of concludes the lecture on observer and so simultaneous controller observer design and also the reduce order observer.

And what is important here is to decide where to place the observers. And one take away from those little graphs we plotted was that. The observer dynamics must be faster or the error must converge to 0 faster than the then the then the controller dynamics right. So, the observer pole should be further to the left then the desired poles of the closed loop plant system ok. So, that is what we had today right. So, in module 11, we will start with some basics of optimal control. And then end up with what is the famous Riccati equation ok.

Thanks for listening.