Linear Systems Theory
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Module - 10
Lecture - 01
Output Feedback

Hello everyone. Welcome to this week 10's lecture on the course on Linear Systems Theory. So, this week we will focus a bit on Output Feedback of the system. So, so far in our design methods of pole placements we saw controller designer via the standard state feedback such that the closed loop system is either stable or it has poles at some desired predefined desired locations.

Usually, the state may or may not be available for measurement in that case we may so, what is available for measurement are the outputs. So, in this lecture we will focus on how to design controllers well not the design will come up a bit later. But, what is the analysis part of it or what are the little concepts that we need to build on before we go to the design process when we have outputs for measurement.

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So this is to do with the concept called a state estimation, ok. So, why do we need this state estimation?

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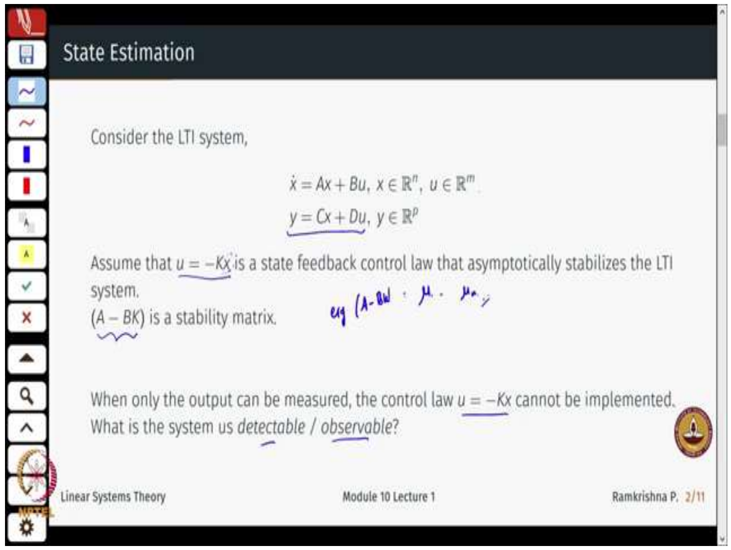

So, we start as usual with the LTI system $\dot{x}$ = A x + B u similarly can also do it for the discrete time versions so nothing really changes. So, the standard control problem which we saw until now was to design a state feedback of the form u = -K x such that A -BK is either a stability matrix or also has eigenvalues at some desired locations.

So, eigenvalues of A - BK could be at some pre determined locations you want till μ_n right. And then we saw a bunch of methods of how to construct the K matrix which was typically called pole placement. We started of techniques using just compare the co efficient of characteristics equations, or look at the controllable canonical form or even towards the end we have derived something called the acronym formula, ok.

So, that is; that was good until you could measure the x right. So, so what is crucial here is measuring this x but in most cases only the output is available for measurement. So, when the output is available for measurement the control law u= -K x cannot be directly implemented on the system, right. Because I really do not have access to all of the x's of the system; however, we know something called observable system and also detectable systems.

So, what we will look at in this lecture is to just to relate these two concepts and see can I design some control law that would still stabilize my system or have my closed loop poles at a μ_1 till μ_n, ok.

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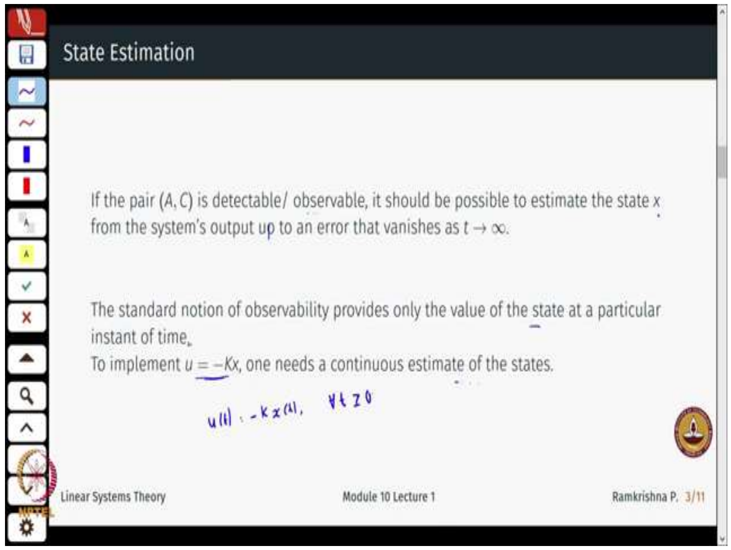

So, what was about state estimation? What is about let us begin with what was about observable or detectable. So, if the pair A, C is detectable or observable it should be possible to estimate the state x from the systems output up to an error that vanishes as t goes to infinity.

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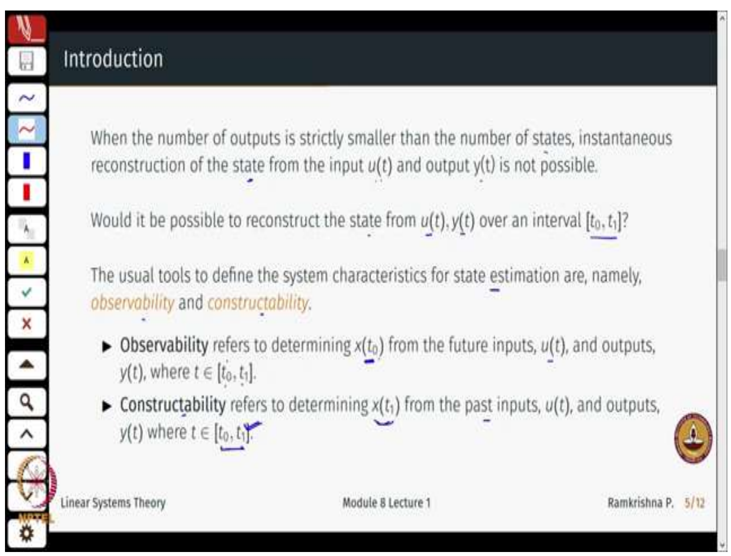

So, this was about the definition right. So, in observability, I was interested in measuring x(t_0) from the future inputs and outputs where you know the interval was given from t_0 to t_1. So, I was measuring x(t_0) right. So, that was my definition of observability and similarly with the constructability ok. So, what was happening here is if you look at x(t_0), I was only measuring the value of a state at a particular instant of time; whereas, the implementation u = -K x, one needs a continuous estimation of all the states, right.

So, just measuring x(t_0) or at initial time does not really solve by purpose, but what I want is a continuous feedback of states u = -K x. So, this if I write it in terms of time this should be something like this -K x(t) right may be for all t if my initial time is 0 right, ok.

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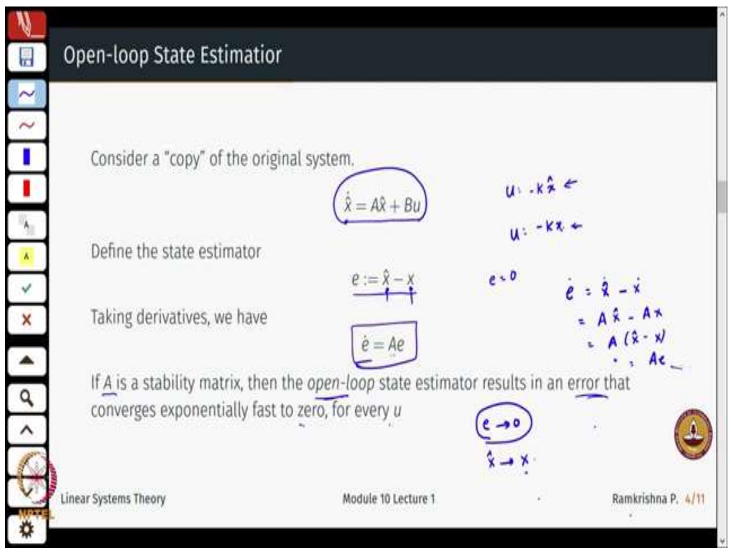

So, what do I do to get an estimate of this or to get an estimate of how the states looks look like. So, we start with what is called as open loop state estimator. And to construct this state estimator I start with what I call as the copy of the original system with new states let me call this x bar right.

So, let us; so, like typically in what I would like to do is to feed back this u = -K$\hat{x}$ instead of x equal to no instead of u= -K x ok; because this is not really available for measurement. But, I kind of construct the copy of the system with new states $\hat{x}$ or states or the copy or the system which is the copy of the system as states $\hat{x}$. In such a way that now I can define the error between the what I call as the estimated state and the actual state.

And what I really want is that the error should be equal to 0; that the estimated states which is being fed back is equal to the actual state of the system needs, right. So, if I just look at the error dynamics it is easy to derive that $\dot{e}$= $\dot{\hat{x}}$ - $\dot{x}$ that will be simply a of x cap minus A of x this simply be A, this x hat minus x. That will be a times c and the B is we just cancel out right.

So, I have a system which now looks like this right. So, $\dot{e}$ = A e right. Now I want to e to go to 0. So, when is this possible when does the error go to 0? Well, it is obvious so, far what we have learnt is that if A is the stability matrix. Then the open loop state estimator results in an error that converges exponentially fast to 0 for every u right. So, there is nothing that depending on u here, right.

So, I can I start with a copy of the system. And I see that as long as my system is stable, it results in an error that converts this to 0 when error converges to 0. I know that the estimator of this state converges actually to the original state, this is want for feedback right, ok.

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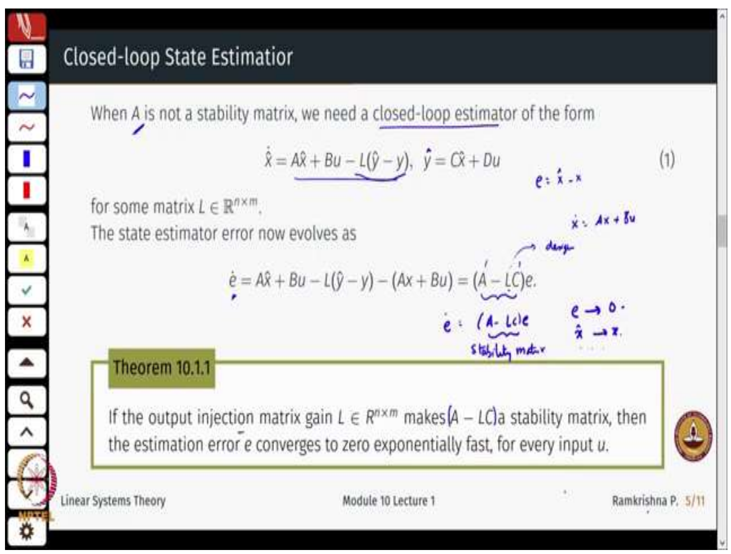

So, problems arise when A is not a stability matrix ok. So, what if A is not a stability matrix then let us construct, what we now call as a close loop estimator which looks like this. So, $\dot{\hat{x}}$= A$\hat{x}$ + B u – L($\hat{y}$ – y) or $\hat{y}$ minus y or y is; so this will also be a hat here, $\hat{y}$ = C$\hat{x}$ + D u, ok.

The error is again of the form what was here and that was $\hat{x}$ - x ok, and if I just rewrite the dynamics of $\dot{e}$ in terms of this $\dot{x}$ and of course, x evolves as A x + B u. Now, what I have is the error dynamics now as a function of A because this is known to me C; is known to me and some matrix L which we can design by this is right like design parameter or a design matrix.

So, if the L sorry, if my A matrix is not stable I can always choose a matrix L which makes A - LC a stability matrix, right. If A - LC is a stability matrix, then again e will converge to 0 and therefore, $\hat{x}$ will converge to x, ok. So, to summarize if the output injection matrix gain L, so I call this output injection matrix right.

So, this is available for design that makes A - LC a stability matrix, then the estimation error converges to 0 exponentially fast for every input u. And therefore, $\hat{x}$ converges to x. So, two things you have seen, to design an estimator like I need an estimate of the state, when the open loop system is stable then it is pretty straightforward right to design this.

Whereas, when A is not stable I need to do a little bit of modification to get an estimate to get a copy of the system which looks like this and the parameter L is free is a design parameter or a design matrix such that, A-L C should be a stability matrix. Now, obvious questions will arise how do I choose the L matrix can I always choose it or not and based on what we learnt in control ability or stabilizability the answer should be exactly guessable. It, you can guess under what condition A - LC is a stability matrix, when I have L as a design parameter or I can choose or I can construct the matrix L.

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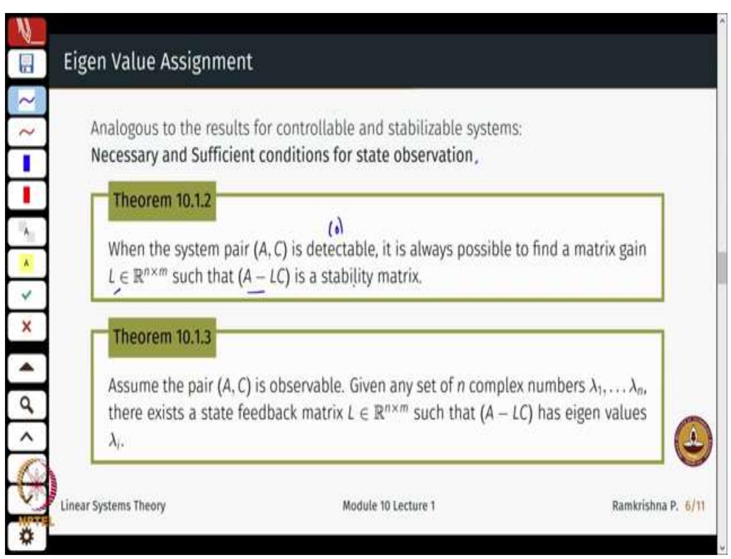

So, we will see what are this conditions. So, analogous to controllable and stabilizable systems we have couple of necessary and condition sufficient conditions for state observation. So, process when the pair A, C is detectable or even observable, right. If the pair A, C is observable or detectable then it is always possible to find gain matrix L such that A - LC is a stability matrix, ok.

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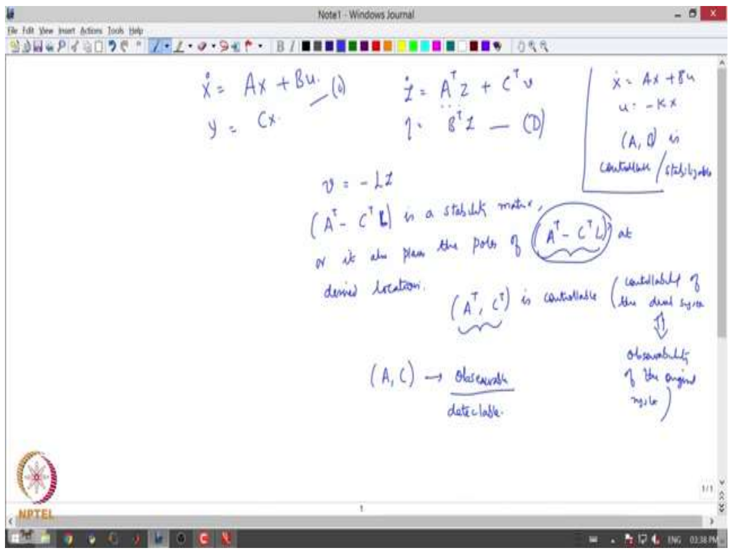

So, first let us try to understand this in terms of a duality and then come back to our system; to the original system. So, I have $\dot{x}$ = A x + B u and say for simplicity y is C x I kind of just ignore the d for the moment nothing really changes just with loss of generality. And say I have now dual system $\dot{z}$=A^Tz + $C^T v$ and say some, ok let us call this some and say some η = B^Tz ok.

So, what I want is that L that A - LC should be a stability matrix. So, here in this case I want to choose v of the form minus K sorry, let us call this L times z such that, A minus L, ok. Let us say A - C^TL is a stability matrix ok, this is L here right. Or in other words I can also place or it also yields or it also places the poles of A^T - C^TL at desired locations, ok.

Now, when is this possible? If I just look at so, this turns out to be like a standard pole placement problem for linear system which we saw earlier. So, what was necessary and sufficient condition for pole placement of $\dot{x}$= A x + B u with u = -K x. Well, that was at the A pair A, B is either controllable or the weaker version stabilizable, ok.

Now, similarly look at this here so, what I require for pole assignment like this or even stabilization thing is that the pair A,B is controllable, ok. Or if I so, control ability of the dual system so, this is controllability of the dual system of this is equal to observability of the original system. So, this is my original system, this is I can call my dual system ok. So, necessary and sufficient condition would now turn for this observer design is A times C is observable or at least detectable.   .

So that is what this result will tell us; when A, C is detectable or observable it is always possible to find gain matrix such that A - LC is a stability matrix. So, these are both necessary and sufficient conditions. Further now, what we saw here is in terms of not only that A minus not only this matrix is a stability matrix it can also place poles at desired locations.

So, under the assumption or if the pair A, C is observable and given any set of n complex numbers λ_1 till λ_n, there will always exist a state feedback matrix L such that A - LC has eigenvalues at precisely this locations. So, very similar to the pole placement problem that we saw while we were doing control design, right.

So, the duality is little a easier to check, we could still do this with standard observability definition and go on checking that these are actually necessary and sufficient conditions. But now, we have the very nice tool in terms of duality. So, so we will we may rather exploit that tool than just going for other regress proof of these two conditions ok.

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So, now how do I design a stabilizing controller now through what I call as output feedback, ok.

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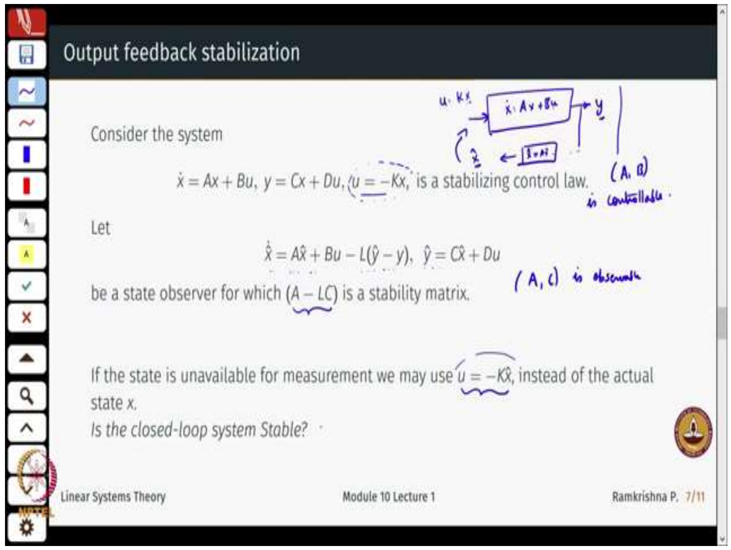

So, let us start again with the LTI system $\dot{x}$= A x + B u, y = C x + D u, and let me assume that u = -K x is a stabilizing control law. Similar things would also exist for discrete time system. So, whatever I am doing here can be directly translated to discrete time system with the appropriate definition of stability in terms of eigenvalues within the unit circle and so on, ok.

So, let me construct this observer of this form $\dot{x}$ or $\dot{\hat{x}}$ = A$\hat{x}$ + Bu – L($\hat{y}$ – y); $\hat{y}$ = C$\hat{x}$+ D u. So, let this be a state observer for which I can design in such a way that A - LC is a stability matrix which also means that I start with the assumption that A, C is observable and that the pair A,B is a controllable.

So, usually the state may not be available for measurement right. So, when the state is not available for measurement, I may want to use u =-K$\hat{x}$ which is the estimated state instead of the actual state, ok. Now once I do this how would my system look like, ok. So, loosely speaking what I m trying to do here is $\dot{x}$ = A x + B u where I would want u to be some K times x.

But I do not have x for measurement, but all I have for measurement is some output y. And based on this can I construct an observer at with this $\dot{\hat{x}}$ = A$\hat{x}$ and so on, which will give me an estimate of $\hat{x}$ and this $\hat{x}$ can be fed back as my input not a very good diagram, but something to just visualize what is happening. I do not have x for measurement, but based on this y, I construct this $\hat{x}$ and then feed this back to the controller via this form.

And now what do I know from previous slides is that; well, if as long as A - LC is a stability matrix $\hat{x}$ converges to x, ok. Now, what can I say about while I do this things there is some loop that is happening and is a close loop system stable, right. So, let us quickly verify that, ok.

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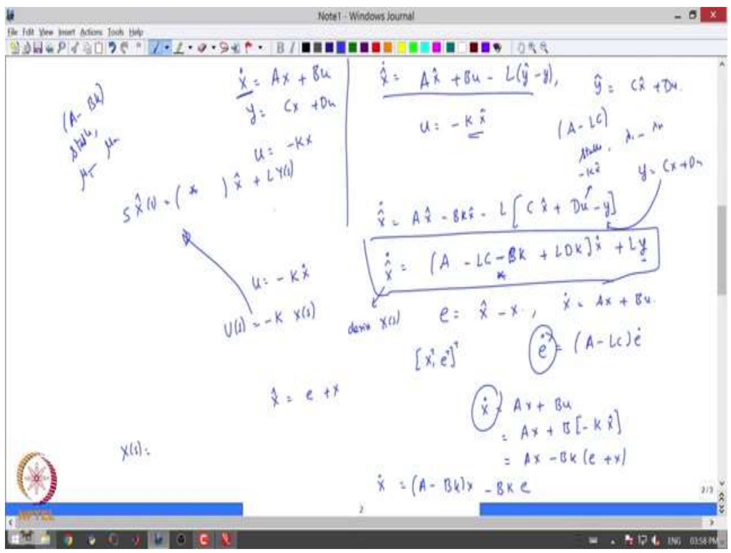

So, what do I have is $\dot{x}$ = A x +B u and say y = Cx + D u and a stabilizing controller of the form u = -K x. And the observer or the state estimator looks something like this A$\hat{x}$+B u – L($\hat{y}$ – y) and output looks something like this, ok. So, if I were to say I feedback u = -K$\hat{x}$ and let me check what happens to the overall system here not.

So, I have two things here right, one is the dynamics evolving in x; which should be such that u such that the pair A minus or the matrix A - BK should either be stable or should have eigenvalues at some locations μ_1 till μ_n ok. And moreover I have also have this observer dynamics in such a way that this matrix A - L C is stable.

And also has a eigenvalue let say some λ_1 till λ_n right. And I have all of this dynamics together must be stable, ok. So, $\dot{\hat{x}}$ = A$\hat{x}$ + B u - L; what is $\hat{y}$? $\hat{y}$ = C$\hat{x}$ minus sorry, plus D u - y, ok. We can also write this equivalently as $\dot{\hat{x}}$ = A - L C. Now, what is u? Right, u= -K$\hat{x}$. So, I can write this as BK$\hat{x}$. So, I will have plus sorry, B minus right u = -K$\hat{x}$. So, I will have a -BK right then, what is y? y = C x + D u I plug this here and I will have another term plus L DK $\hat{x}$, ok.

So, I start with the dynamics of $\hat{x}$ substitute for u = -K$\hat{x}$ and I have now something like this. $\dot{\hat{x}}$ = (A - LC - BK + LD K)$\hat{x}$ + Ly. So, what will remain is this L I can just put this as L times y. So, in this u also I substitute u = -K$\hat{x}$, ok. So, I just let this y be as it is, ok. So once I do this, I now want to look at how my error dynamics looks like right; e, which was $\hat{x}$ - x and I am also interested in how many original dynamics themselves perform right which is $\dot{x}$= Ax + B u, ok.

So, the total states that I am interested now are x , e let us do $[x^T\ e^T]^T$ right. So, first what do I have now about e? Right so, $\dot{e}$= (A- LC) e that is what we derived earlier ok. Moreover $\dot{x}$= A x + B u this is A x + B u is -K$\hat{x}$.

So, what is $\hat{x}$ in terms of e? $\hat{x}$ =e + x, so I will have this A x -BK times e plus x. So, this is x dot = (A – BK)x + - BKe and I will have -BK e. So, the overall system in terms of the states x and e will now look something like this, right.

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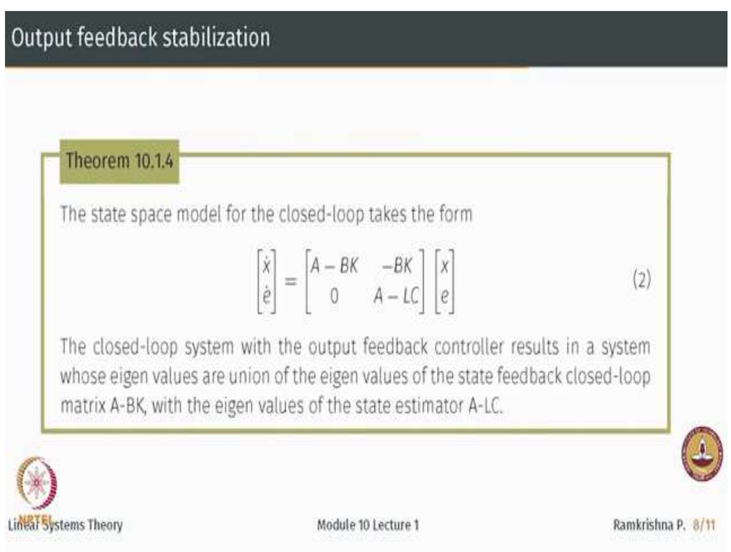

So, the state space model for the closed loop system takes this form, ok. Now well, I just want to look at the stability of this right. So, the closed loop system with the output feedback controller results in a system whose eigenvalues are the union of the eigenvalues of what do I do for the pole placement with the state feedback law. And also I am looking at the eigenvalues now of the state estimator with A - L C, ok. So, this will be the total eigenvalues of the closed loop system, right.

So, and therefore, the conclusion of stability will just be on where are my eigenvalues of A - BK and A - L C, right. So, the closed loop system is of course, stable because we start with the assumption that A B is a controllable or at stabilizable and A, C is observable or adverse it is detectable.

And therefore, the closed loop eigenvalues which are depending on eigenvalues or the union of the eigenvalues of A - BK and A - LC will form the eigen values of the total system. And therefore, the closed loop is actually stable, right. So, the answer here is yes. So one interesting thing at look at is when I introduce an observer in this system, right. So, starting with the state feedback u = -K x, I just have $\dot{x}$= A - BK and I can do a bunch of things from stabilizability to pole placement and so on.

Now if I put on top of it an observer, does it really affect my original system, right; does it also interfere in the in the poles of my original system of A - B k? So, whatever I do with the observer should not change really original design procedure, right. So, let us see what happens in that case, right. So, what is the effect of the additional observer on the closed loop system, ok. So, let us just try to derive few things here, ok.

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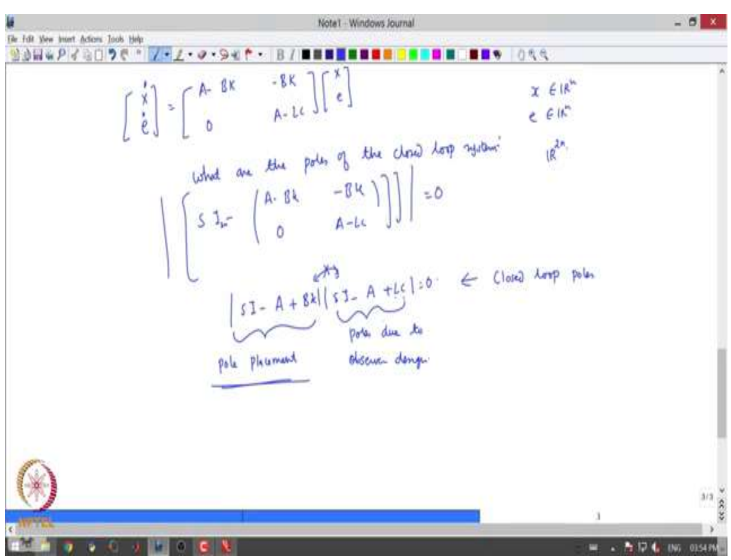

So, I have the closed loop dynamics in the state and the error in the following way, $\begin{bmatrix} A - Bk & -BK \\ 0 & A - LC \end{bmatrix}\begin{bmatrix} x \\ e \end{bmatrix}$. Now ok, what are the poles of this system? So I just do as sI – $\begin{bmatrix} A - Bk & -BK \\ 0 & A - LC \end{bmatrix}$ x sorry, S I minus A; the determinant of this equal to 0 will give me the characteristics equation ok.

So, remind you that x is in R^n and the error is also in R^n. So, the overall system will have of will be of dimension R^{2n}, ok. So, if I just solve for this it is easy to check that I am essentially solving for this one S I so, this will be this identity will be I to n right. So, the identity of two cross of dimension two, 2 times n so ok. So, I will have |sI - A + BK|sI-A+LC| = 0, ok.

So, this will give me the solution to this will be, will give me my closed loop poles. So what are the closed loops poles; well, the first n poles will be the poles which come as the result of pole placement, ok. And these are the poles due to observer designed and what we see is that this two do not interfere with each other, right. So, there is nothing here. So, whatever happens with the L, the C, the pole placement component will remain same and vice versa, right.

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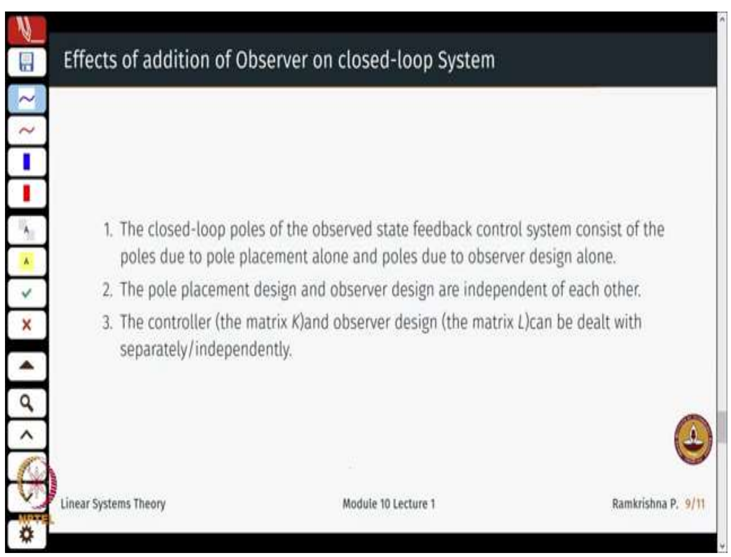

So, whenever I do simultaneous or design an observer and the controller, the closed loop poles of the system consist of the poles due to poles placement alone and poles due to observer design alone, nothing really changes right in the design, or the poles remain poles due to pole placement remain unchanged by the poles due to observer design and vice versa, ok.

So, therefore, we can design the controller matrix K and the observer design matrix L independently. So, one would possibly think that oh, when I am actually I design a controller and I just bring it to you as a designer observer for me. Does it change in have any change in the design procedure? No, I can actually design both of them separately though both need not be done simultaneously, right.

So, I design a controller, I design a observer plug them separately right and then everything works not that L will have some effect on K or K will have some effect on L. So, nothing like that happens, ok.

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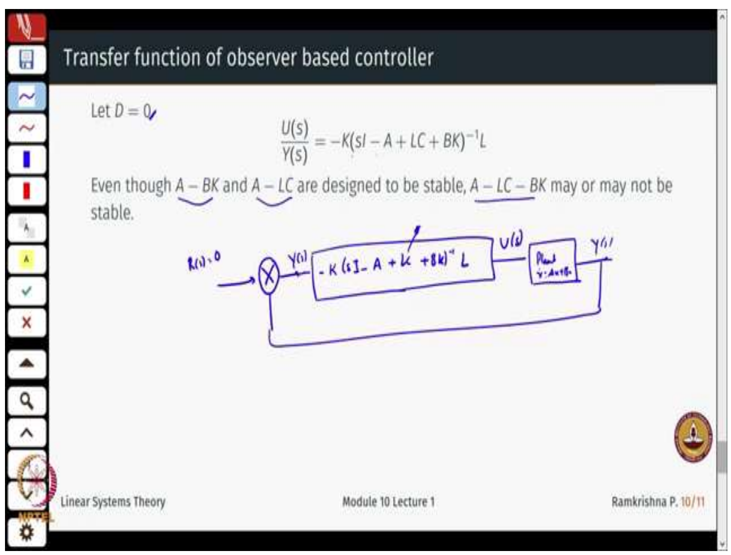

So, to end with we just look at what is the transfer function now of the observer based controller? Let us start with D equal to 0 and I just derive let us quickly see if we can derive this, ok. So, the way we derive the transfer function is look at this expression here, together with u = -K$\hat{x}$ or also mean that U(s) = -Kx(s).

From this expression I can derive x(s), what will x(s) be? X(s) here will be well you can just take the Laplace of this I will have s$\hat{x}$(s) is entire matrix. So, let me call this as star here $\hat{x}$ + L Y(s) ok. And then I just plug in for U over here and then just get an expression for relating y and u and it turns out that. Simple exercise may be just for simplicity assume D = 0 that $\frac{U(s)}{Y(s)}$ = -K$(sI - A + LC + BK)^{-1}$L, ok.

Little interesting thing just as a passing comment is that even though A - BK and A - LC are designed to be stable, this A- LC - BK need not be stable all the time ok. Let us just say how a typical block diagram might look like. So, let us say I have some reference signal say possibly 0 and I have say Y s here. So, this is essentially so, this entire thing will sit here. So, K$(sI - A + LC + BK)^{-1}$L may be with, ok and then this will be U(s) and this U(s) will go to my plant, which is of the form $\dot{x}$ = A x + B u.

And this will give me some measurement Y(s) and be fed back in this way ok. But even though ok; so, even though this may this need not be stable by itself; the overall closed loop system is of course, stable that is what we saw over here and this actually is a complete stability matrix.

So, I just next time we will do a little block diagram version of the each of these components of what is measured? What is computed? What is fed back and so on when we do the actual design?

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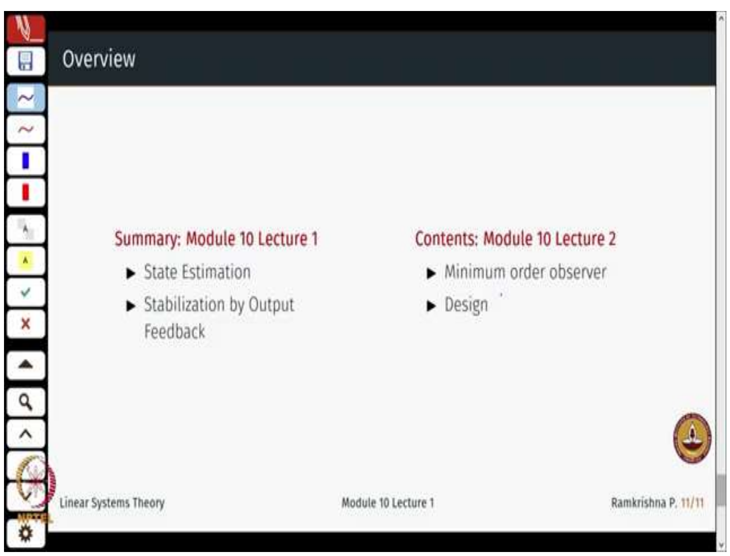

So, far in this lecture we saw about state estimation, we also saw how to stabilize via output feedback ok. In the next lecture we will do in addition to solving design problems which involve both designing the controller and the observer at the same time. We also look at what is called as a minimum order observer. So, that will come up in next lecture.

Thanks for watching.