Linear Systems Theory
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Module - 09
Lecture – 04
Tutorial for Modules 09 and 10

Hello everyone, in this tutorial, we will discuss some problems relating to the content module 9 and 10. So, we will try to solve about three or four problems. And we will also upload an exercise which will consist of about 9 to 10 problems which you can solve for yourself.

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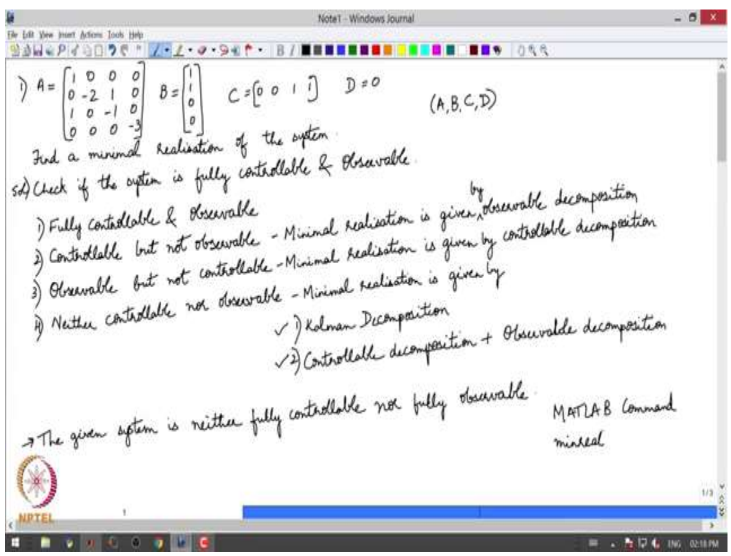

So, going to the first problem which is as follows. Given a state space model with the following matrices a $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$. So, this is a state matrix and then the input matrix is $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ which means there is a single input, and the output matrix is given by [0 0 1 1], and D we take it to be 0. So, given this system, we are asked to find a minimal realization. So, what is the realization, we saw that if any system is represented in this manner, A, B, C, D matrices, then it is called one of the realizations.

And we have seen that any system which can be realized from a transfer function, it will have infinite possible realization. So, one realization is already given to us. And using this realization we are asked to find out another realization which is minimal. So, even minimal realizations there could be many we are asked to find one of them. Now, how can we go about this, first we need to check if the system is fully controllable and observable.

So, once we check this, we arrive at four cases. The case one being the system being fully controllable and observable; second one being the system being controllable, but not fully observable; third one being a system being observable, but not fully controllable; the fourth one being system being neither fully controllable nor observable.

So, in the lectures we have seen that when the system is fully observable and uncontrollable, then the given realization is the minimum realization because we know that the minimal realization always is in fully controllable and observable form. But in the cases when the system is not fully controllable or not fully observable or neither of them is being satisfied, then we need to find a lower order system which is minimal. So, now, when the system is not fully observable but it is controllable, then simple way of finding minimal realization is given by the observable decomposition.

So, you just take a system decompose it into observable and unobservable parts, then the observable part will be the minimal realization. So, in this case, when the system is not fully controllable, then the minimal realization is given by the controllable decomposition. So, now, what if happens that the system is neither fully observable nor fully controllable, then minimal realization is given by you can do this in two ways, one is using Kalman decomposition, but finding the basis for a Kalman decomposition is very difficult. So, we have an alternate way of doing controllable decomposition plus observable decomposition. So, we will do one after the other.

So, if we take the system, this is the given system is neither fully controllable nor fully observable you can check this using the controllability and observability matrices, the rank of those matrices by tell you that the system neither fully controllable nor observable. Now, so we need either need to find the Kalman decomposition or we have to do first a controllable decomposition and then an observable decomposition. So, as a system is fourth order system instead of doing this manually we will try to solve this using matlab.

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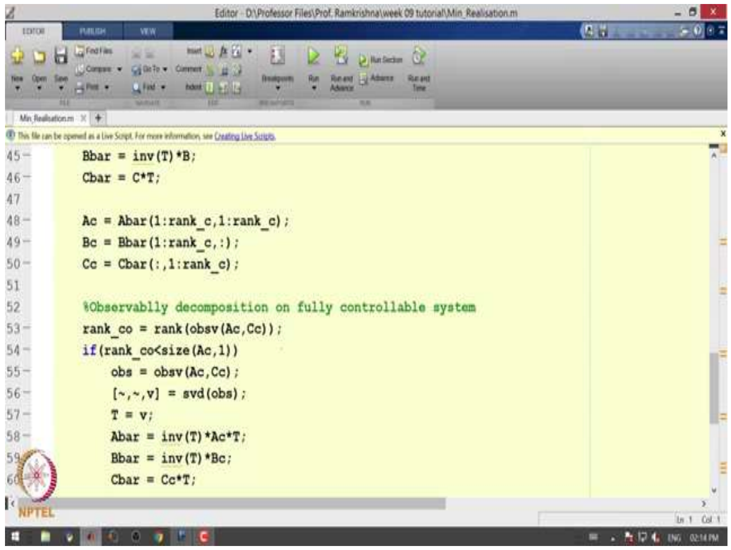

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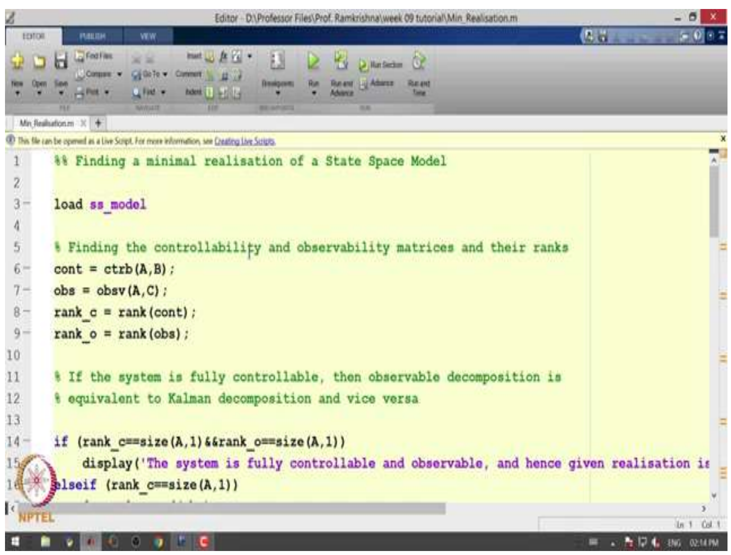

So, in this case, I have written a code which will give you the minimal realization of any given state space model. So, the first part of the code loads the model and then we are finding out the controllability and observability matrix using this command ctrb and obsv, and then we check the ranks.

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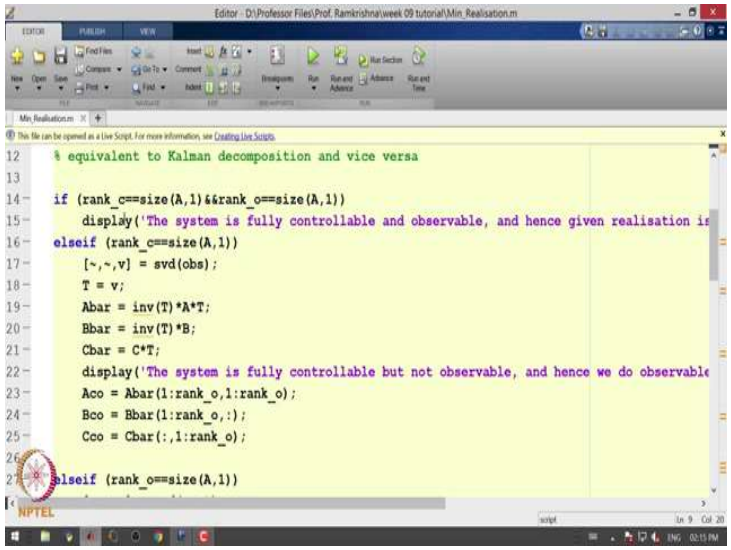

So, now the code goes like this. If both ranks are full, then we know that the system is fully controllable and observable hence the realization is given to be minimal. But, what is the controllability matrix rank is full, but not the observability rank. So, in this case we follow the observable decomposition. So, this is what we are doing. To find the observable decomposition, we take the svd of the observability matrix and then use the eigen vectors in V as the model matrix for transformation, and we do the following transformation and then we get this.

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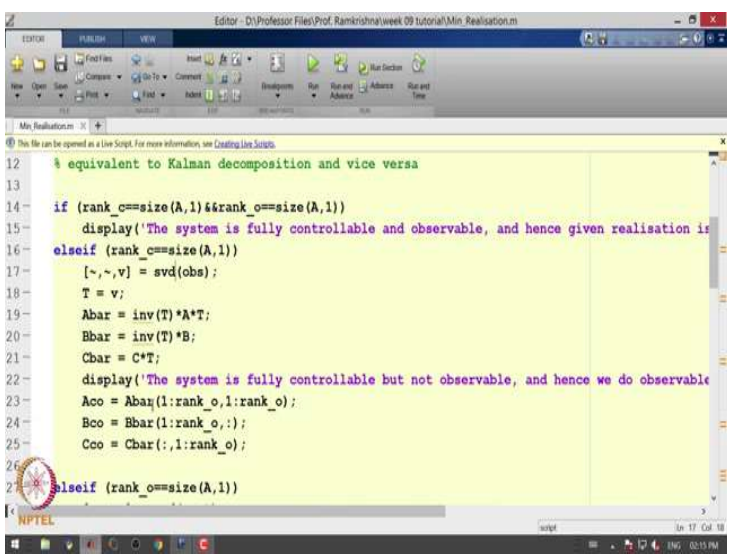

So, now the system is in minimal form because we already know that it is fully controllable.

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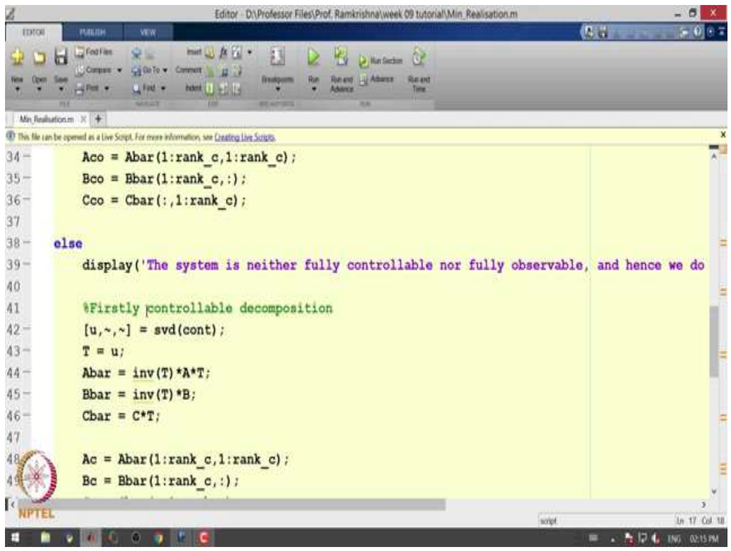

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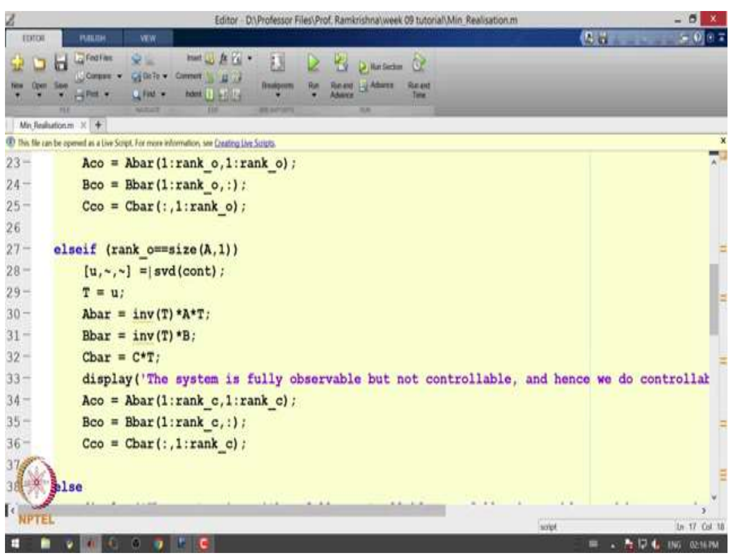

What if it is otherwise the system is fully controllable, fully observable, but not controllable, then we do the reverse we find the svd of the controllability matrix, then using U we transform the system into a form which will give us the controllable observable system which is minimal in nature.

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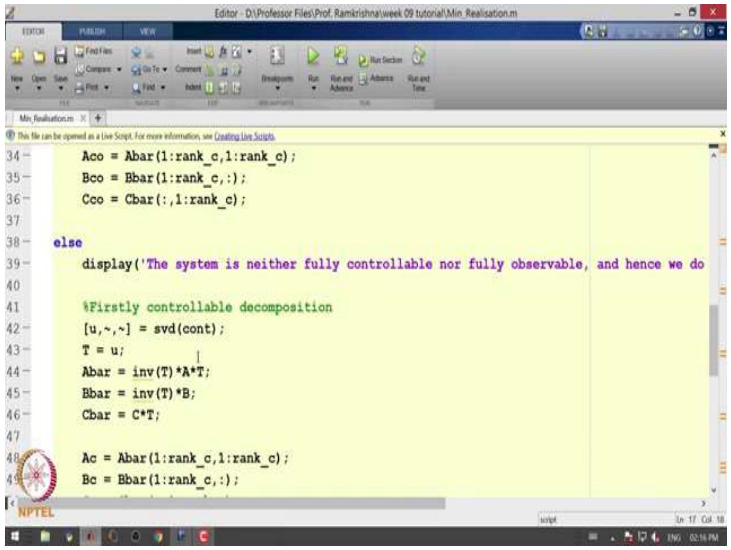

So, but our case is the third case where it is not fully observable and not fully controllable. In this case, what we will first do is we will take the controllable decomposition and then we transform it into a controllable form.

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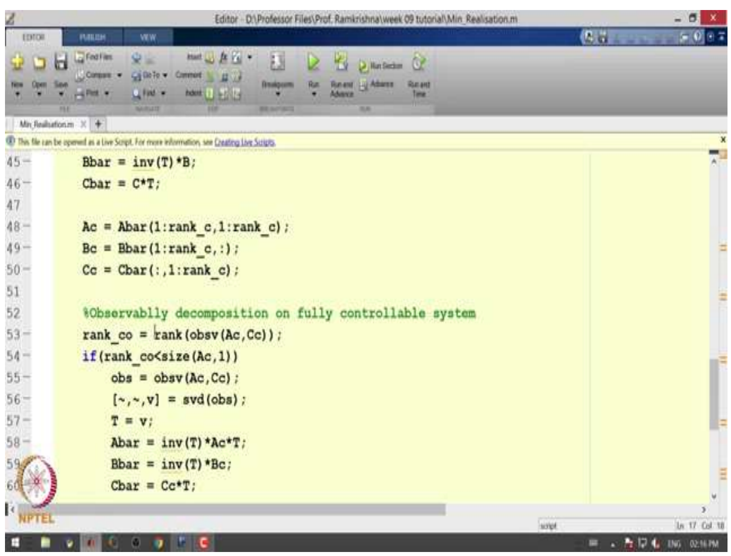

We will take the controllability part. And then we will take an observable decomposition on the fully controllable system. So, this is what we do.

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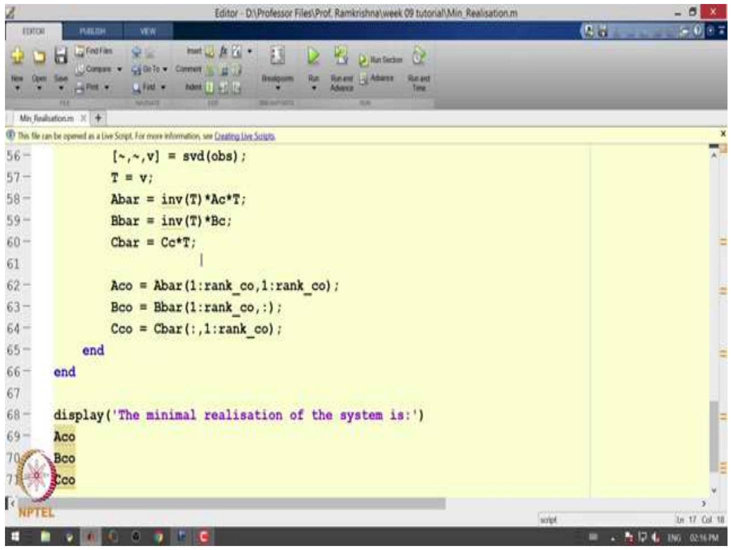

And if the rank is less than the full rank, then again we do again we find out the observability matrix then we take the svd of the observability matrix and then transform is follows. So, this code will run the whole thing, I already copied the whole model into this variable call ss underscore models. So, I will run this code and you will see what happens.

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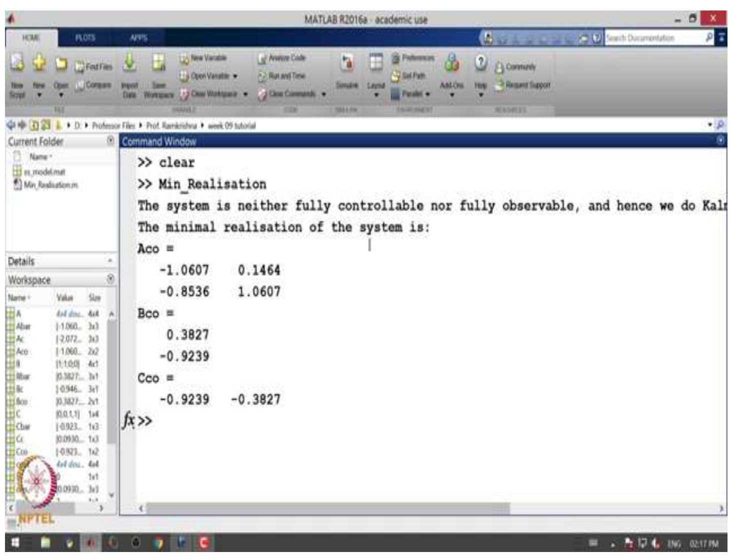

So, as you can see the system is neither fully controllable not observable and we will do Kalman decomposition, so which is alternately done using first the controllable decomposition and then the observable decomposition and after doing that the minimal realization is given by this matrices. So, I will upload this code, you can try it out on different matrices. And so this is the way to understand how minimal decomposition is done. You can also use a command called min real. So, there is a MATLAB command min real, which will also give you the minimal realization, but it might not be the same as what you obtained using this way. But you can still verify that both the minimal realizations will lead you to the same transfer function. So, I will leave that for you as an exercise.

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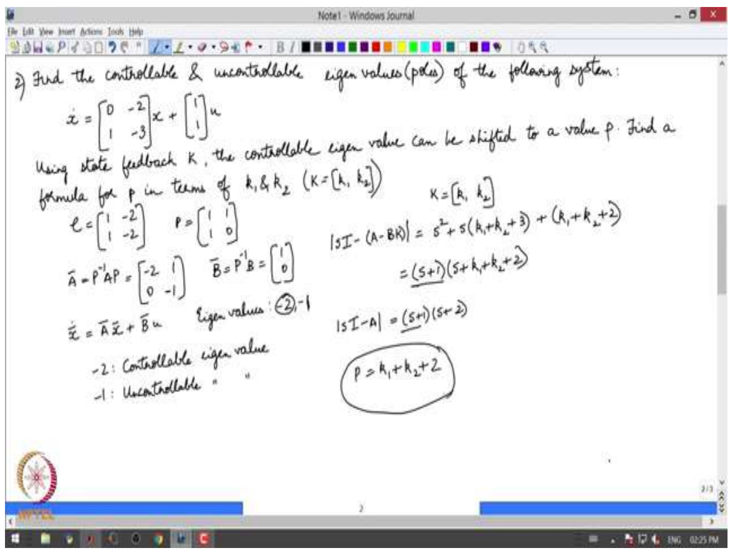

So, going into the second problem, the second problem we are asked to find the controllable and uncontrollable eigenvalues which are nothing but the poles of the following system $\dot{x}$ dot is equals to which is the A matrix A x plus B is $\begin{bmatrix}1\\1\end{bmatrix}$1 1 times u a single input. And then after finding out the controllable and uncontrollable eigenvalues, using state feedback K the controllable eigenvalue can be shifted to a location to value p. And we are asked to find a formula for p in terms of k_1 and k_2, these are the variables. So, K is $[k_1\ k_2]$. So, it is already said that there exists a controllable and uncontrollable eigenvalues. So, we can always find one pole to be controllable and other pole to be uncontrollable. So, how can we find out? So, a simple way is to actually find out the controllable decomposition of the system.

So, you can always find out the matrix ok. I will find out the controllability matrix, you can just verify it will be $\begin{bmatrix}1-2\\1-2\end{bmatrix}$. So, its rank is 1. So, I will take a model matrix p, it consists of the first column from here and another column which is linearly independent of the first one, it could be 1 0. So, now, using this, I will find out $\bar{A} = P^{-1}$ AP. And if I do that I will get $\begin{bmatrix}-2 & 1\\ 0 & -1\end{bmatrix}$. And similarly $\bar{B} = P^{-1}$B in this I will get it to be $\begin{bmatrix}1\\0\end{bmatrix}$.

So, the transform system is $\bar{A}$x + $\bar{B}$u. So, now, you can see that only the first state is controllable which means the first eigenvalue is controllable while the second eigenvalue is not controllable sorry this minus 1. So, what are the eigenvalues; what the first eigenvalue and second eigenvalue are clearly -2 and -1, because this is a upper triangular matrix. I can always write the diagonal matrix diagonal elements as the eigenvalue.

So, now the first eigenvalues controllable because one is present in u in the first element in $\bar{B}$ and the second eigenvalue is not controllable. So, now, how can we find out a formula for P in terms of k_1 and k_2 to control this eigenvalue -2. So, -2 is the controllable eigenvalue or the pole -1 is uncontrollable eigenvalues, so which means that I can always change -2, but I -1 always remains to be a pole of the system that cannot be controlled, ok.

So, now we will take this K, K = $[k_1\ \ k_2]$ and using state feedback, I find out the characteristic matrix of the system with state feedback I can say it A - B K. So, if you find out the characteristic equation of the system, it will turn out to be like this s^2 + s($k_1 + k_2 + 3$) + (k_1 + k_2 + 2). So, we can just substitute A, B and k and find out this characteristic equation. Now, if you observe, this can be written as follows, (s+1)(s+k_1+$k_2 + 2$). Now, you can see that the original characteristic equation sI - A will be (s+1)(s+2) because -1 and -2 are the eigenvalues.

Now, once I introduce state feedback this S plus 1 still remains, but (s+2) gets replaced by (s+$k_1 + k_2 + 2$) which means that now I can change my values or choose my values k_1 and k_2 such that I can place this -2 anywhere I want. So, the formula for P will be k_1 +k_2+2. So, this is how we solve this problem. Now, going into the next problem, so this problem pertains to the tenth module it this is about observer design. So, this how the problem goes.

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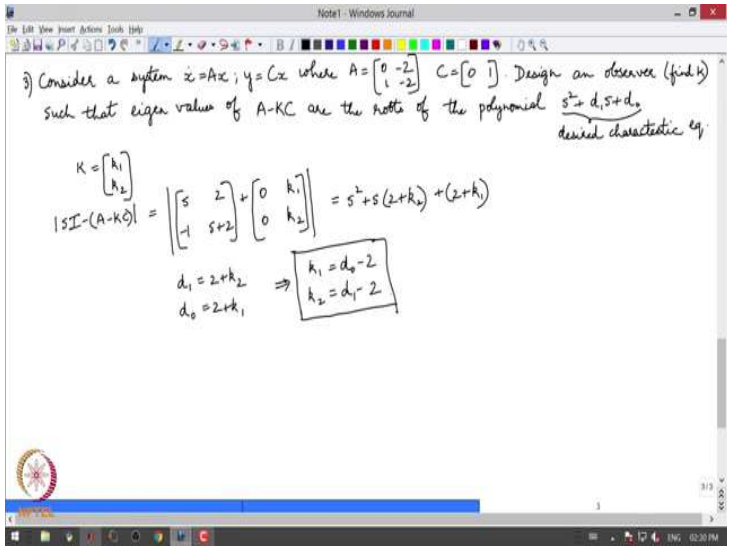

So, in this problem we are asked to design an observer which means we need to find an observer gain matrix K such that we have a system $\dot{x}$=A x, and y = C x. And we want to place the eigenvalues of the observer a -K C such that they are the roots of the polynomial s^2 + d_1s + d_0. So, this is the desired characteristic equation.

So, this is very similar to the controller design that we did in the previous module. So, how do we go about it? So, as per the dimensions K needs to be a column vector $\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$. So, now, what we will do is we will find out the characteristic equation which I can write it as now if I will expand this I will be getting s^2 + s(2 + k_2) + (2 + k_1). So, we have a characteristic equation coming from A K and C. And now if we compare it with this given characteristic equation, then we can simply say that d_1= 2 + $k_{2,}$ d_2 = 2+k_1 which implies that k_1 is equals to sorry not d_2 this is d_0. So, k_1 is = d_0- 2 and k_2 = d_1- 2.

So, this is how we can place the eigenvalues or the poles of the observer system such that they are equal to the roots of this given polynomial. So, this brings us to the end of this tutorial; as I said at the beginning of the tutorial we will be uploading some exercise problems which are somewhat similar or somewhat similar to the content that is covered in modules 9 and 10. You can go through them and post any queries in the forum if you have.

Thank you.