Linear Systems Theory
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Module - 09
Lecture - 03
Controller Design using Pole Placement

Hi everyone. So, in this lecture of lecture 3 of week 9, we will look at explicit Design methods for arbitrary Pole Placement. So, and then we will also eventually prove that a necessary and sufficient condition for arbitrary pole placement is that the system should be completely controllable, ok.

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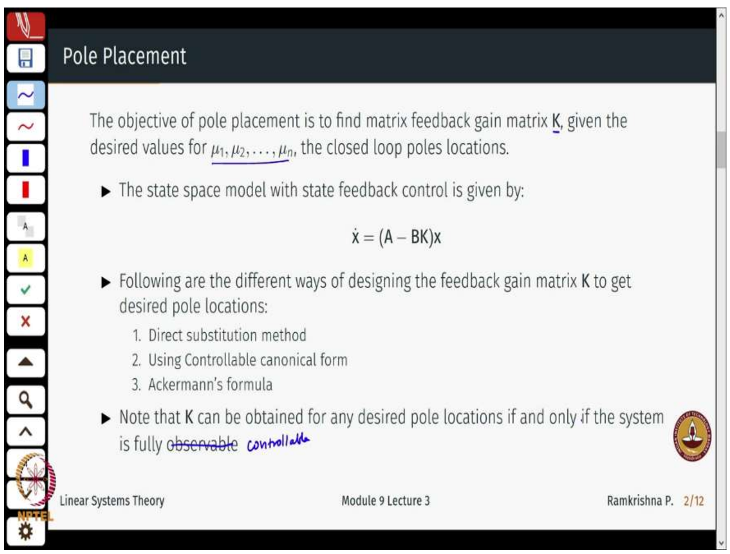

So, let us start with one of the very basic methods of pole placement, ok. So, what is the objective again? The objective is to find a feedback matrix K, such that the close loop poles have the desired locations from μ_1 until till μ_n, ok. So, we will look at these 3 methods. So, should be controllable here, ok. So, towards the end of the lecture we will prove this that the K can be obtained if and only if the system is completely controllable. So, we will first start with the direct substitution method, ok.

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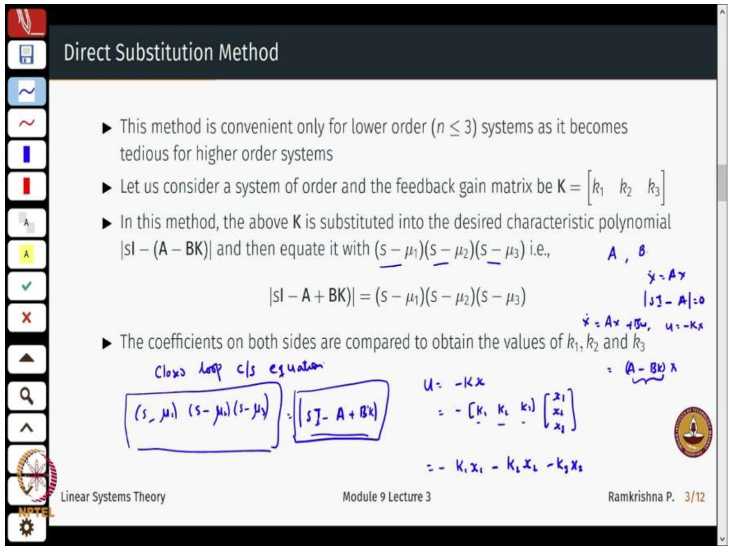

So, let us say I am looking at a as a third order system. So, my feedback law u -Kx will be of the form say $-[k_1 k_2 \; k_3]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, it will be given as $-k_1x_1$ -k_2x_2-k_3x_3, ok.

So, what is known to us? I know A, I know B, additionally what do I know is that, additionally I also know the closed loop characteristic equation, ok. Where does the close loop characteristic equation come from? That comes from the location of the desired poles. So, I have (s - μ_1)(s-μ_2)($s - \mu_3$), ok. So, the everything in this expression is known to me, ok.

Now, what is unknown? Unknown is these 3 numbers k_1, k_2, k_3, ok. Now, what is the characteristic equation of the close loop system? That will just be the determinant of this (s I -A + Bk), ok. Where did this come from if I just look at $\dot{x}$ = Ax I am looking at the characteristic equation of this form (sI – A) = 0.

Now, if I have input to the system $\dot{x}$ = Ax + B u with u of the form - Kx, then my system close loop system takes this form A - BK x, and the poles will now correspond to the characteristic equation which is obtained by this matrix A - B k. And therefore, the close loop characteristic equation will be of this form (sI - A + BK), and all the unknowns now are in this matrix k, ok.

So, now this kind of looks on the left hand side I have a equation of all the everything is known to me, on the right hand side I have equations of 3 unknowns and I can just solve it by via some simple simultaneous equations, ok.

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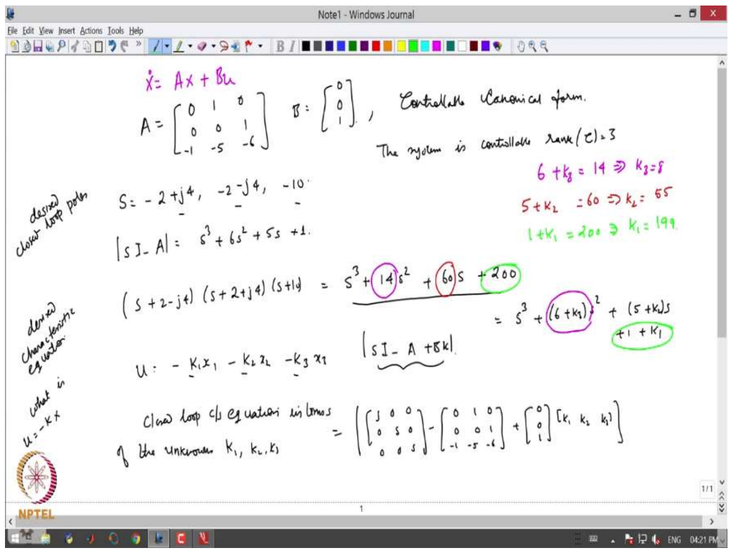

I just do a very small example on this. So, I have $\dot{x}$ = Ax + B u, ok. I will go back to the black colour, ok, where A = $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}$, B is also of the form $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. And let us say easy to check that this is actually in the controllable canonical form, right.

So, let us say I want to place my close loop poles in the following locations. So, these are the desired poles. So, one pole is decided to be at 2 + j 4, the other would obviously, then be at 2 - j 4 and the third pole would be at -10, ok. So, first is, ok, check if the system is controllable. Well, the answer is yes, a the system is actually controllable.

I leave the steps for you to verify that you know you can gate easily check that the rank of C is equal to 3 and also because I have it in the controllable canonical form of the system is actually controllable, ok. And the characteristic equation of the uncontrolled or the open loop system would just be of this form s^3 + $6s^2$ + 5s + 1, this is what is given to me.

So, now based on the close loop poles can I find out what is the desired characteristic equation? Ok. That will be from, (s + 2 - j4)(s + 2 + j4) (s + 10), right, from here. So, this is how the close loop characteristic equation knows and I know what is μ_1, μ_2, μ_3, which are essentially these 3 numbers here. And this looks in the following this is s^3 + 14s^2 + 60s + 200, ok. So, now, what do I need to find? So, I know, I know this, right. Now, I need to find what is u = -kx that is useless. So, u will be of the form -k_1x_1 -k_2x_2-k_3x_3 or I am looking at the characteristic equation of the close loop system which is (S I - A + BK), ok.

Now, if I compute this, if I compute this in terms of this unknowns k_1, k_2 and k_3 it turns out that the close loop characteristic equation, in terms of the unknowns k_1, k_2, k_3 is I am just looking at the determinant of this. So, this will be, $\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix}$ - $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}$ + $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}[k_1 k_2 \ k_3]$, ok.

I can expand this and then this is s^3 + (6+k_3)s^2 + (5+k_2)s + (1+k_1). Now, I can actually look compare the coefficients now, right. The coefficient of s^2 here is 6 +k_3 which must be equal to 14, and therefore, k_3, so with this to I will have 6 + k_3= 14 which means k_3 = 8.

Similarly, I can look at the coefficient of S, it is a 5 k_2 times S is 60 and therefore, k_2 is sorry, there should be a, this should be (5 + k_2)s, 5 + k 2 is say 60, therefore k_2 = 55 and the last expression says, 1 + k_1 = 200 and therefore, k_1= 199, ok. That is looks pretty neat and straightforward, right. I can do everything with hand ah. But where the drawback is if as n becomes much larger these computations become tedious to do by hand, ok.

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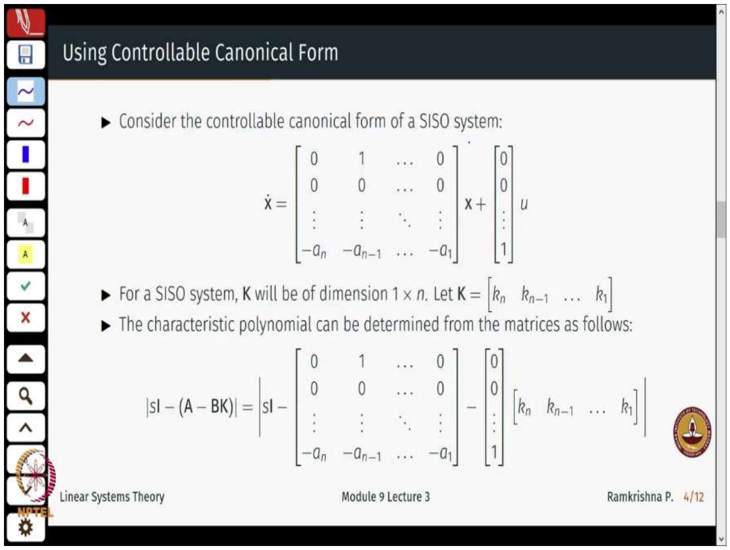

Now, what is the second method? The second method comes via the controllable canonical form, right. So, assume that my system is in the controllable canonical form, if it is not in the controllable canonical form I actually know how it works, right. I will tell you how to even deal with the system which is not in controllable canonical form. At the moment I assume that let the system be in controllable canonical form, ok.

Again I will do it for the case of the SISO system. And again my problem is to find these unknowns, K which is from k_n, k_{n-1} till k_1, right. Again, I am now looking at the close loop characteristic equation, again obtained by sI –(A – BK), ok. Let us actually do this the do this computations and check where we are, ok.

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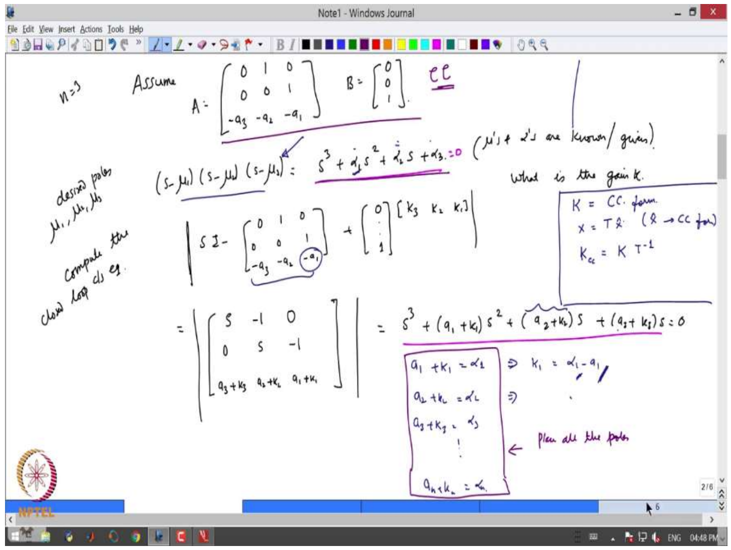

Where do I start from? So, I start from assume that the system is in the controllable canonical form that, and also I assume that n equal to 3, so that my computations become easy and they also look a little neat, and would not look too tedious. So, I have a $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}$. This will be my A matrix, B will be of the form $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, ok.

Now, what do I know? I know the following, right I know the close loop poles are the same steps like here, right. So, I know the desired poles. So, with the desired poles let them be at some arbitrary μ_1, μ_2 and μ_3 such that (s - μ_1)(s -μ_2)(s - μ_3) expands this one s^n + alpha 1. So, for the case of n = 3 this will be s^3+ $\alpha_1 s^2$, α_2s + α_3. Let us say all these are known to the μs and αs are known. Now, they are given to us, right, ok.

Now, what should I find is what is what is K, right. So, I compute close loop characteristic equation which for the case the 3 by 3 case will just be like this as I have sI - A which is $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}$, plus have a B times K here $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $[k_1 k_2\ k_3]$, ok. So, how will this look like? So, this will be S here 1 to the minus 0 0 S, another minus 1 and here will be $a_3 + k_3$, here we will have a_2 + k_2, here we will have a_1 +k_1, ok.

And then I am looking at the determinant of this. And this turns out that it will be something like this I have s^n + a_1 +k_1. So, in this case I am looking at s^3; s^3 + (a_1 +k_1)s^2+(a_2 +k_2)s+ a_3 +$k_3 = 0$. So, I just look at these two equations now, right. So, this which, ok, this is a characteristic equation this and then I go about comparing the coefficients, ok.

What do I have now from the first term is that a_1 +k_1 = α_1, ok. I know what is what is a 1 that comes from my given system, I know what is α_1 that comes from by desired roots, so therefore, I can compute k_1 as $\alpha_1 - a_1$. Similarly, I will have $a_2 + k_2$ by equating the coefficient of S here, with the coefficient of S that is $\alpha_{2,}$ $a_3 + k_3$ = α_3 and so on, if I have a system of dimension n, ok.

Now; so this is ok. What is a subtle difference between here? Right. So, here if I know, ok. What is I know; do I know a 1? Well, a 1 is directly given from here. Do I know α_1? Well, the answer is yes, ok. So, give me this and give me this I can directly compute what is the k 1, similarly for k_2 and so on. A little distinction from here is that we were trying to explicitly compute the close loop characteristic equation in terms of k_1, k_2, k_3 and so on.

But here, just give me these two numbers and so, this can be can be obtained α, the αs can be obtained by the close loop characteristic equation a_1, a_2, a_3 are known to me, ok. So, a little caveat and which is not very bad for us is that the system is assumed to be in the controllable canonical form. So, the K, I obtained here is in the controllable canonical form, ok.

And let me assume now that well I do system is not in the controllable canonical form, but I get it into the controllable canonical form via some transformation x = T$\hat{x}$, and the system in $\hat{x}$ is in the controllable canonical form, ok. So, the K I compute here, the K I compute in the controllable canonical form would just be related to the original K via just this transform, ok, that is that is an easy process to check.

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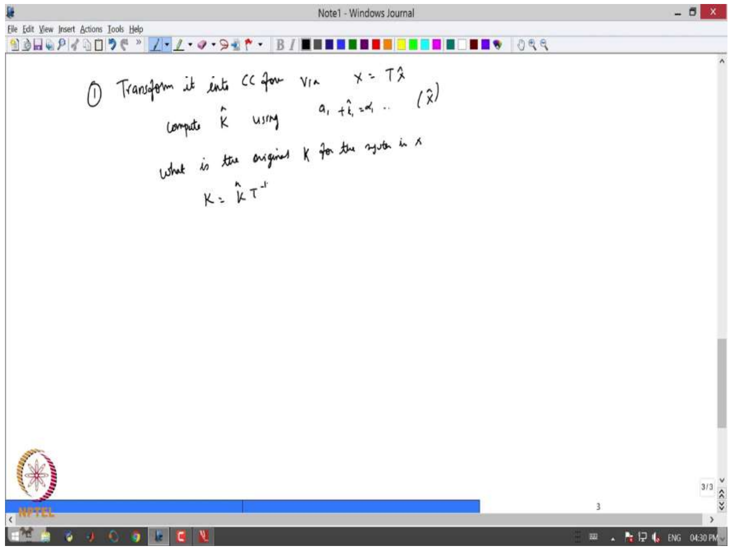

So, if I do not have the system in the controllable canonical form. So, we decided this distance, right. So, if I do not have the system in the controllable canonical form first transform it into this controllable canonical form via x = T$\hat{x}$, ok. Now, compute K, let me call this $\widehat{K}$ because I am looking now at the controllable canonical form using these formulas here, right, that, ok, what did I have; a_1 + $\widehat{k_1}$ = α_1 and so on, ok.

Now, what is the original K? Original K for the system in x well this all this was in the x cap, right well the k turns out simply to be $\widehat{K}T^{-1}$, ok. It is a very straightforward thing once you understand how one can arrive at the similarity transformation, right starting from any given system, ok.

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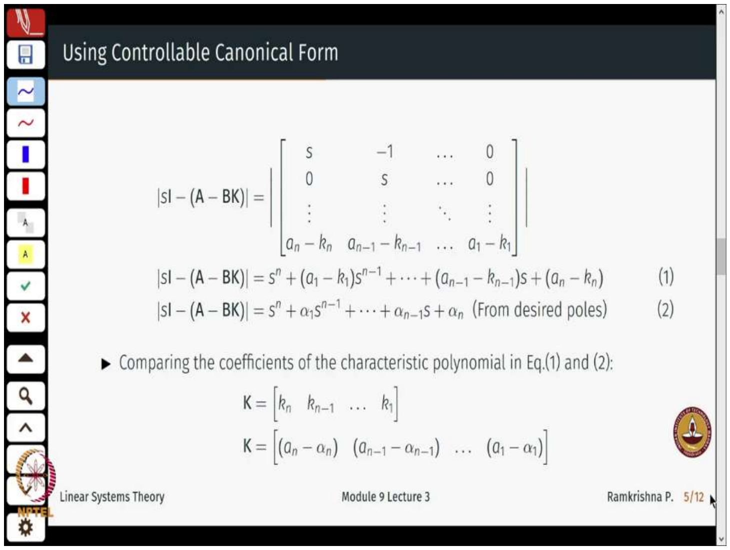

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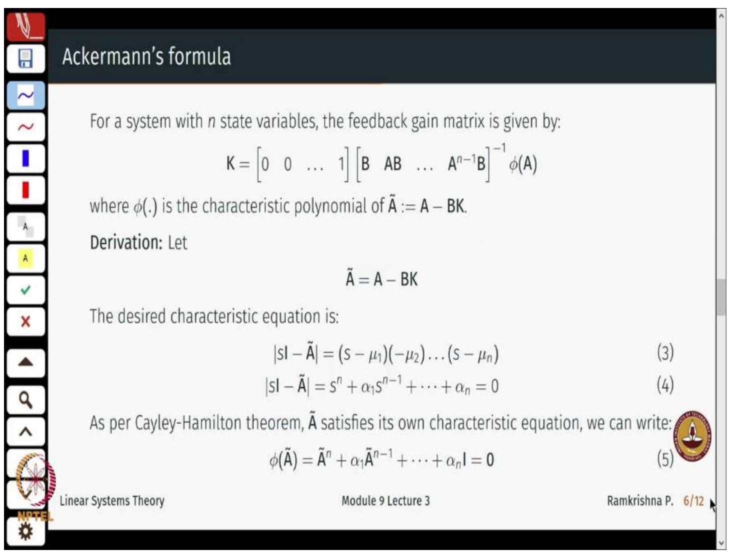

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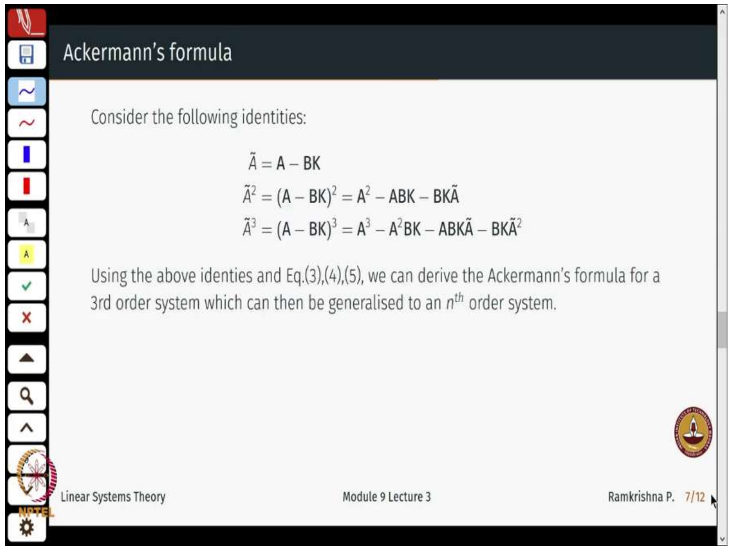

So, that is pretty easy and neat ah. So, I will just do that the last thing which is a little tedious, but it is it is nice to know the Ackermann's formula, ok.

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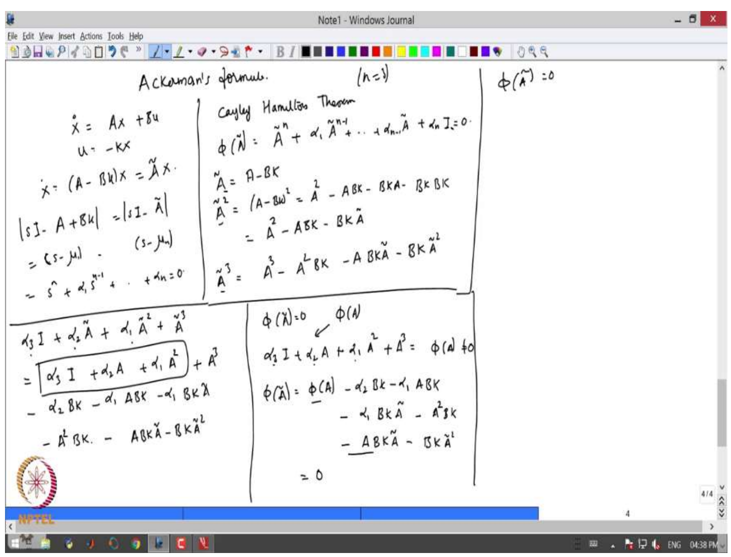

This might look a little ugly and tedious, but let us still do it ah. So, I have $\dot{x}$ = Ax + B u which I transform via u = -Kx to $\dot{x}$ = (A-BK)x which I also call as $\tilde{A}$x, ok.

Now, what is a characteristic equation of the close loop system? Characteristic equation of the close loop system with (sI - A + BK) is (sI - $\tilde{A}$) that will be again (s - μ_1)(s -μ_2) . .

. . . (s - μ_n). So, all these close loop poles are given to me, and I can expand this and write it in terms of the coefficients of the polynomial in S, ok. Now, ok so this is good.

Now, let me go to the Cayley Hamilton theorem and what it tells me is $\Phi(\tilde{A})$ is that a matrix satisfies its own characteristic equation $\tilde{A}^n$ + $\alpha_1\tilde{A}^{n-1}$ + . . . $\alpha_{n-1}\tilde{A}$ + α_nI = 0. This is the n x n identity, ok. So, what do I know that $\tilde{A}$ =A - Bk, A tilde square is what I just expand this $(A - BK)^2$ square and I can write this as A^2 -A BK - BKA – (BK)(BK).

I can also write this in a little simplified way as yeah sorry, A^2- ABK - BK$\tilde{A}$. Similarly, A^3 can be written as A^3 - A^2BK - A BK$\tilde{A}$ - BK$\tilde{A}^2$, ok. I will just do the proof for n = 3 again for simplicity and then to just to avoid ugly looking computations, ok.

Now, I put this all these values into the characteristic equation and what I get is the following. So, I am looking at $\alpha_3 I$ + $\alpha_2\tilde{A}$ + $\alpha_1\tilde{A}^2$ plus sorry, $\tilde{A}^3$, ok. So, I will skip some steps and just write down how it looks in front. So, I will have some terms to do with $\alpha_3 I$ + $\alpha_2 A$ + $\alpha_1 A^2$ +A^3all this in A, then I will have -α_2BK -α_1A BK -α_1BK$\tilde{A}$ - A^2BK, ok.

So, we have again that $\Phi(\tilde{A})$ = 0, right. And, what can then I what can I then say in terms of $\Phi(A)$? $\Phi(A)$ which looks like $\alpha_3 I$ + $\alpha_2 A$ + $\alpha_1 A^2$ +A^3 is $\Phi(A)$ and this is not 0, right because a this α s are the coefficients for the characteristic equation corresponding to $\tilde{A}$ and, therefore they will not be the coefficients corresponding to the matrix A and therefore, $\Phi(A)$ will not be 0.

But I can write $\Phi(\tilde{A})$ now as, ok, see the entire thing here is $\Phi(A)$, ok. So, this is $\Phi(A)$ - α_2BK -α_1A BK -α_1BK$\tilde{A}$ - A^2BK, ok. So, we will have two more terms here -A BK $\tilde{A}$ - BK$\tilde{A}^2$, because I will just put them up here to A BK $\tilde{A}$ - BK$\tilde{A}^2$, ok. Now, this $\Phi(\tilde{A})$ = 0, ok. Now, so, I can, so this entire thing will equate to a 0 here, ok. So, $\Phi(A)$ minus all these terms will equate to 0.

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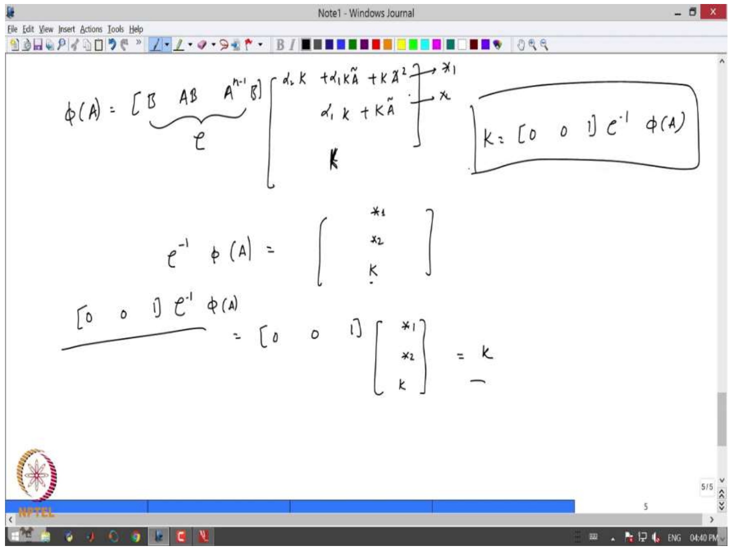

And this we will simplify to some nice looking term here that, ok; I just take $\Phi(A)$ to the left and what I will be left is [B AB $A^{n-1}B$], and allow some terms here, right α_2 K and then K, ok, so ok. Something looks nice here, right. I just want if I just can do some magic to just find what is this K then I am done, ok. I will just skip, I will just keep skipping all the all these steps here, ok.

So, for controllability this matrix should be should be invertible if I am looking essentially in the single input single output case. So, let me call this just C and I can now write this equivalently as $C^{-1}\Phi(A)$ is this entire matrix here with some term star 1, some term star 2 and the K, ok, I call this star 1, I call this as star 2, ok, nice.

So, I just again want to do something here, right. So, if I just multiply to the left of both sides by [0 0 1], this will be [0 0 1], times star of 1, star of 2, K, ok. What is this? This is just equal to K and I just now look at this and this it turns out that the K is simply [0 0 1] $C^{-1}\Phi(A)$, ok. This is the what is called as the famous Ackermann's formula, ok.

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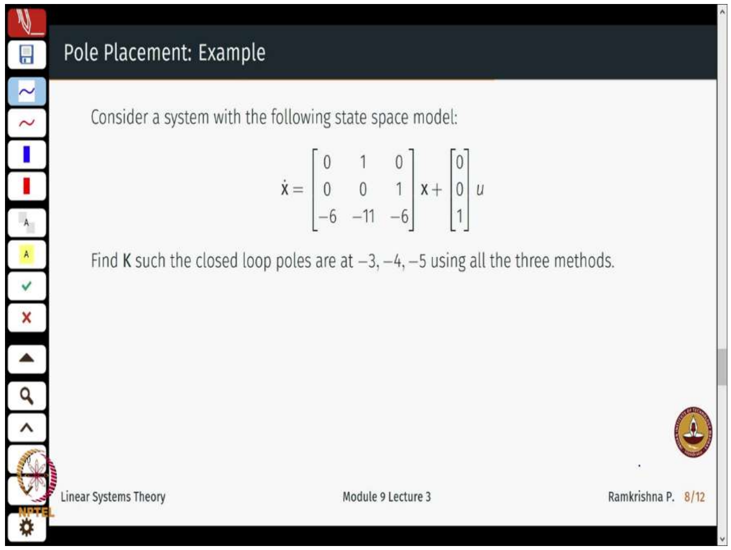

So, that is, so that is what is we derive through the Ackermann's formula, ok. So, I just have this example I just live for you to solve this. It is again same application of the methods that we had over here, right, ok. And I will maybe put up the solutions when I when I upload the slides, ok, right.

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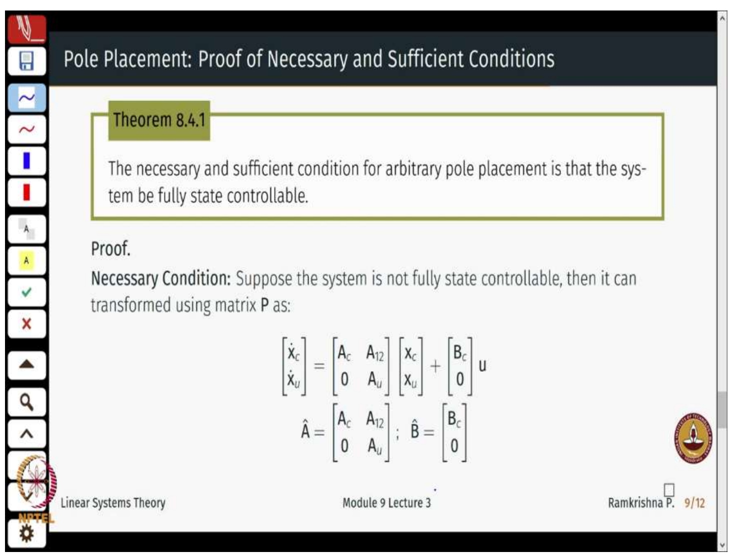

Before I started this lecture we said that we should or; there were some hints that controllability should be some kind of a necessary condition for pole placement, right. So, you can start from the controllable canonical form, something similar was also seen in the

Ackermann's formula where you where you needed inevitability of the controllability matrix. So, let us spend some time and prove the following, right that the necessary and sufficient condition for arbitrary pole placement is that the system is fully controllable and observable, ok.

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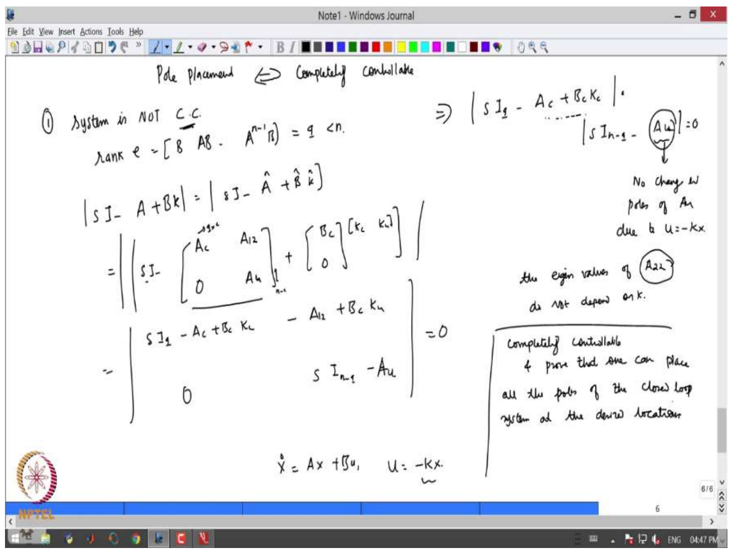

So, then system is, so what does it mean? So, if I can place the pole placement is equivalent to system being completely controllable, ok. So, let us start with part 1 of the proof, right. Let us say system is not completely controllable, which means at the rank of C which is [B AB $A^{n-1}B$] let it be some q < n, ok.

In that situation, what I know is I can transform system into its controllable part and its uncontrollable part, right, ok. So, so I assume that there exists a transformation that will take my system from it is starting from it is original form to the to the controllable decomposed form, right, ok. Now, this is easy to check that if I have the characteristic equation in A and B, this will be the same as having the characteristic equation in $\hat{A}$, where $\hat{A}$, $\hat{B}$ and all they correspond to be a controllable canonical form, ok. Now, how does this expand to?

So, I just look at this term here this will be sI minus, ok. What is A? A looks something like this $\begin{bmatrix} A_C & A_{12} \\ 0 & A_u \end{bmatrix}$ plus B will be $\begin{bmatrix} B_C \\ 0 \end{bmatrix}$, some $[K_C \; K_u]$, ok. So, this will be now of the forms I am looking at the determinant of this. So, this will be sI_q. So, this A is a q x q matrix here right. So, this will be sI - A_c + B_cK_C all these controllable terms. Here I will have a -A_{12} + B_cK_u, 0 here I will have the S n corresponding to this A_u would be of dimension n - q. So, the identity of sI_{n-q} - A_{22}, the determinant of this should be 0 which essentially means that I have (sI_q - A_c + B_cK_c)(sI_{n-q} -A_{22}) = 0, ok.

So, I start with a system which is of the form $\dot{x}$= Ax + Bu apply a control law of the form u = -K x and I assume that the that the that the system is not completely controllable which means I can decompose it into this controllable an uncontrollable forms and I look what happens with the application of control to the close loop characteristic equation, right.

So, what I know is that the close loop characteristic equation will tell me, where the close loop poles are, ok. So, look at this, right. So, what are, what is the difference between the close loop poles of A_u and its open loop poles? Right. So, they are the same, right, open loop sorry, I should call this A_u, right this to be consistent with the notation here, right. So, this should also be A_u, ok. So, this control law does not alter the poles of A_u, there is no change in poles here due to u = -K x, ok. Or other words the close loop eigenvalues of A_u do not depend on K and therefore, if the system is not completely controllable I cannot place the poles of this part of the uncontrollable part, ok. So, this is this completes one part of the proof ah.

The second part of the proof would say that, start with a system which is completely controllable, start with the system and prove that one can place all the poles of the close loop system at the desired locations, ok. So, I just go back I know do the proof not repeated over here again, but what we saw say for example, here in this example is that, ok.

So, if I have a system which is in this form this system is of course, completely controllable, ok. It is in the controllable canonical form, if it is not in the controllable canonical form I always know how to get it into the controllable canonical form. So, what was the conclusion here is that if the system is in this form then I can always place the close loop poles according to this formula. I can always compute k_1 till k_n, right.

So, what does this mean? Right. I start from a completely controllable system and I show that I can place all the poles or a solution to this exercise, starting from this given characteristic equation to this unknown equation always exists, right. So, that is a also proof of saying that well start from a completely controllable system, then you can place all the poles of the close loop system at the desired locations. I will not write down the steps, but I think this is kind of very obvious from what we saw over there, ok.

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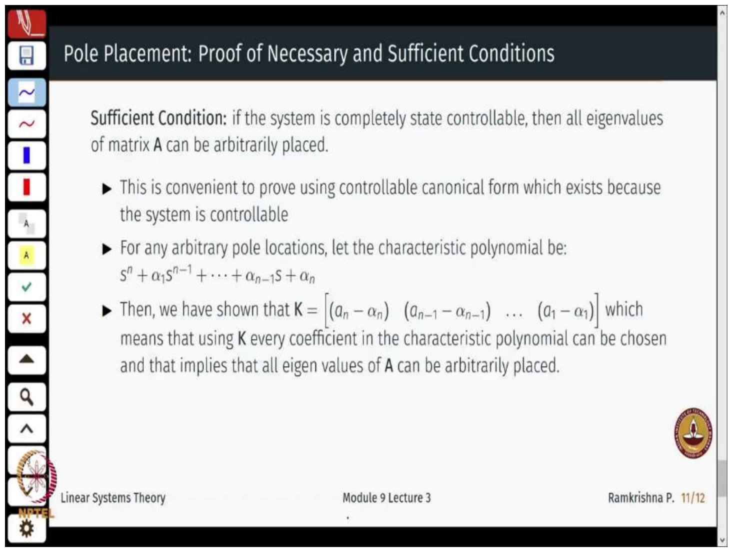

So, that is kind of kind of writing down the text over here again.

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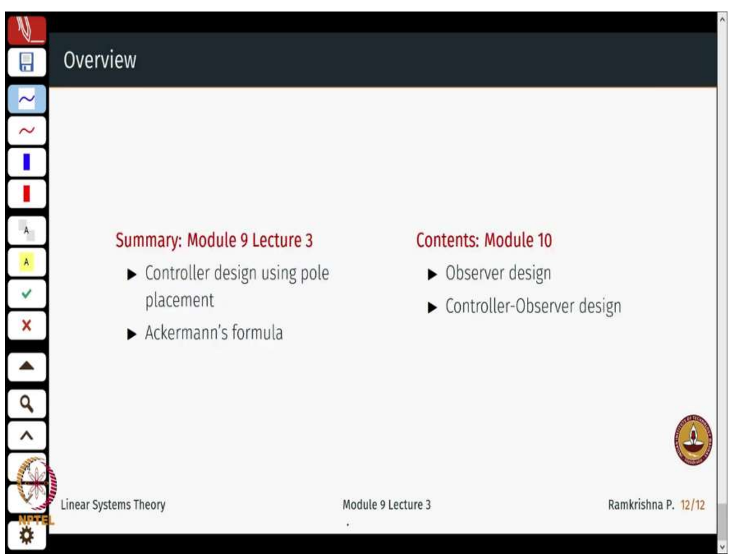

So, what we just saw is 3 methods of placing the close loop poles one from directly comparing the open loop and close loop characteristic equation. Second is write down in the controllable canonical form, so that the formulas become much easier. And the third was a little more involved one of the of the Ackermann's formula, ok. So, what we will do is also have a little tutorial session where we solve a bunch of problems including partial pool placement, how to even handle these pole placement problems via MATLAB and a bunch of other practice problems that we will post online. And I hope that helps a bit in the in the learning process.

So, this concludes a lecture on design controller design assuming the system was full is like, I all the states of the system were available for measurement, so that I could just do u= -K x. The question is what if all the states of the system are not observable sorry, are not yeah are not observable. That would lead to then say, then I need to do something else can I do an observer design, can I add on top of some I just have the output measurements, can I make use of the outputs to construct the states and then feedback those states to get the appropriate control design, right.

So, that will come up in next week's lectures in terms of a observer design. And we will also look at simultaneously how to design control and observer, and also see if designing an observer does it really affect the controllable, control part or and vice versa, right. So, we will see this both together. That is, that will come up next week.

Thanks for listening.