Linear Systems Theory
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Module - 08
Lecture - 04
Observability Decomposition and Detectability

Hello everyone. So, welcome to this lecture number 4, for week 8 on the course on Linear Systems Theory. So, it will be a little little short lecture, where we will just list down the results for observability related concepts, essentially to do with the observable decomposition and so on. Very analogously to what we saw in the controllable decomposition we defined a weaker notion of a controllability called stabilizability.

And then, we derived results based on the requirement that I can stabilize the system, if and only if the uncontrollable part of the system is a stable matrix or it is a stable part. So, essentially we could decompose the system into its controllable and the uncontrollable part and much of the design was related to the condition that if and only if the uncontrollable part is stable, I can do some kind of a eigen value assignment and so on with stabilizable systems. So, we will do something very equivalent with the concept of observability.

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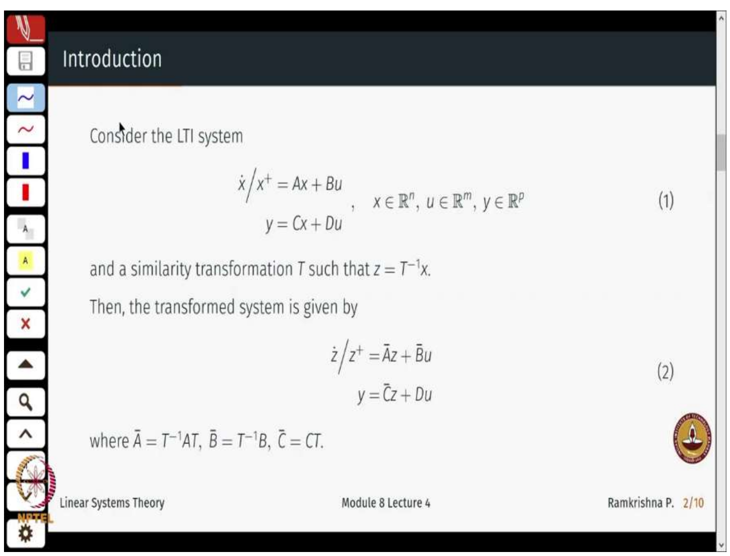

We start as usual with an LTI system could be in continuous in time or discrete in time and which via a certain transformation T takes my matrix A to some equivalent $\bar{A}$. So, and similarly with A, B and C being transformed to $\bar{B}$ and $\bar{C}$ via this transformation C.

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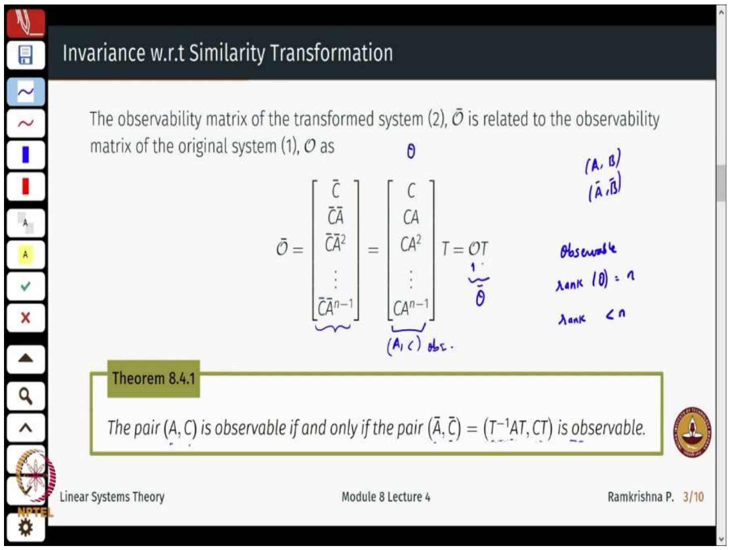

So, the result which is again easy to derive is the following that the pair A C is observable if and only if the pair $(\bar{A}, \bar{C})$, where $\bar{A}$ and $\bar{C}$ come as a result of this transformation T and. So, sorry, the pair $(\bar{A}, \bar{C})$ which result as which come as a result of the transformation T is also observable. So, that is like easy to check, right say assume that this A C is observable and it has a certain observability matrix, $\bar{C}$ $\bar{A}$, the observability matrix of this can be written simply as the, let me call this O, as O T.

Now, if O is full rank, then definitely $\bar{O}$ is also a full rank because T I know is this is a full rank matrix. So, this is like easy to check. So, where we also had the similar result on controllability, where A B was controllable if and only if this pair $(\bar{A}, \bar{B})$ was controllable, right. So, I can; the proof is pretty straightforward, ok.

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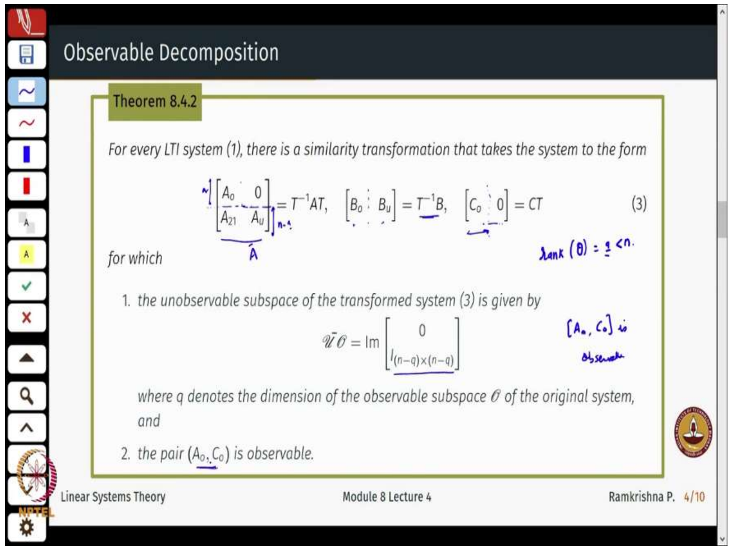

So, for every LTI system, so again we will ask our self, so the system is observable when the rank of this observability matrix is n, ok. So, what if the rank is less than n? Right. So, that is what we will answer now. So, for every LTI system there exists a similarity transformations that takes the system into a form which looks like this, this is my new $\bar{A}$ via a certain transformation T. B is decomposed into two parts, B_o and B_u which is again the result of this transformation and C looks something like this, ok

So, whenever I have a transformation like this where the rank of C, sorry rank of this O is now q which is less than n. So, under such conditions the unobservable space now is of dimension n minus q times n minus q. This is my observable space. So, what I have done is via this transformation which comes as a result of this O, I could decompose my system into the observable part here and the unobservable part, ok.

In such a way now that the pair A C, sorry A_o C_o is observable, ok. So, the A matrix decomposes this way, B decomposes into its own observable unobservable components, ok. This structure is not important, but what is important here is that the outputs now can be very easily decomposed. So, I have a, so C_o here and A_o here.

You can also write down the dual version of this and check how the controllability properties of the dual system or the controllable decomposition of the dual system translates to the observable decomposition of the original system, ok. So, I will use it as a very small exercise, ok. So, what we know now is the decomposition into the observable and unobservable space and that the observable pair is A_o C_o, ok.

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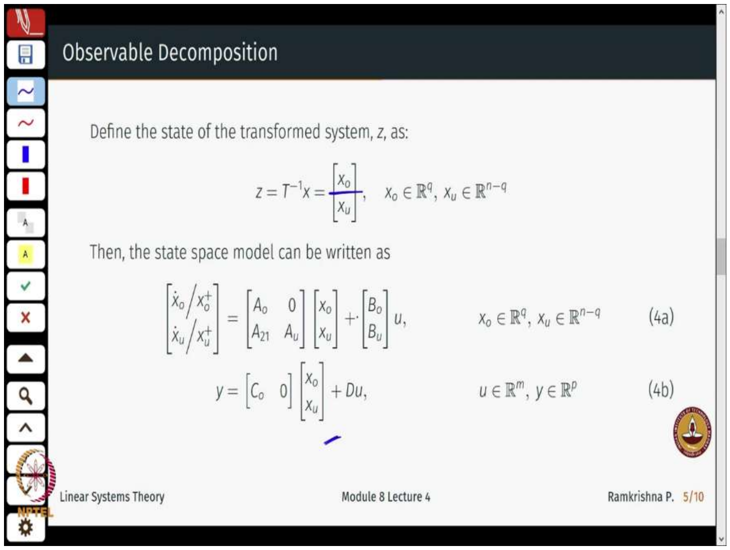

So, this can, again just come as a result of for certain transformation where I can divide my states into the observable part and the unobservable part with a state space model which looks like this, right. I am just writing down these equations plugging them into the appropriate x, u, y matrices and so on, ok. So, the discrete time notion is also very similar, ok.

Now, since we know that the system is not a completely observable, can we define a weaker notion of observability? Right. I only know that certain states are observable. So, maybe a q states are observable and the remaining n - q are not. So, can I define some weaker notion of controllability, and in that definition what are the things that I need to be careful of, ok.

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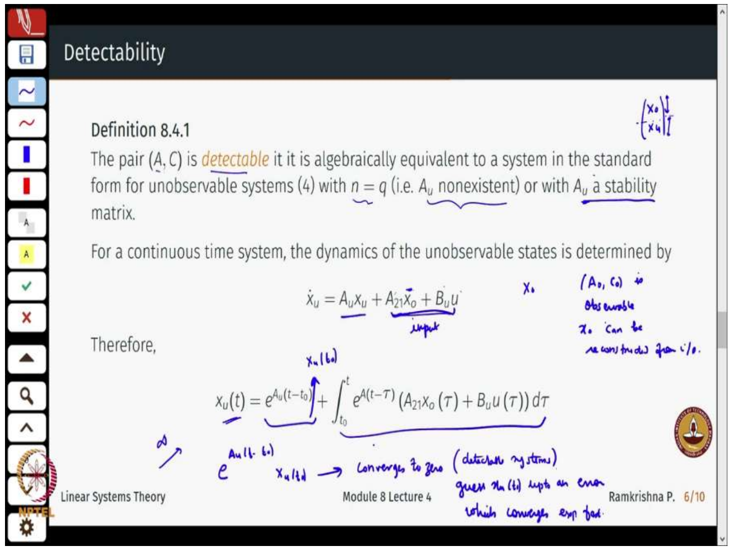

So, the definition goes something like this. That the pair A, C is detectable, if it is algebraically equivalent to a system in the standard form with n = q, right. So, when n = q, now it would mean that the system is completely observable, right, which means this guy does not exist anymore and then the unobservable space is just the 0, ok.

So, and, what if A_u does not exist, it is a easy case it is just a standard observability case. What if there is A_u, what is what if n ≠ to q then the requirement is that A_u is a stability matrix, this one. Now, this one must be a stability matrix. So, what does it mean here is; so, I have x_o and x_u, ok. Now, these are the observable state, so I can construct them, these are the unobservable states. So, if I if I look at the second equation I can write this as $\dot{x_u}$ = $A_{21}x_o$+ $A_u x_u + B_o u$ and so on, right. So, the expression becomes something like this, right.

Now, what do I know is I know I can construct x_o, right, so and I can write down the solution for the observable component. So, it turns out that I can write, so or I can also equivalently because x_o is observable, I can look at this entire entirely as some kind of input to the system. This be determined, this is already known from the structure of the system, u is known, B is known. I just want to see what happens with this $A_u x_u$ which is unobservable, right.

So, I can write down the solution now in terms of. So, there is something missing here. And this is this will be x_u at t_o, ok. So, taking this as the input my x_u takes this form, right. Now, what do I know is that A_o, C_o is observable, ok, which means that I can reconstruct is, ok, that this x_o can be reconstructed from the definition of observability, right, reconstructed from the input output pair, ok. And therefore, I can kind of construct this entire term within the integral because everything is known.

Now, for detectable systems. So, when the system is detectable, we will look at what happens to this term $e^{A_u(t-t_o)}x_u(t_o)$. What is it for detectable systems? That A_u is a stability matrix. So, when A_u is a stability matrix this term eventually converges to 0 and this is for detectable systems, ok. And therefore, one can guess $x_u(t_o)$ up to an error which converges exponentially fast, ok. So, what this means, ok, if I look at the opposite case where A_u is not a stability matrix. So, this guy will go up to infinity, right and therefore, this is some kind of instability within the system which is undesirable, ok.

So, the conclusion here also is that if the system is not completely observable, there is some hope of doing some design with the system if and only if the unobservable part is stable, right. And this is a little illustration for that, right, ok.

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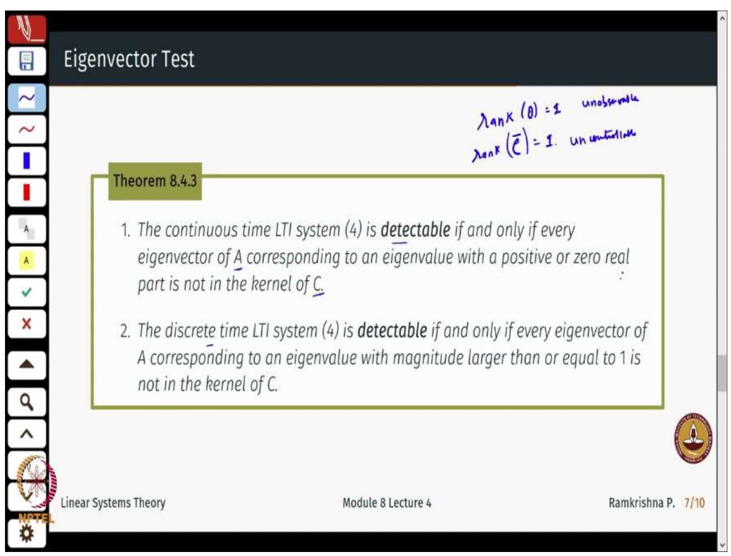

So, the rest of the proofs are exactly the same as they look for the controllable decomposition case. So, the system is detectable if and only if every eigenvector of A corresponding to an eigen value with a positive or real part is not in the kernel of C. And similarly for the detectable case offer discrete time systems. You can do this proofs from scratch or you can also write down the dual system and just compare with the controllable properties of the controllable decomposition.

So, here if the rank of O is q in the dual system, the rank of C the controllable matrix of the dual system which we I think denoted by $\bar{A}$ will be q, right. So, here the unobservable modes will show up here as the uncontrollable modes and then and so on. So, I think that the duality is kind of a very very useful, useful concept here. So, I can just rewrite every result in terms of duality and then verify this or in I can also do it from scratch, right, ok.

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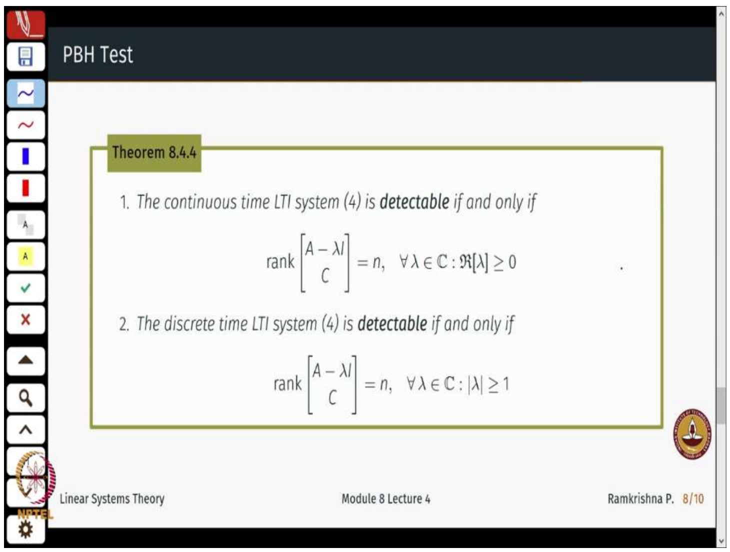

Then you have a similar version of the PBH test for detectability. Again, it looks exactly like very similar to what happened in the controllable case or the controllable decomposition or the stabilizability case.

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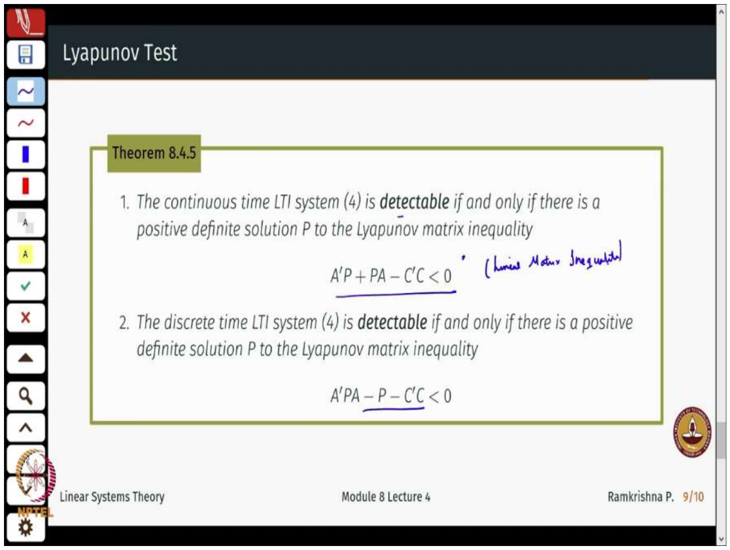

So, the last result here which again I will just; I will just state it again the proofs are kind of pretty easy to work out, once you know the proofs for the controllability decomposition or the stabilizability case. So, the continuous time LTI system is detectable if and only if there is a positive definite solution to the matrix inequality which looks like this for the continuous time and which looks in this way for the discrete times, ok.

So, just a little note on this thing. So, this kind of inequalities are usually called linear matrix inequalities, ok. So, I am just giving you conditions on whether or not the system is detectable or stabilizable or controllable and so on. But I am not really telling you how to solve this at the moment, right. So, towards the end of the course on week 12, we will spend a lot of time looking at computational aspects of these kind of inequalities which are also called as linear matrix inequalities.

So, we spend a lot of time there, just to look at all these inequalities which we see in terms of Lyapunov stability, A^TP + PA < 0, and so on. So, we will spend some time actually finding out ways to solve for this equation. So, maybe introduced to you to some MATLAB packages which actually solve this kind of equations. So, if you are; if you are confused, so what do I do with these equations?

Just hold on, I will give you some very easy methods to check a numerically how this how these works. So, if I am give, if I give you of say a 10 cross 10 system, it might be very difficult to check with hand, but there are some really beautiful elegant MATLAB packages which you can use and I will eventually introduce you to those.

So, to conclude this week. So, we did a bunch of, proved a bunch of results starting with definitions of observability and detectability, very similar to what we had for the controllable case. From the next lecture we will start slowly building up on design concepts. A bit of introduction was given to design when I was talking about placing the eigenvalues at certain locations, less than mu, that was somewhere in the previous lecture, previous weeks lectures. So, we will do all those procedures a little more constructive way, right, so which will be called as pole placements. So, that is coming up starting a next week.

Thanks for listening.