Linear Systems Theory

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Module - 08
Lecture – 03
Observability for Discrete Time Systems and Observability Tests

Hello everybody is so, welcome to this lecture number 3 of week 8 on the course on Linear Systems Theory. So, we were talking in the last two lectures about the way we define the concept of observability, why we need the concept of observability and so on. so, we will have this short lecture just to understand the discrete time analogue of the observability as and most of the steps will be very similar to what we did in the discrete time systems for controllability analysis.

So, I would skip much of those steps here. So, they might just be redundant to write to solve over again and we will equivalently try to derive the eigenvector test, the PBH test and so on. so, what I know now is the concept of duality right that if a b so, we had a system written in A B C and D and A^T C^T $B^T$and D^T and then we had some relation between controllability of one system with the dual of it and then observability of one system and the and the dual of it right. So, we will make use of that of that results to derive the remaining tests for controller for observability ok.

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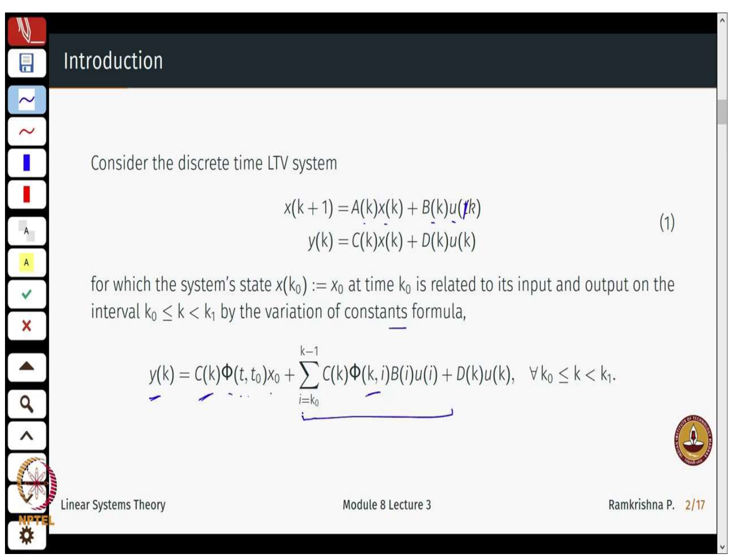

So, we start with discrete time systems. the discrete time case well everything is looks pretty similar. So, I have x(k+1) = A(k)x(k) + B(k) u(k) and so on. similarly for the output again so, the way we construct the solutions via the variation of constants formula by defining the discrete time state transition matrix, I just substitute for the outputs at y(k) it looks as with C(k)Φ(t, t_o) x_o and then the bunch of terms here. So, we did derive this in one of our earlier lectures ok.

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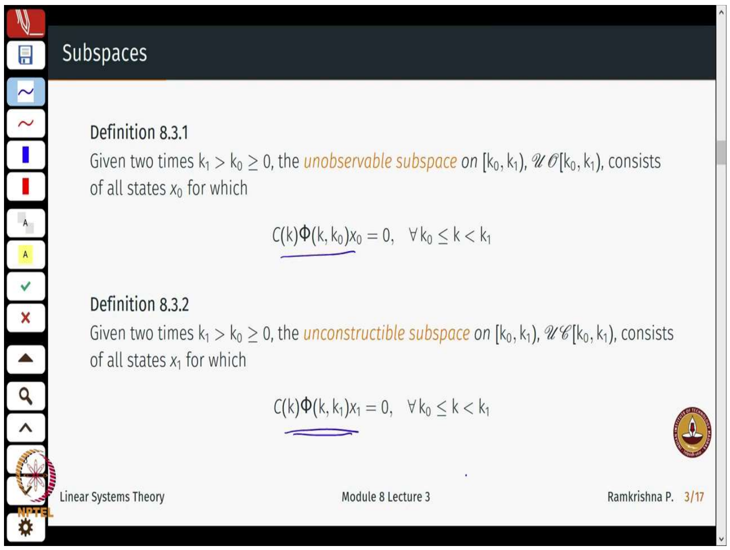

So, the definitions go very similar to the continuous time counterpart. So, given two times k_1 and k_o the unobservable subspace consists of all space all states x_0 for which like this holds right, it is very similar to what we had earlier and similarly the unconstructible subspace consisted of all states this one. So, that the definition of observability and constructability still remain the same ok.

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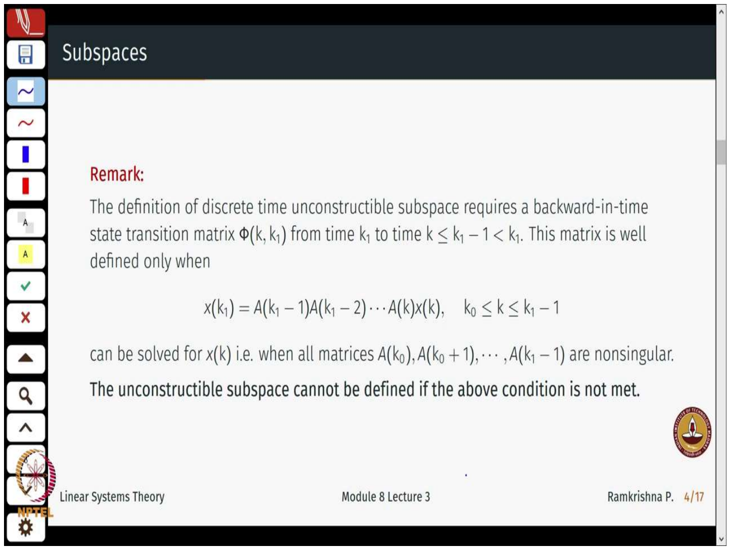

So, similarly in the case of discrete time controllability there is a little distinction that the notion of reachability and controllability were same for continuous time LTI systems but the discrete time equality required some extra conditions on the matrix A to be invertible right.

So, similar results hold also for the for the observability test for discrete time systems ok. So, so the definition of discrete time system requires a backward in time state transition matrix from k_1 to time k right. So, and then this matrix is defined. So, x(k_1) is defined with x (k). So, this can be solved for x(k) if and only if all these A's are invertible or all this A's are non-singular. Otherwise the un-constructible subspace cannot be defined if this if this condition is not met. Again, I will I will skip the steps here, but you can just refer to lectures on week 7 for controllability. It will be exactly the same ok.

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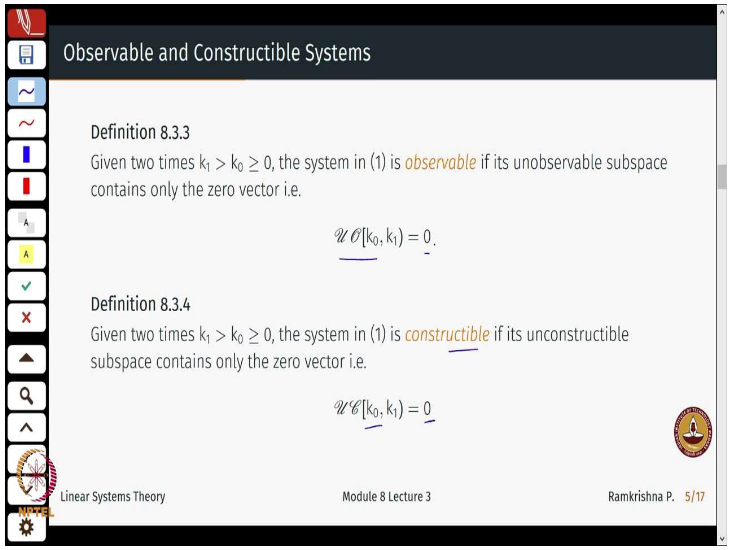

So, again when is the system observable? The system is observable even the discrete time when it is some unobservable subspace consists only of the 0 vector and similarly it if it is constructible if its un constructible subspace contains again only of the 0 vector right. So, so this is the very similar definitions to what we had before ok. So, when this holds we will first try to construct an equivalent of the Gramian for discrete time systems.

So, I just define my Gramian here the observability Gramian again depending on phi c c transpose and so on right. So, again very very similar to what we had seen earlier, and the results also translate beautifully here that given any two times the unobservable space is just the kernel of the observable Gramian and similarly the unconstructable space is just the kind of kernel of the unconstructable Gramian.

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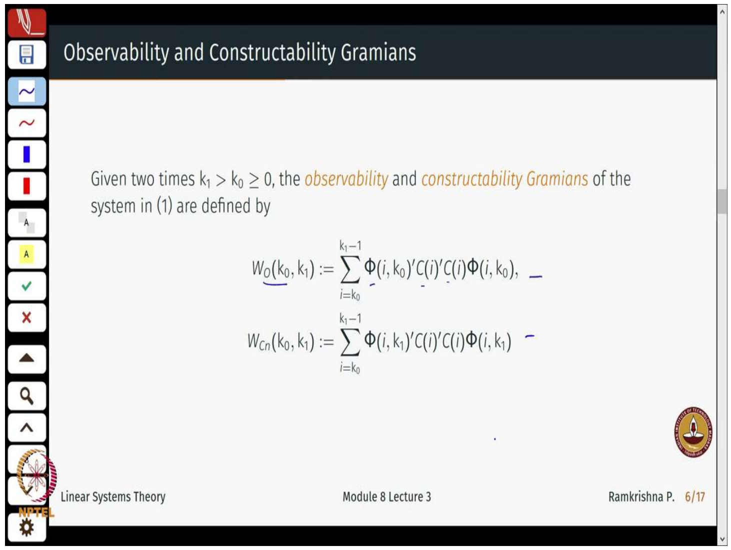

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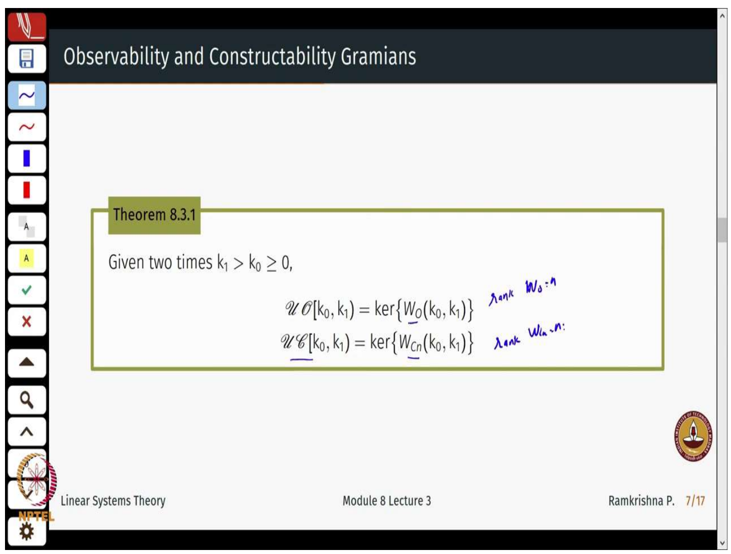

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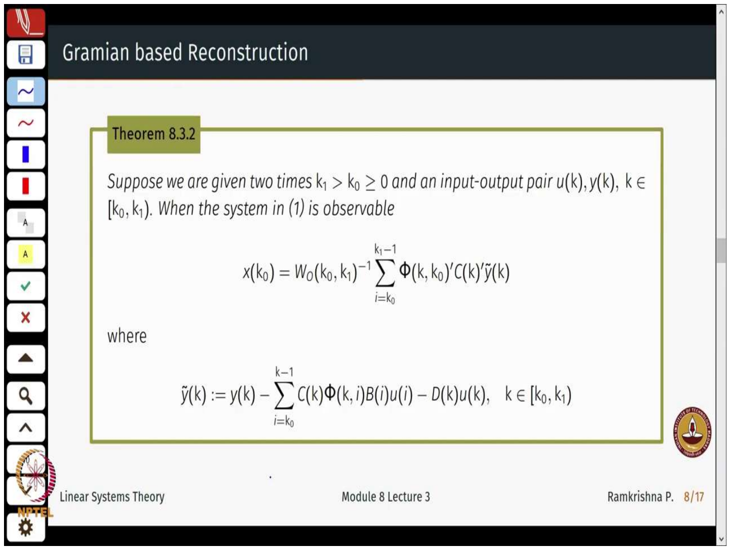

It is so, again we can prove quite easily of course, so the rank based condition still hold right. So, whether the system is observable if and only if this rank of W_0 is n similarly the system is observable if the rank of W_{cn} is n ok. So, nothing changes here. a similarly I can do the Gramian based construction. So, having known the Gramian how do I construct the state it again starts from the definitions over here and then using the Gramian I can just write down this expression here right.

This is this is very similar to what happened in the continuous time case and loosely speaking just the integral replaced by a summation here so to speak ok. So, these are all just plug in put it over here and they all translate quite beautifully except one condition of equivalence between unconstructable and the unobservable space or is the or the equivalence between constructability and observability right. similarly I have a Gramian base reconstruction for the for the constructible case ok.

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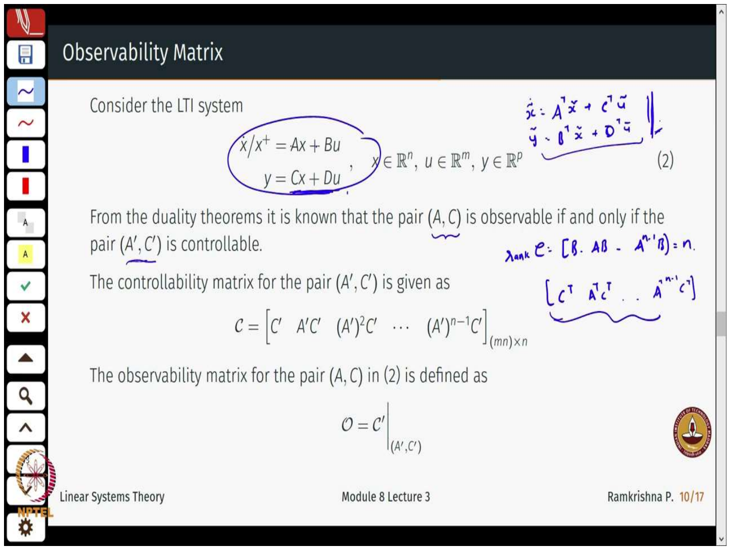

Now we go to the observability tests. So, first is the rank based condition right. So, if I look at so I will just do the derivations from the duality thing. So, from the duality theorems, it is known that the pair A, C from the duality theorems we know that the pair A C is observable if and only if the pair A^T , C^T is controllable ok. So, what was the let us write down the dual system here. So, I have $\dot{\tilde{x}} = A^T\tilde{x} + C^T\tilde{u}$, $\tilde{y} = B^T\tilde{x} + D^T\tilde{u}$.

So, whenever A this system is observable it is equivalent to saying this system is controllable ok. Now how do I derive the observability conditions for this system based on the controllability properties of this of the dual system, because I know that observability of this system is equal to the duality to the controllability properties of this system. So, what do I know that the rank of for controllability in general is at the rank [B AB . . . . A^{n-1}B] should be n. So, this is the controllability matrix the rank of it should be n ok.

Similarly, now I can define the controllability matrix for this guy. So, the controllability matrix for this guy would have $[C^T \;\; A^T C^T \ldots\ldots {A^T}^{n-1} C^T]$ right. So, this will be the controllability matrix for the dual system will be the observability matrix for the original system ok. So, I just then write down the observability matrix in this format.

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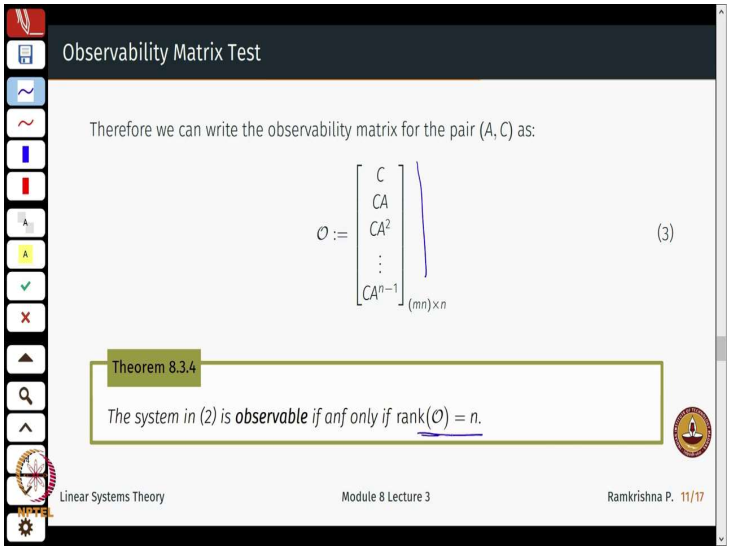

So, the observability matrix just translates to this one and then therefore, I can conclude that so, so the observability for the pair A C is defined as this is the transpose of C right; so, if you just look at the definition over here or the way we derive the observability matrix here ok. So, observability matrix is now defined in this way $\begin{bmatrix} C \\ CA \\ CA^2 \\ . \\ CA^{n-1} \end{bmatrix}$ and so on and then the system is observable if and only if the rank of the observability matrix is n ok.

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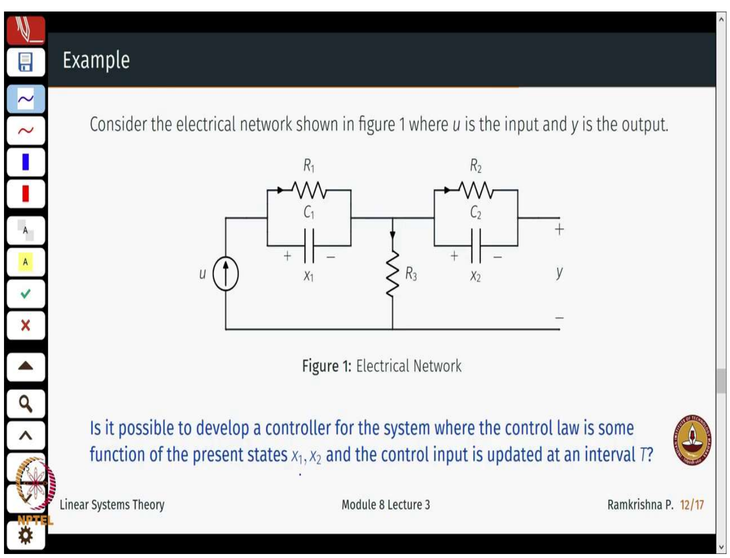

So, we start with the example that we that we had in the in the first lecture right so, the question there was is it possible to develop a controller where the controller is some function of the states x_1 and x_2 and the control law is say updated at some every interval T ok. So, this and then what I have for measurement is only the output y here.

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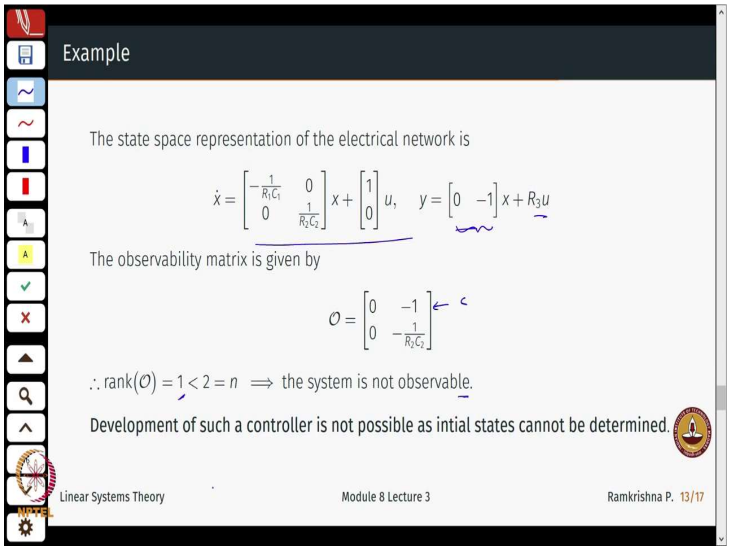

So, if I write down the state equations for this so, I will have something like this where y is this as this 1 plus some d here ok. So, I just look at the observability matrix which is just given by C. So, this is the C and then A C. So, I can just look at this matrix and say that the rank of this is not equal to 2; it is actually equal to 1 and therefore, the system is not observable and therefore, the problem here which says can I develop a control law which is based on this present states and then and the control is in is updated as an interval T, the such a controller well is not possible as the initial states cannot be determined and so, because the system is not completely observable right ok.

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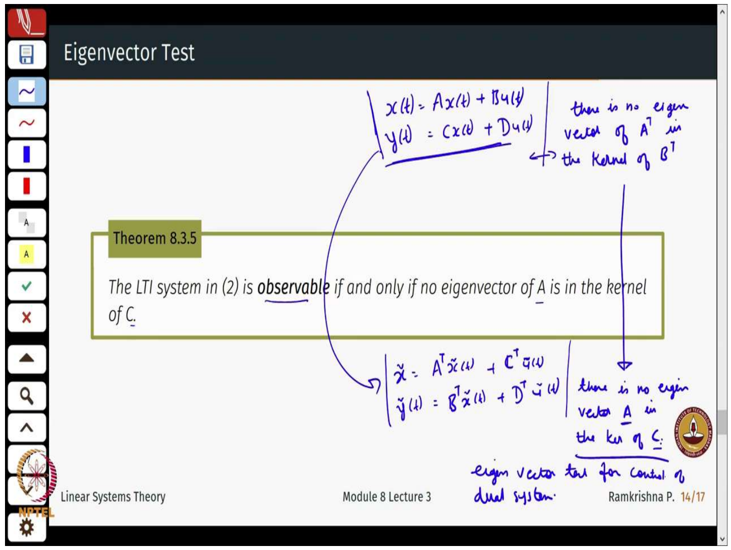

So, now some tests which are very similar to what we did for the controllability. So, the first results is that the system 2 is observable if and only if there is no eigenvector of A in the kernel of C^T ok. I will not do the proof of it, but I will explain you this with the help of the duality concept. y(t) = Cx(t) + Du(t) then have a dual system A^Tx +C^Tu(t), y the new output is sorry this will be C^T then y = B^Tx (t) + D^Tu(t) ok.

So, this system is I am looking at the observability of this system and this is related to the controllability of this one right ok. So, what is controllability here? So, controllability would say that there is no eigen vector of A^T in the kernel of B^T that was the controllability for this system ok.

Now, what about observability of the system? The observability of this system is equal to saying controllability of the dual system. Now let us write down the eigenvector test for the dual system. So, analogously the system is controllable if and only if there is no eigenvector, so, this A^T here becomes A in the kernel of C.

This is the eigenvector test for controllability of the dual system and this condition is exactly equal to the observability of the original system and therefore, I can say that the system this LTI system two is observable if and only if there is no eigenvector of A in the kernel of C. Like I can do a constructive proof the way we did it in the previous one, but then since you already know the duality we can just make use of the tools there and then exploit the relationship between the controllability of the system with the observability of the dual system and so on right.

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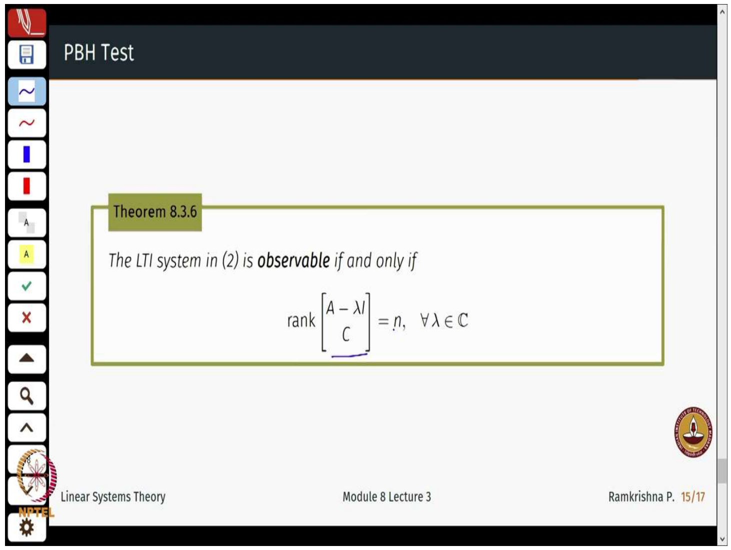

Similarly, I will have the PBH test; again, I will not do the proof of it or you can just write down the analogous thing for yourself. The LTI system 2 is observable if and only if this rank is equal to n. Again, I will not touch upon the proof or even with the duality.I would like you to work on it a bit.

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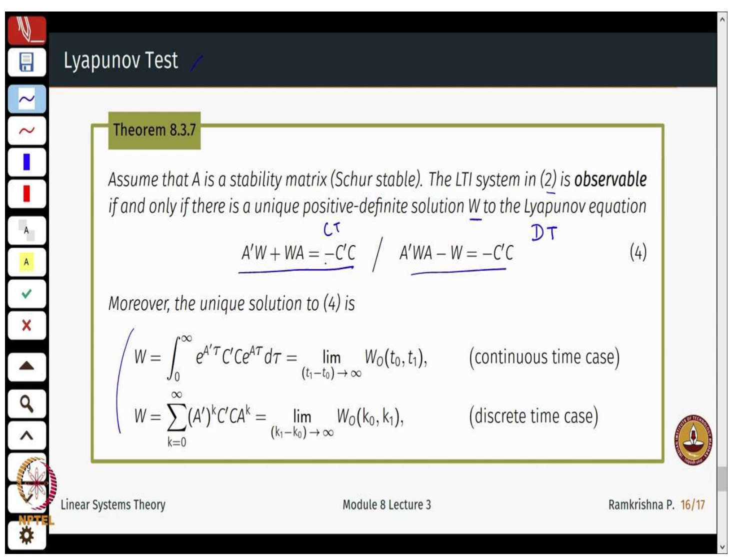

Similarly if I have that A is a stability matrix so, I am looking at the at the equivalent of the Lyapunov test. Assume that A is a stability matrix the LTI system 2 is observable if and only if there is a unique positive solution W to the Lyapunov equation, this one right. Moreover, the solution is given by these two here; one is this is for the continuous time, this is for the discrete time.

Again this they look very similar to what we had in the case of controllability again all these derivations you can just do by exploiting the duality relationship between controllability of a system and the observability of the dual system and vice versa ok. just for your own benefit or just for a little more learning, you can write down the proofs starting from 0 right, assuming there that you do not know the relation between the duality you do not know the duality relationship between controllability and observability and so on.

And it will just help you to kind of get more and more familiar with proof techniques. I will leave I will not do that here I will just leave that to you for some for some of your own or own learning right and if you have any difficulties, you can always post that in the forum and we will be more than happy to help you out ok.

So, so that concludes today's lecture. So, next lecture we will look at the observable decomposition very similar to what we had done in the case of controllable decomposition and then we will have a weaker version of observability called detect detectability which is again you can relate it to stabilizability, which was a weaker version of controllability. So, that is coming up coming up next.

Thanks for watching.