Linear Systems Theory
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Module - 08
Lecture - 02
Gramians and duality

Hi everyone, in today's lecture we will build upon the concepts that we had defined for observability and constructability and we will try to derive conditions to verify observability and constructability. So, much of the techniques that we will use or much of the a basics or the foundational tools that you will use will be similar to that we used to derive methods to identify or verify controllability or even reach ability.

So, some of the things may not be surprising that the concept of controllability sorry the concept of observability and constructability co inside for LTI systems whereas, for discrete time. So, they co inside for continuous time LTI system that is for discrete time systems they may not actually co inside. So, there are some conditions that need to be verified, very similar to what happened while we were doing the controllability analysis for continuous time LTI and discrete time LTI systems ok.

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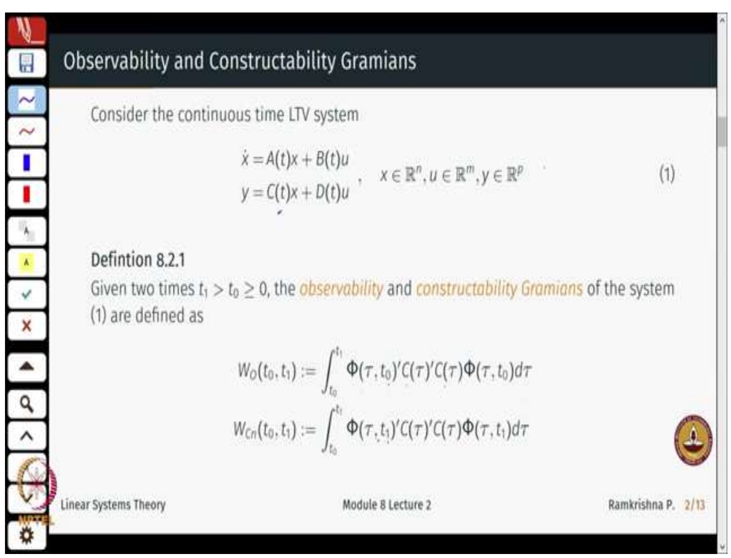

So, we start with again the definitions of the Gramians. So, I start with a general LTV system of course, we will start with the continuous time formulation. So, given two times

t_1 and t_0 the observability and the constructability Gramian defined in the following way right. So, they depend on the state transition matrix and the observability or the output matrix C this one here is similarly for the constructability again it is. So, again it just defined depends on the way they are they defined with respect to t_0 and with respect to t_1.

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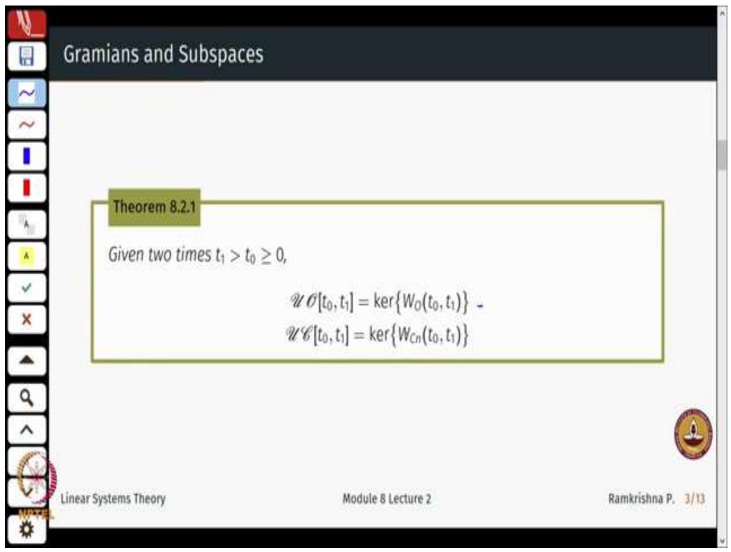

So the first result so, given this definition of the Gramians so, given two times the unobservable space is the concern is just the kernel of the Gramian defined over here the observability Gramian similarly the un constructible space is just the kernel of the constructability Gramian defined here ok.

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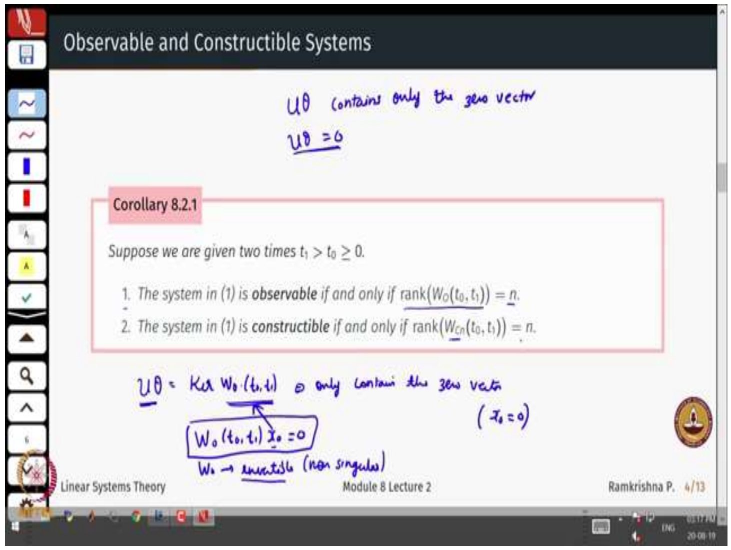

We will quickly run through the proofs of proof of this.

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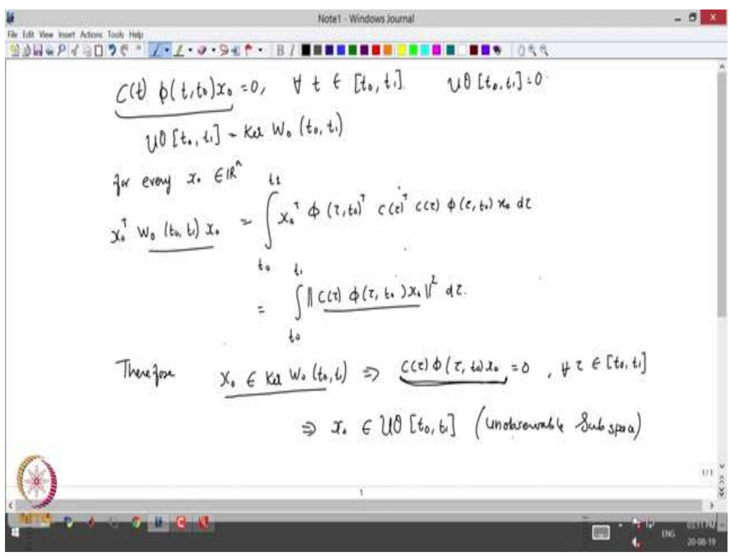

So, before that let us quickly recall how the unobservable subspace was defined. So, the un observable subspace was $C(t)\Phi(t, t_0)\, x_0 = 0$ for all t in t_0 and t_1 ok. And then the system goes observable if $U_o\ [t_0, t_1] = 0$ ok. So, what we want to prove now is or show now is the relation between the un observable space and the observability Gramian ok. So, the proofs are pretty straightforward. So, we just show one proof and then the other one will be a quite obvious t_0, t_1 is the kernel of $W_0(t_0, t_1)$ ok. So, from the definition of the observability Gramian so, for every x_0 in R^n.

We have ${x_0}^T W_0(t_0, t_1)\, x_0$. So, this is I just plug in the definition of observability Gramian. So, I have integral $\int_{t_0}^{t_1} {x_0}^T\, \Phi(\tau, t_0)^T \mathrm{C}(\tau)^T \mathrm{C}(\tau)\Phi(\tau, t_0)\, x_0 d\tau$ ok. So, this is simply t_0 to t_1 then writes it as the following $\int_{t_0}^{t_1} ||\mathrm{C}(\tau)\Phi(\tau, t_0)\, x_0||^2 d\tau$. And therefore, x_0 belonging to the kernel of the observable space would mean that.

So, x_0 belonging to the kernel would mean the left hand side would be 0 and therefore, the relation in the integral also will go to 0, which means $\mathrm{C}(\tau)\Phi(\tau, t_0)\, x_0 = 0$ for all τ within t_0 and t_1 ok. Now look at this so, this x_0 when it belongs to the kernel of W from these 2 steps it is also means that this x_0 satisfies this relation and now we go back here whenever x_0 satisfies this relation this means that this x_0 belongs to this ok.

So, what is this x_0 sorry what is this subspace? This subspace is the unobservable subspace ok. So, that concludes the first part so, whenever x_0 is in the kernel of W_0 or or the observable Gramian it also belongs to the unobservable subspace.

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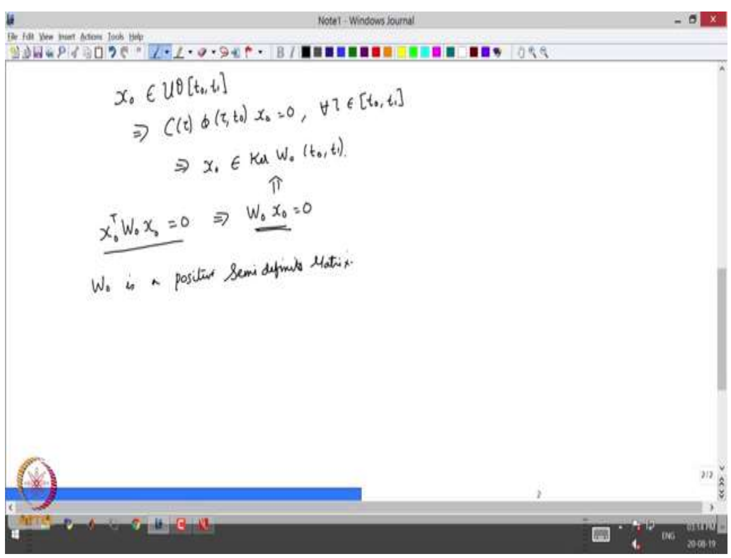

Now the second part of the proof conversely we start with assuming that let x_0 belong to the unobservable subspace. So, if x_0 belongs to the unobservable subspace this also means that $\mathrm{C}(\tau)\Phi(\tau, t_0)\, x_0$ is 0 τ here for all τ in t_0, t_1 ok. So, now look at the definition of the

Gramian which was here and say I just pre and post multiply this W_0 by x_0. So, ${x_0}^T$ here and x_0 here that would mean that ok.

So, this would imply that x_0 would also belong to the kernel of $W_0(t_0, t_1)$. So, I am just using this (Refer Time: 08:22) so, I multiply x^TW x so, with a. So, this will be equal to 0 implies that $W_0\,x_0$ is 0 and essentially this would mean this one ok. So, why is this true this is true because, so this kind of equality relating to this one is true if and only if this W is positive semi definite matrix we just know our case right ok. So, that is the that concludes the proof and then on very similar lines you can prove the second relation that the constructible space is the kernel of W_{cn}.

Now what is the definition of observability then right. So, we just recall the definition that we had for observability and it said that the system is observable if it is unobservable subspace contains only the zero vector right or in other words this set is simply 0 ok. Now so, what how do we check the or how do we now develop the or derived the rank conditions for this.

So, the first statement which is direct consequence of what we said here in the previous slide is that the system 1 or the LTV continuous time system is observable if and only if the rank of the unobservable subspace is equal to n ok, why is this true. So, we are so, what we said is that the unobservable space is also equal to the kernel of W_0. So, this is the observability Gramian ok.

Now so, this should only contain the 0 vector which means $W_0(t_0, t_1)$ if x_0 is in the kernel of it this should be valid only when $x_0 = 0$ for which it means that this W_0 should be invertible right. So, this x_0 whenever this relation is satisfied this means that this x naught is in the kernel of this of the observable Gramian.

Now this kernel is equal to the unobservable subspace and for observability this un observable subspace should only contain the 0 vector ok, which means this x_0 should be the 0 vector. So, this is a 0 vector if and only if this W is invertible or it is non singular. So, in other words this also means that invertibility of this W_0 means that it is observable if and only if the rank of $W_0(t_0, c)$ is equal to n. Similarly with the constructible system or to check constructability of the system is only if and only if the rank of the constructability Gramian is equal to n.

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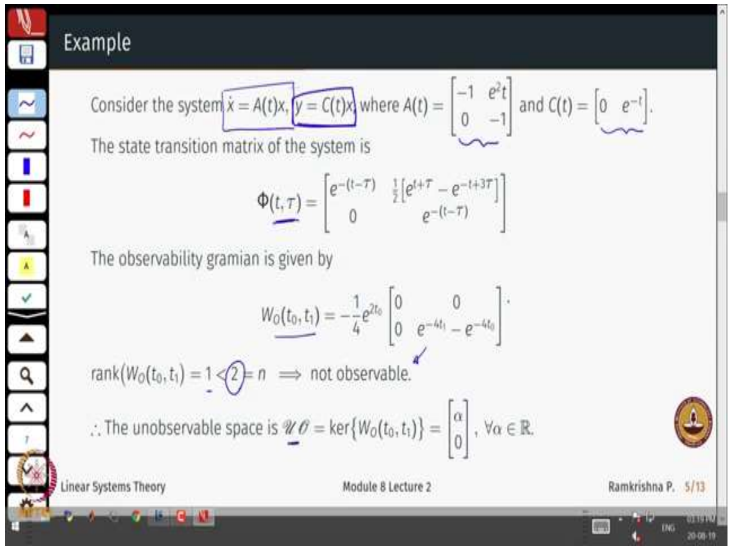

Little example nothing really special here and just computing the Gramian. So, and you just recall some tools what we learned earlier. So, I have $\dot{x}$ = A(t)x(t), y = C(t)x(t). So, in most cases I would not really need information on the input. So, whenever most texts when they talk of observability they kind of neglect the input in the computations this just for obvious reasons because what I am really interested in how do I measure the state or x_0 from the output equation right.

So, the input becomes a little redundant because all those qualities can be computed and this can be added later on. So, we just ignore the input and then just deal with systems like this and all the results which are valid here are valid for the systems with input ok. So, my A(t) is of this form C is of this form I can easily compute the state transition matrix we had several methods learnt in week 4 of this lectures.

So, from this phi the state transition matrix I can compute the Gramian. So, once I have the Gramian I can compute the rank and the simple computation show that the rank is 1 whereas, the full rank is 2 and therefore, the system is not observable. And I can actually construct the observable subspace as the kernel of this one which is any vector α, 0 will be in the observable subspace of the system. So, therefore, the system is not observable.

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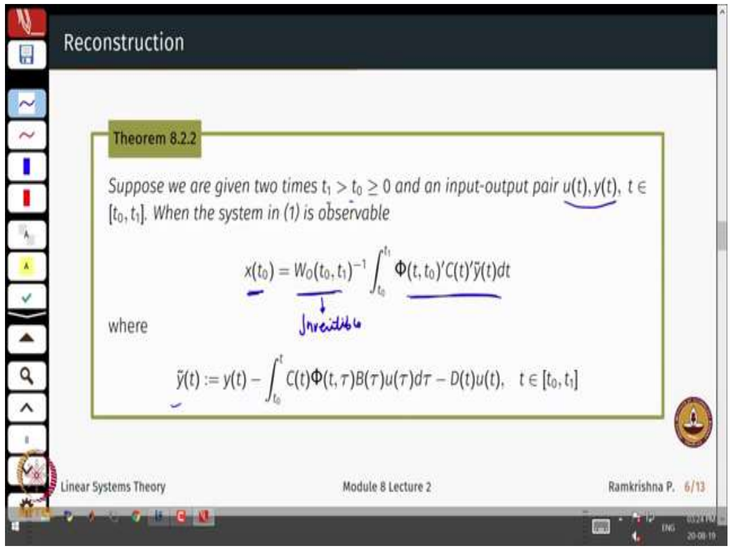

So the next thing we will see shortly is how to given the Gramian how do you reconstruct the state. So, it turns out we will do the computations and then come back to this result here.

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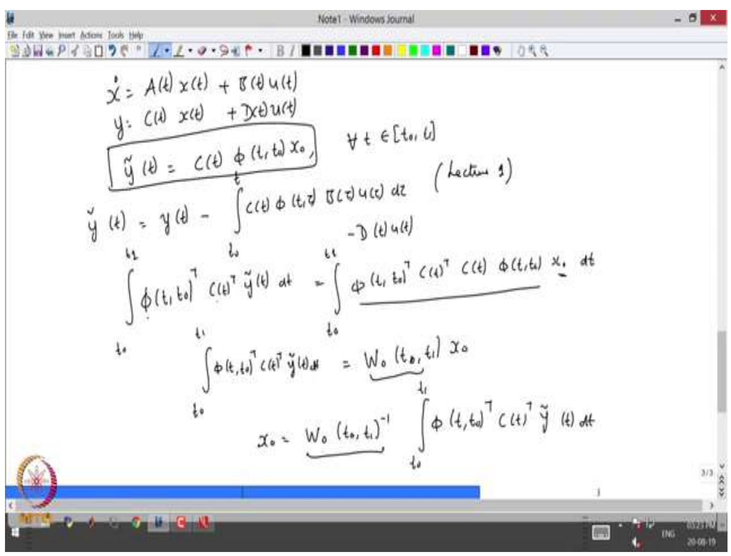

That I start with $\dot{x}$ = A(t)x(t)+B(t)u(t), y = C(t)x(t)+Du ok. So, we had this computation rather than right. So, we had this $\tilde{y}$ and so on. So, we have this $\tilde{y}$ defined as $C(t)\Phi(t, t_0)\ x_0$ for all t in t_0, t_1 with $\tilde{y}$ = y(t) - $\int_{t_0}^{t_1} C(\tau)\Phi(\tau, t_0)B(\tau)u(\tau)d\tau$.

So, this we just refer to lecture 1 of this week on how we get this derivation and then you have a – D(t) u(t) also here ok. Now take this expression here and what I do is we pre multiply this by Φ^T and C^T integrate this from t_0 to t_1 with I getting this equation I pre multiply this with $\Phi^T C^T$ integrate this from t_0 to t_1 and what I get on the right hand side is t_0 to t_1 again I pre multiply this by $\Phi^T C(t)^T C\Phi$ ok.

So, this entire term here in the integral without the x_0 is just $W_0(t_0, t_1)\, x_0$ right this is my observability Gramian in this entire term here with integral without this x_0. So, this is what you have and on the left hand side I still have $\int_{t_0}^{t_1} \Phi(\tau, t_0)^T C(\tau)^T \tilde{y}$ dt. Therefore, are now given this I can compute x_0 as again I can compute x_0 if and only if this W is invertible right.

So, whenever the system is ccobservable which means this W_0 is invertible I can reconstruct the street from the Gramian right from this and this ok. So, now, just the result just says the following right. So, suppose we are given two times t_0 and t_1 and an input output pair u(t), y(t). So, whenever the system is observable I can reconstruct x(t_0) with this inverse of the Gramian and with this $\Phi C\tilde{y}$ and $\tilde{y}$ is just this expression right. So, again we use make use of the for that whenever the system is observable this W should be invertible ok.

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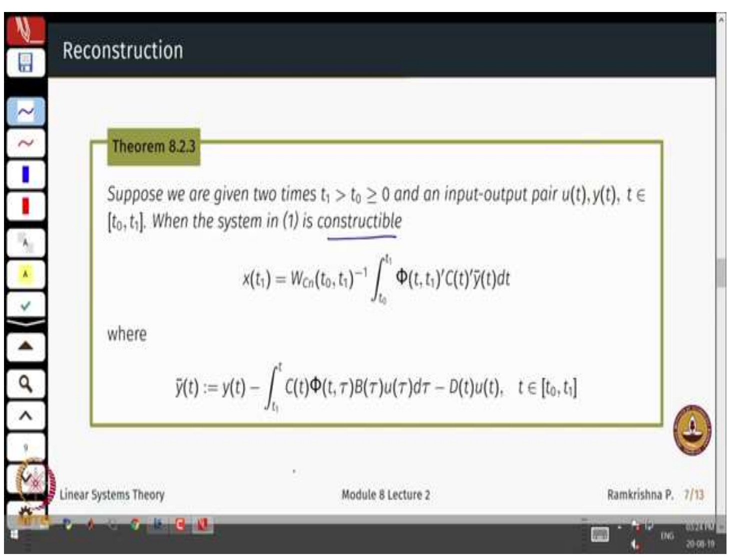

And similarly I can do it for the constructible to check for constructability right. So, I will skip the steps these are exactly similar to what we do in the observable system ok. And a nice interesting thing before we go into the proofs say like the Eigen vector test or the pbh test or the Lyapunov test for observability would be to just check if there is some kind of duality between controllability and observability right.

So, we define Gramians there we define Gramian also so here, we are something to do with the subspaces over there, the reachable subspace the of the unobservable subspace here and so on right. So, we will see if we can make a nice connection between what we derived week 7 and what we are about to derived now ok.

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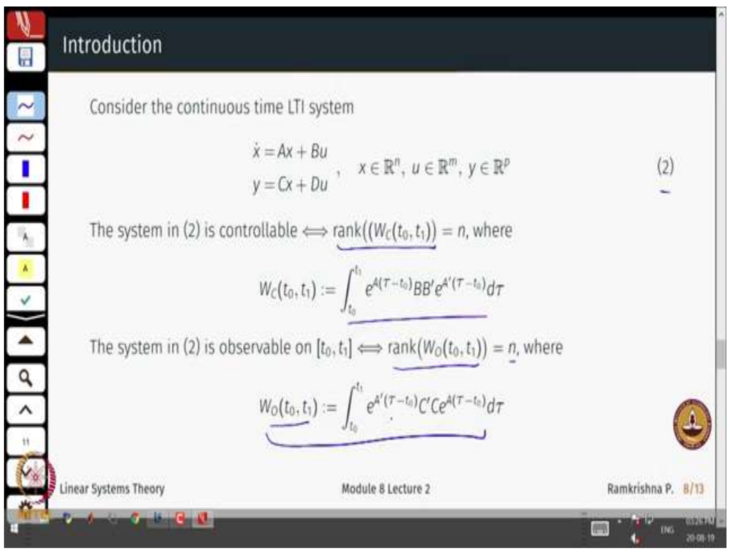

So let me say I start with system continues time LTI system $\dot{x}$ = Ax+Bu, y = Cx +Du ok. So, whenever the system is controllable it means that the rank of the controllability Gramian is n where the controllability Gramian is this is given by this integral over here. Similarly, this system 2 so, this guy is observable if and only if the rank of the observability Gramian is equal to n, which means something like this holds true right. So, which means the this should be invertible and the Gramian looks something like this ok.

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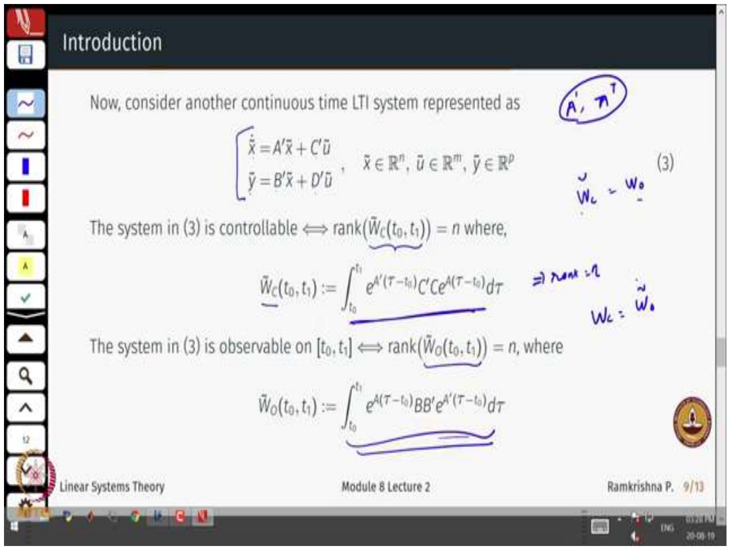

Now let me define alternatively a system in say some other coordinates $\tilde{x}$ with $\dot{\tilde{x}}$= $A^T\tilde{x}$ + C^Tu, $\tilde{y}$ = $B^T\tilde{x}$ + $D^T\tilde{u}$ or so, this sorry these are all transposes right A^T this is C^T ok. So, alternatively we will use sometimes we use this notation sometimes we use transposes. So, both of them actually mean the transpose sorry about this confusion of notations, but essentially this prime also would mean transpose.

So, $\dot{\tilde{x}}$= $A^T\tilde{x}$ + C^Tu and $\tilde{y}$ = $B^T\tilde{x}$ + $D^T\tilde{u}$ ok. So, controllability of this system would again mean that the rank of W_c equal to n where the controllability Gramian takes now this expression this rank should be equal to n. Similarly, this system 3 is observable wherever the rank of this observability Gramian given by this expression is equal to n ok.

Now look at this expression here the way the controllability Gramian is defined here and the way the observability Gramian is defined here well, they look the same right. So, the W_0 = $\widetilde{W_c}$, similarly look at how the observability Gramian is defined here the observability Gramian here looks exactly similar to the controllability Gramian here ok.

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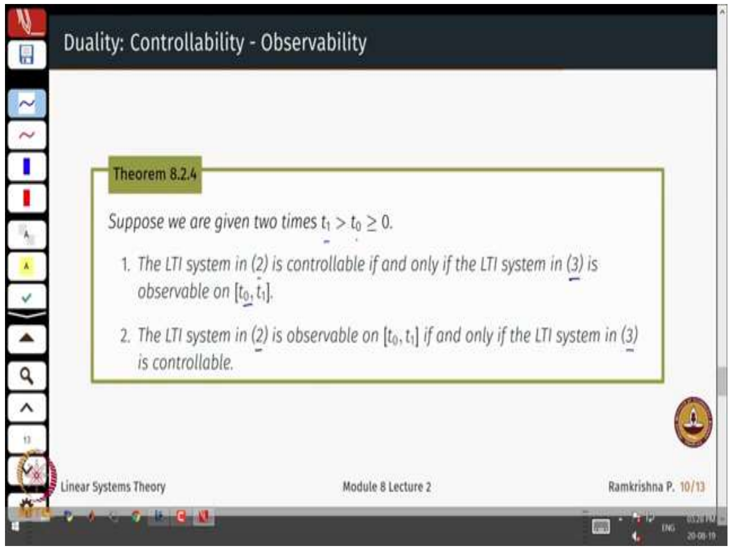

So, there is something nice here right something that we can conclude saying that if you are given again 2 times t_1 $t_0 \geq 0$. This system to $\dot{x}$= A x + B u is controllable if and only if the system 3 is observable again on t_0 t_1 it is one. So, this is system 3, so the controllability condition over here was equal to the observability condition in the previous slide and similarly the controllability condition over there was equal to the observability condition here.

So, I can just write down the steps here. So, I can just conclude that controllability of 2 is equal to the observability of 3. Similarly if 2 is observable then 1 is also controllable and it is both way side. So, 2 is observable if and only if 1 is controllable right.

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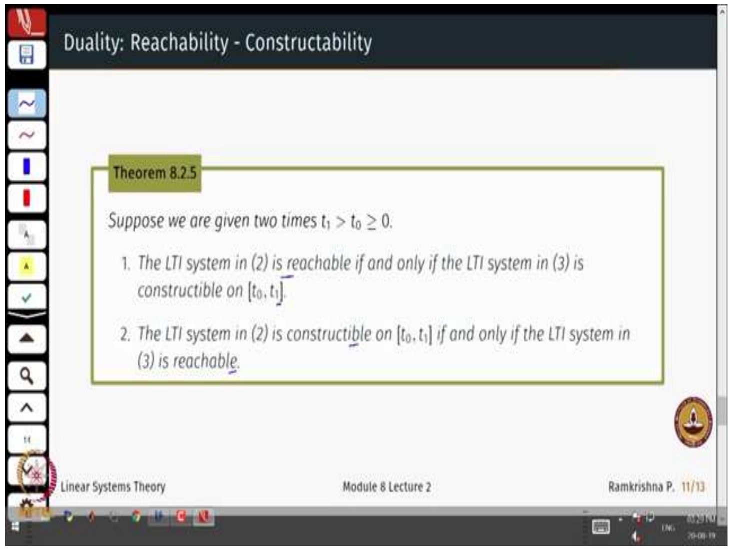

Similarly, I can do it for the reachable and the constructible Gramians right. So, system 2 is reachable if and only if the system 3 is constructible and vice versa. The system 2 is constructible if and only if the system 3 is reachable you can again a check by writing down explicitly the Gramians right ok.

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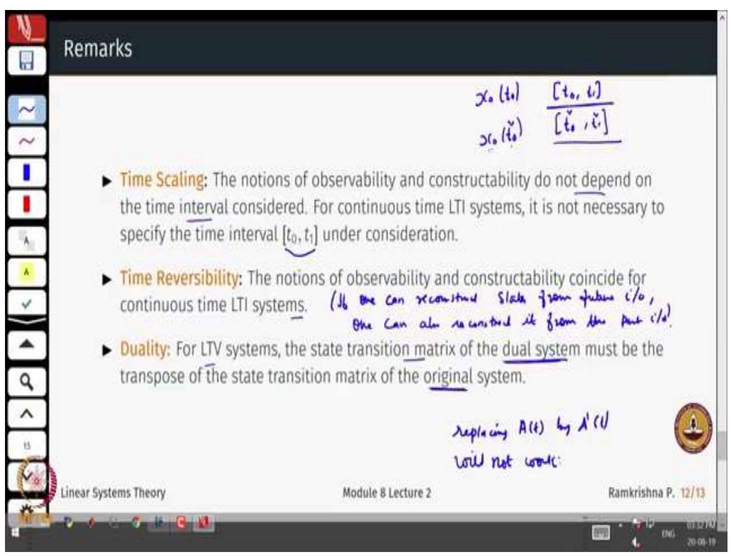

So, again some little observations. So, the notions of observability and constructability do not depend on the time interval considered like for example, it is not really necessary to specify the time t_0 t_1 under construction, which essentially means that if I can uniquely reconstruct the state $x_0$at t_0 right considering the interval t_0 and t_1 for which I know the inputs and the outputs, then it is also possible that I can construct t_0 at some $\widetilde{t_0}$ given any other interval t naught tilde to $\widetilde{t_1}$ right.

So, it is so, when t say it does not really depend on time interval considered it means this one is if I can reconstruct the state in this interval I can also do it in this interval. So, I can find out x if I can find x(t_0) I can it is also equal to so I can actually find x($\widetilde{t_0}$) also ok. So, the notions of observability and constructability coincide for continuous time LTI systems. So, which means that if one can reconstruct the state so, if one can reconstruct the states from future input and output one can also reconstruct it from the past inputs and outputs ok.

It is similar argument of saying that the notion of disability and controllability were similar in the continuous time LTI systems ok. So, a little caveat here because a lot of things we did here just by looking at the transpose right. So, $\dot{\tilde{x}}$ was A^T and then in the Gramian I just replace A by A^T and everything works beautifully. Whereas, a little care must be taken while I am dealing with LTV systems in the sense that for LTV systems the state transition matrix must be the transpose of this the state transition matrix of the dual system must be the state transition matrix of the original system.

So, simply replacing A by A^T will not work. So, this is a little care we may need to take while we are dealing with duality for LTV systems ok. So, that is that concludes this lecture where we saw Gramian based conditions for defining unobservable subspace or even under what condition is the system observable and under what conditions is it controllable.

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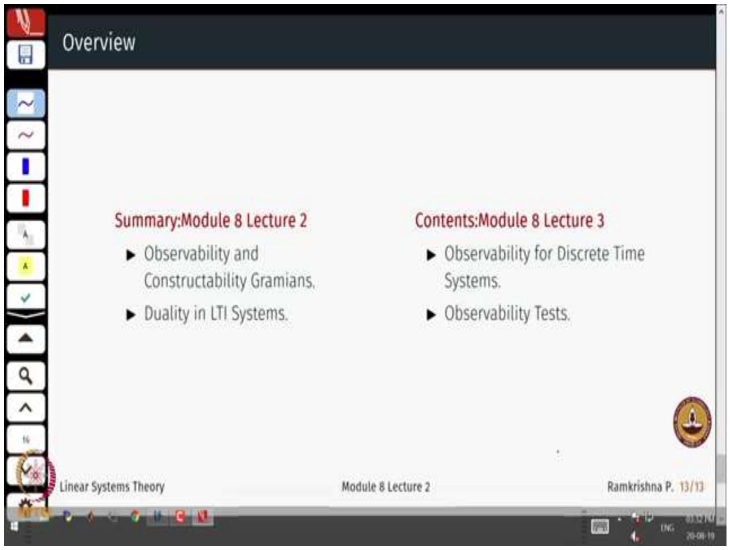

In the next lecture we will quickly run through the concepts of observability for discrete time systems and also do some very basic observability tests.

Thanks.