Linear Systems Theory
Prof. Ramkrishna Pasumarthy
Department of Electrical Engineering
Indian Institute of Technology, Madras

Module – 08
Lecture - 01
Observability

Hello everybody. So, welcome to this week 8's lectures on Linear System Theory. So, we; so, this lecture will be a very brief introduction to the concept of Observability and Constructability. So, let us start with a motivation of why we need to define the concept of Observability.

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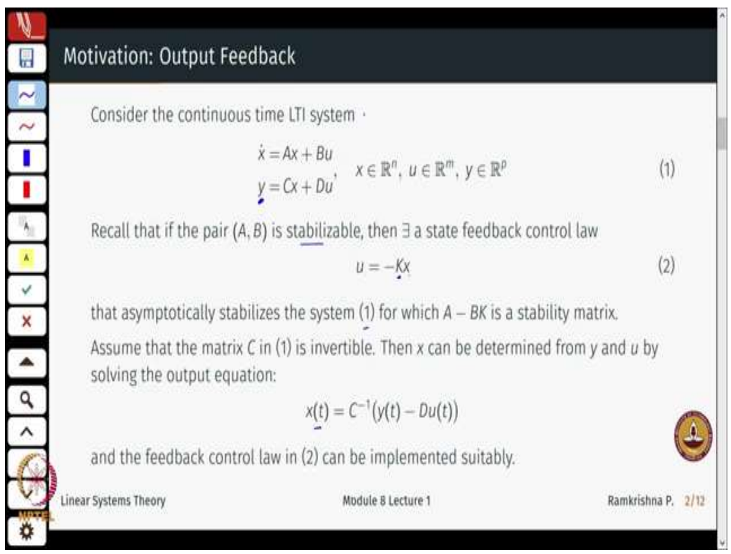

So, a quick recall for of what happened in the previous week's lectures. So, we were talking about LTI systems, you talking about their controllability verification, we were talking of stabilized ability and so on. And towards the end we had a notion that if the pair A, B is stabilize able then there exists a feedback control law of the form u equal to -Kx that stabilizes the system one for which A - BK is a stability matrix, ok.

Now, the point is ok, how do we get this x? So, what do we have for measurements is the following. So, I measure the outputs, these are what are available for measurement. So, assume that if C is invertible then I can determine this x instantaneously and can directly feedback, right. So, this u can become directly computed as u of t is -Kx of t. And the feedback control law the implementation of that is this kind of free will, it is very straightforward.

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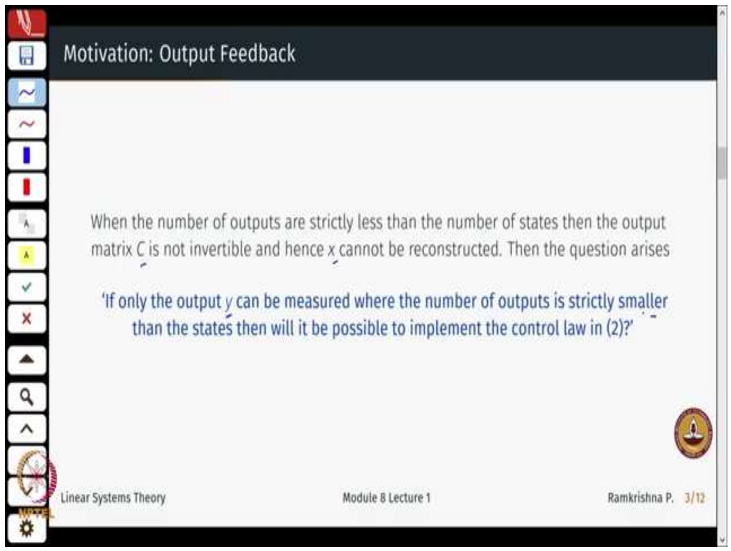

However, what happens is if the number of outputs are strictly less than the number of states then the output matrix C is not invertible, and hence x cannot be instantaneously reconstructed. So, an interesting question to be asked is if only the output y can be measured when the number of output is strictly smaller than the states will it be possible to implement a controller which is u = -Kx, ok. I do not know at the answer at the moment, that I would like to say well let me construct a system where C is invertible, but it is not always possible to measure all the states of the system because of physical constraints and then and so on.

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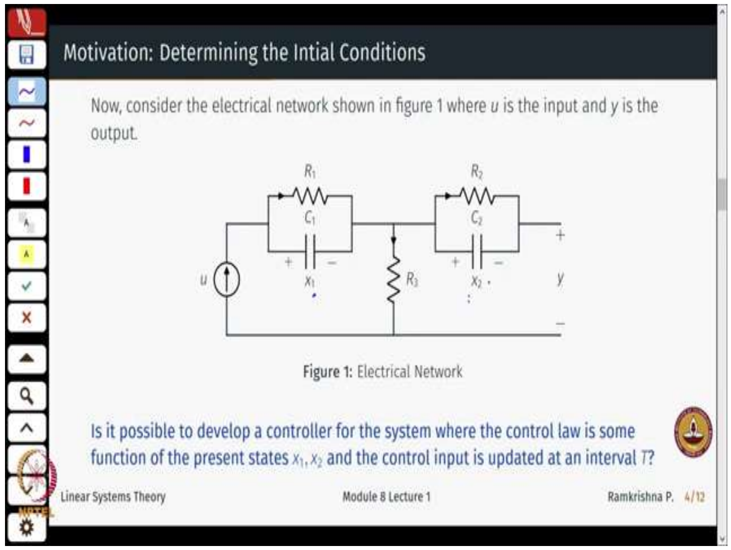

So, how does an example look like? So, I have an electric network which looks something like this, where I have an input u, output is a voltage measured across here. So, is it possible to develop a controller for the system, where the control law is that I want to control x_1 to some voltage, x_2 to some voltage and I just have one output for measurement. So, with this one measurement can I control x_1 and x_2 or even stabilize x_1 and x_2 independently? So, that is what we will try to answer, how with the measurement of outputs can I reconstruct the state? Ok. So, this leads us to define what are called as unobservable and unconstructible subspaces. So, how does the definition go, ok.

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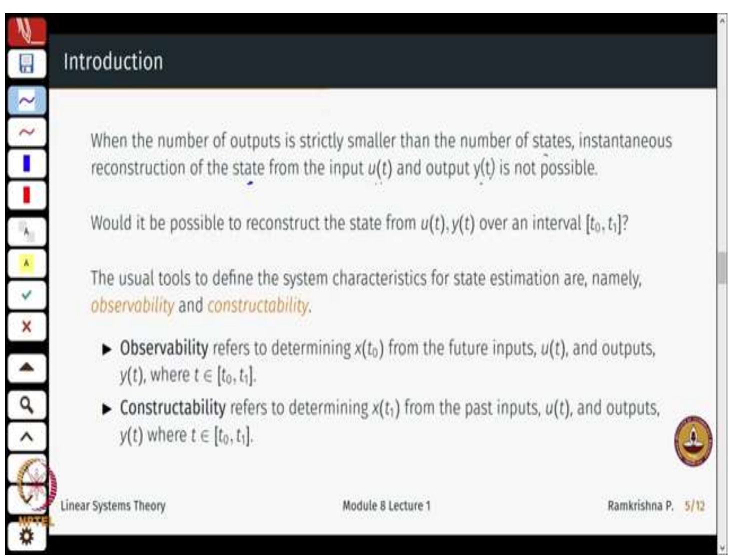

So, when the number of outputs is strictly smaller than the number of states, we cannot instantaneously recollect, sorry reconstruct the state from the input and the output, right. So, that is, what does it mean by instantaneous reconstruction? Can I always do this when C is not invertible? Well, the answer is no, ok. Now, I ask a different question, what it be possible to reconstruct the state given some measurements over (t_0, t_1)? I know u between (t_0, t_1), I also know y(t) over this interval, ok. So, based on this information that I have, I characterize to two notions here of state estimation and because I really want to know what the xs are and these are the notions of observability and a constructability.

So, observability refers to determining x(t_0) from the future inputs u(t) and y(t), right. So, I know all these measurements u t between t_0 and t_1, I know y t between t_0 and t_1. Can actually, actually construct x(t_0)? Ok. So, conversely or it will differently constructability refers should, can I determine x (t_1) knowing the past inputs? So, if I know all inputs from t_0 to t_1 can actually determine what is the state at the time t_1? Ok. So, this are the questions that we will try to address, not really address, but at least define problems related to, how to formulate this based on the information that I have so far, ok.

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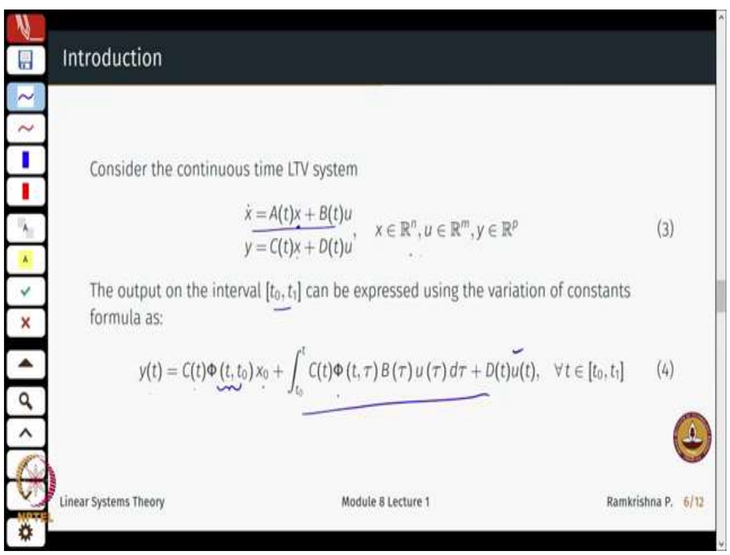

So, the first thing is ok, let us start with a continuous time LTV system with equations of this form. The output, ok; so, we spend a lot of time on how to compute the solutions of x given these equations, right from the state transition matrix and so on. So, I just substitute the value of x here and I get the following, that the output on the interval t_0 to t_1 can be expressed well based by the solutions in this form. So, y(t) is C with the state transition matrix, the initial condition and this terms in the integral and of course, D (t) u(t), ok. I will not I will not go into the details of this. This was done extensively in lectures on week 4.

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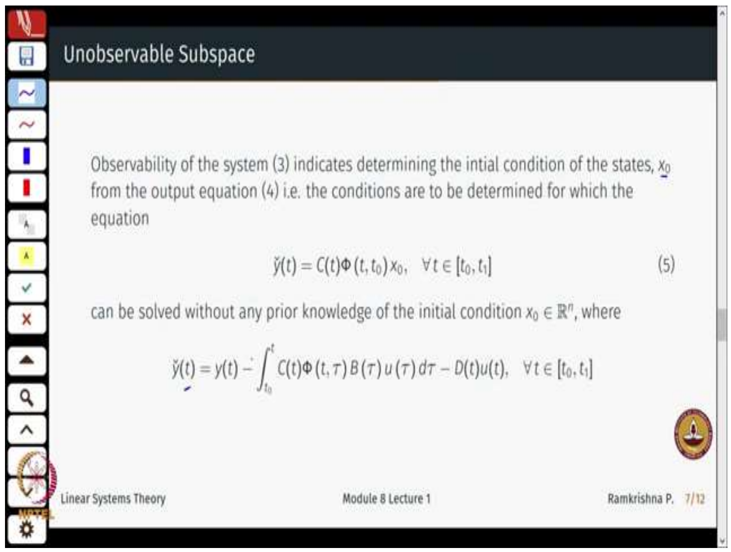

So, observability of the system indicates determining the initial states x naught from the output equation, right. So, this is the output equation, sorry; so, this was the output equation and I want to see if I can determine x_0 from the output conditions, that is, ok. So, I want to determine x naught having known this y, this $\tilde{y}$ here, ok. So, I just; so, $\tilde{y}$ is just. So, I call this as y tilde and then they just turns out to be y(t) minus this item here, ok. Now, so can I solve this problem? Ok.

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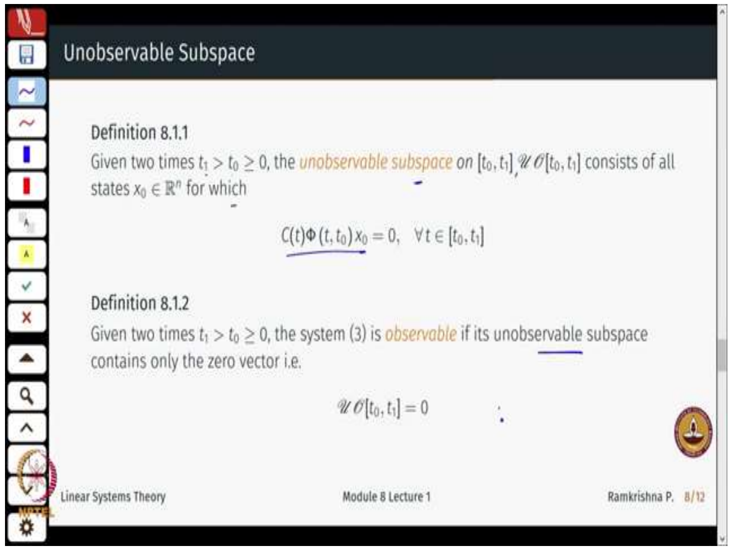

So, what is the definition? So, given two times t_1 and t_0, the unobservable subspace that I denoted by u of this to naught t_1 consists of all the states x_0 in R^n for which CΦx_0= 0, ok. So, this is the, I would really talk of what is x_0 at the moment. I just I am looking for solutions of this equation at, coming from here, ok.

Now, a system is observable if its unobservable subspace contains only the 0 vector, right, which means the unobservable space, right which is all the street so x_0 contains only the 0 vector, ok. So, some nice properties here.

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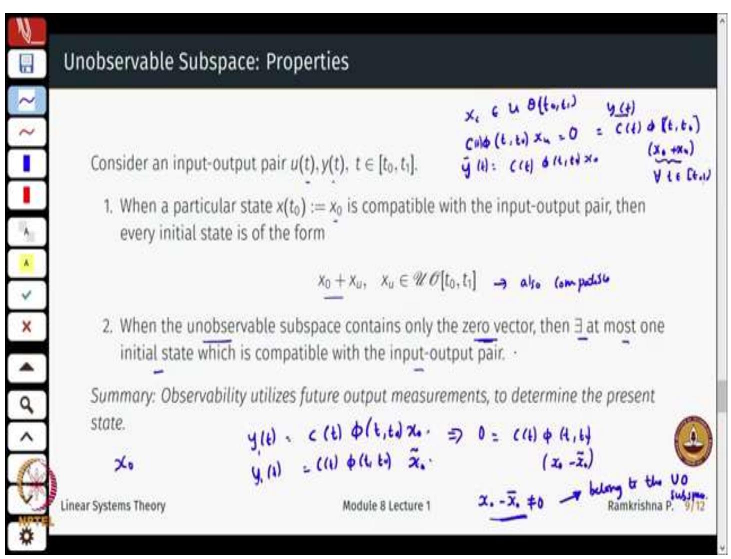

So, if I have an input output pair u(t), y(t), right for all t_0 in t_1 when a particular state is compatible with the input-output pair then every initial state of the form is also a compatible with the input output pair. So, this is also compatible, ok. So, why is this true? Let us say that if x_u is in the unobservable space t_0 to t_1 which means that C$\Phi(t, t_0)$ x_u is necessarily equal to 0, ok. And then what is with the other input with x_0.

So, there will be some y, such that this y of t is C of t, this is also be a function of t here, C(t)Φ(t, t_0)x_0. Then if I add these two what will I have is that y(t) = C(t)Φ(t, t_0), right. And then what do I have here is x_0 plus x_u, this is for all t in t_0 t_1. So, this is also compatible, right, this and this, where the C and Φ, ok.

Then the next step. So, when the unobservable subspace consists only the 0 vector, then there exists at most one initial state which is compatible with the input-output pair. Now, this will be important for this way, right. So, then the unobservable subspace consists of only the 0 vector we see what happens, ok. Now, let us do a little proof by contradiction let us say y(t), let me call this a some y_1, the C(t)Φ(t, t_0) x_0, ok. Now, the unobservable subspace contains only of the 0 vector, then there is at most one initial state which is compatible with the input output pair, ok.

Now, let me see that this is also the same y_1 for which there exists another x_0, say $\widetilde{x_0}$ again coming from the C and Φ, ok. So, there is the same y_0 which comes from x_0 same y_1 sorry, same y_1 which comes from x_0 the same y_1 which comes from $\widetilde{x_0}$. So, this implies that 0, I just do the separation C(t)Φ(t, t_0)($x_0 - \widetilde{x_0}$), ok. What do I know? That the unobservable subspace consists of only the 0 vector and therefore, x_0; so, this when it is not equal to 0, right, this would necessarily belong to the unobservable subspace, ok.

So, this is a contradiction here, right. So, what was the claim? That the, when the unobservable subspace consists of only the 0 vector, so the definitions come from here, right. When the unobservable space consists of only the 0 vector then there exists at most one initial state which is compatible with the input output pair.

So, I assume that there are actually two of them, right. I just do prove by contradiction. If there are two of these initial states, right then I just do the subtraction and I essentially claim here that the 0 is CΦ(x -x_0), which essentially means that x $-x_0$ would have to belong to be unobservable subspace which means it is not true, right. And therefore, x_0 is necessarily equal to $\widetilde{x_0}$. And therefore, when the system is observable I can uniquely can construct x_0, that is what this statement means over here, right, ok.

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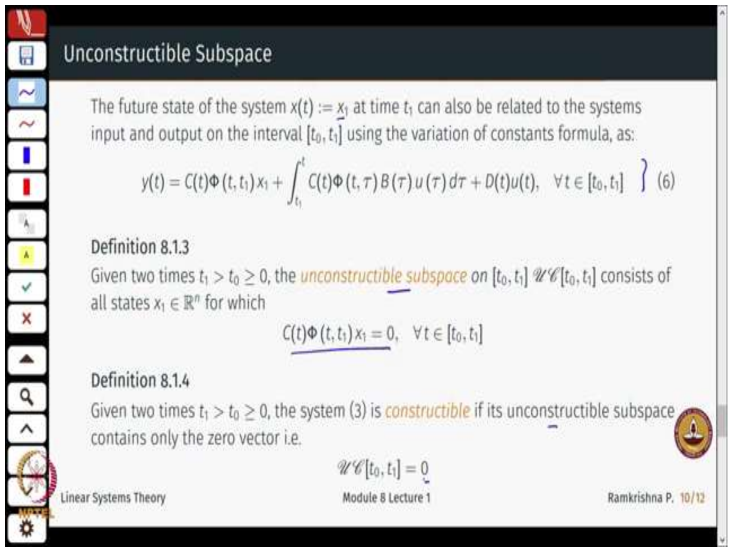

So, similarly with the unconstructible subspaces so, what am I; what is a definition? So, when I talk of constructability I am looking at finding this t_1 given the set of measurements between u and y, between t_0 and t_1, ok. So, the future state of the system x(t) at x(t_1) can also be related to the systems input and output. Again, you use the same formulas that we derived earlier I will not go into the derivation of this, ok. So, what is the definition say? So, if I have two times t_0 and t_1, the unconstructible subspace consists of all states for which this holds, ok.

Now, the system is constructible if its unconstructible subspace contains only the 0 vector, right. So, there is only one; so, if it unconstructible subspace has only the zero vector, similarly to the observable thing, right. So, when system was observable when its observable subspace consisted only of the zero vector.

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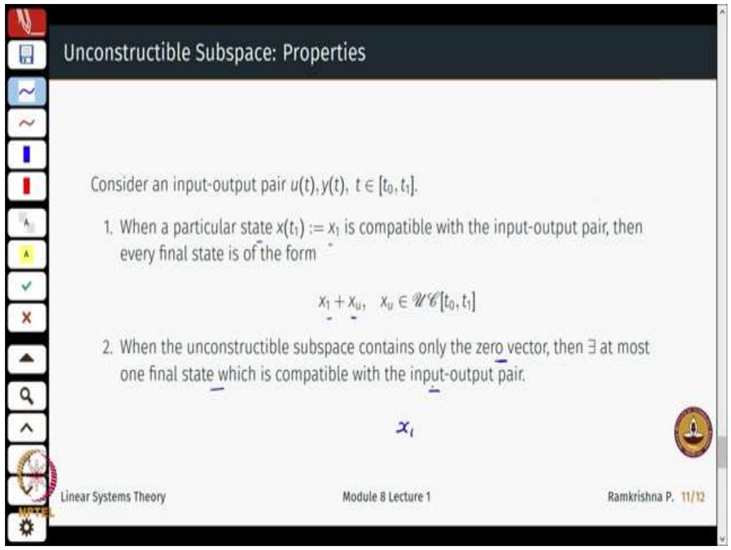

And of course, these two properties are now very obviously, right, as in the previous case. So, when a particular case, when a particular state x_1 is compatible with the input output pair then $x_1 + x_u$ is also compatible, ok. And then when the unconstructible subspace consist only the 0 vector then the x_1 is unique, right then there exist at most one final state which is compatible with the input output pair which is compatible with the measurements that we do between this interval, the measurements y and then the inputs u which are also a measurable. So, this was a brief introduction to the characterization of observability and constructability.

So, we will do a very small example before we end for today.

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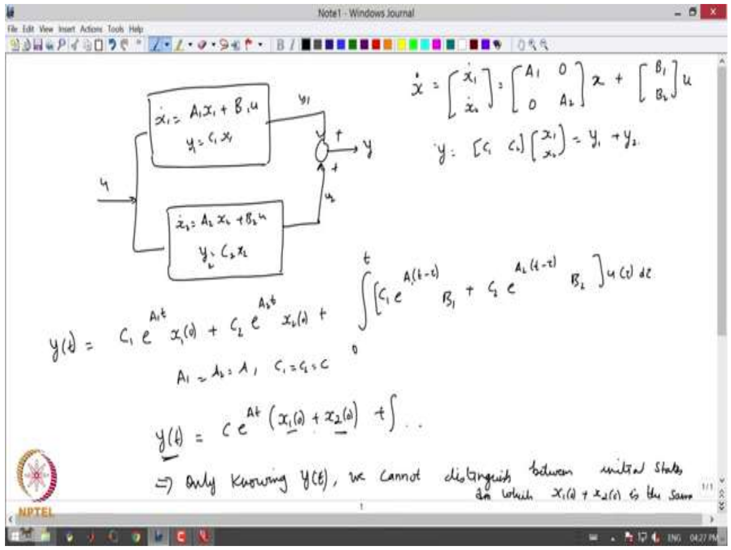

What do, these things actually look like? So, if I have a system, ok. So, let me call this $\dot{x_1}$ is a little bigger one, $\dot{x_1}$ = A_1x_1 + B_1u, and then y = C_1x_1, ok. This is one system and then there is another system here which is $\dot{x_2}$ = A_2x_2 + B_2u y is output here is C_2x_2, ok. And then, I will call this y_1, I will call this y_2, both systems are driven by the same input u, I take both the outputs here y_1 and y_2 and together I measure y (Refer Time: 15:47), ok. So, this is; so we have two systems. So, I think I can we will just write it down. So, the overall system will look something like this, $\dot{x}$ = $\begin{bmatrix}\dot{x_1}\\ \dot{x_2}\end{bmatrix}$=$\begin{bmatrix}A_1 & 0\\ 0 & A_2\end{bmatrix}$x + $\begin{bmatrix}B_1\\ B_2\end{bmatrix}$u. And y = $[C_1\ C_2]\begin{bmatrix}x_1\\ x_2\end{bmatrix}$, that is $C_1x_1 + C_2x_2$ that is essentially $y_1 + y_2$, ok.

So, y of t which is the overall output is given as C_1. Again, I use the same definitions of the C transition matrix the way I compute the solutions and so on. C_1 this will be $C_2e^{A_2t}x_0$+$\int_0^t(C_1e^{A_1(t-\tau)}\,B_1 + C_2e^{A_2(t-\tau)}B_2)$u(τ)dτ ok. When in a special case when A_1, A_2 are all equal to A; C_1, C_2 = C, note that I do not really say anything about the B, because that is not really important here. So, y of t now takes the form of C $e^{At}(x_{10} + x_{20})$ plus all the terms in the integral, ok. So, what is; what does this go on to such?

So, the problem of observability was can I measure just y; I am not measuring this in the individual rate, I am just measuring this and I want to get and get an estimate or what is x_{10} and x_{20}. So, by this expression we can conclude that by only knowing y of t we cannot distinguish between initial states, ok, for which, ok. So, I cannot really measure x_{10} and x_{20} just from the information given to me, ok.

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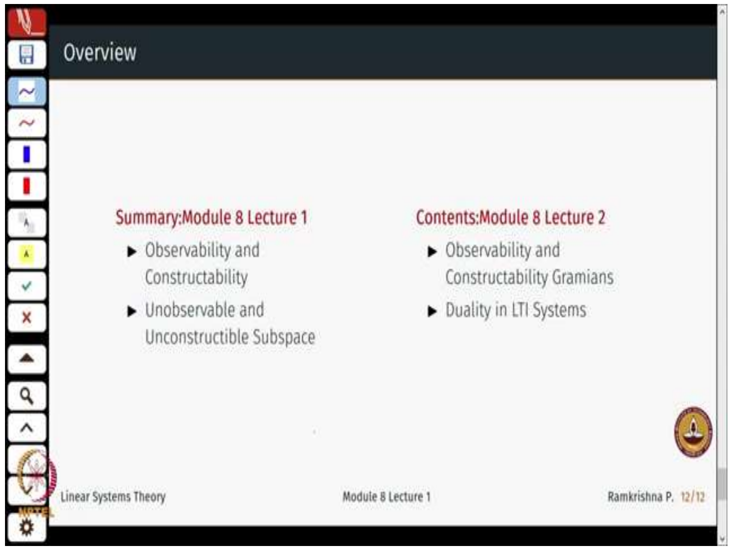

So, what we will do in the next lecture is to find out or to derive conditions by which I can measure observability and constructability. And then, we will also derive a nice equivalence between controllability and observability and see that these two are the dual of each other, ok. So, that will be coming up in the next couple of lectures.

Thanks for listening.