Linear Systems Theory
Prof. Ramkrishna Pasumarthy
Department of Electrical Engineering
Indian Institute of Technology, Madras

Module - 07
Lecture - 05
Controllable Decomposition

Hi welcome to this lecture 6 on week 7 of the course on linear systems theory. So, this lecture we will discuss some important aspects of what if the system is not controllable and how to even identify post the controllability rank conditions of what are the possible modes that are uncontrollable.

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So, we will use the property of invariance with respect to a similarity transformations ok.

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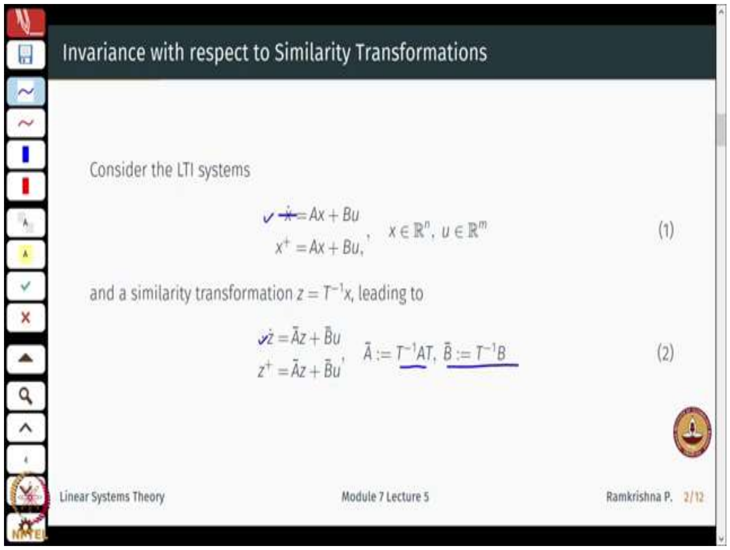

So, we will be as usual interested in both continuous and discrete time systems and what we saw in earlier lectures is that if there is a similarity transformation z = T^{-1} x which takes the system to from x coordinates to the z coordinates. I have a new system $\dot{z}$ = Az + Bu with A new A and A B which look like this.

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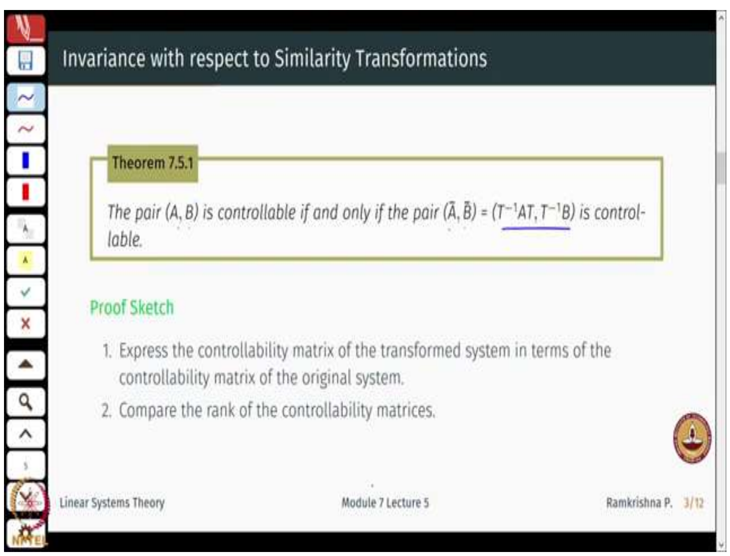

So the first thing that we will look at it is assume that the pair A, B is controllable what happens to the pair $\bar{A}$,$\bar{B}$. So, a quick result says that this pair is controllable or when I say the pair A B is controllable I am essentially looking at controllability of the system ok. So, when this system is controllable then so controllability of this system automatically implies controllability of the transformed system ok. So, it is a very quick proof let us just write down the steps.

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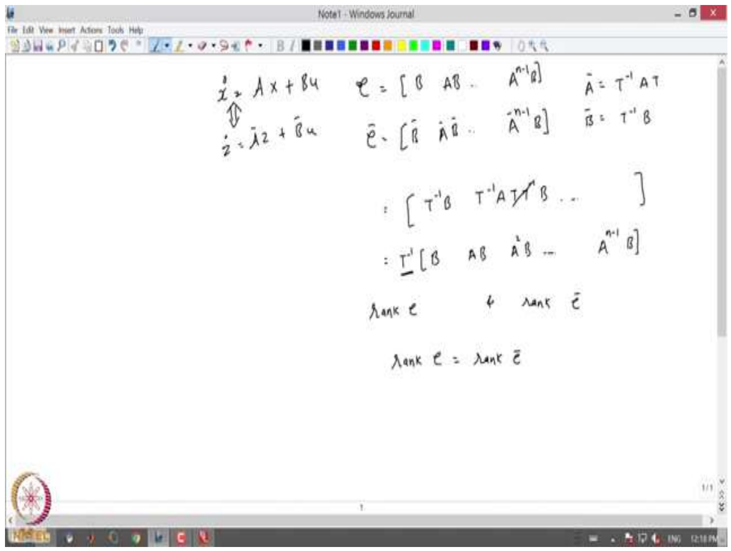

So, what do I have that if I have $\dot{x}$ = Ax + Bu the controllability matrix looks like this. So, we have [B AB . . . . A^{n-1}B] ok. Now I have $\dot{z}$ = $\bar{A}$z + $\bar{B}$u ok. So, what is the controllability matrix here. This will look like $\overline{[B}\ \bar{A}\bar{B}$ . . . . $\bar{A}^{\,n-1}\bar{B}$] ok. Now what is $\bar{A}$ that is T^{-1}AT, $\bar{B}$ is T^{-1}B ok. I just substitute those here. So, what is $\bar{B}$? $\bar{B}$ is T inverse B. What is $\bar{A}\bar{B}$? $\bar{A}\bar{B}$ is T^{-1}ATT^{-1}B right and so on.

So, if I just write this down what I will get here is I just take this T^{-1} out I have A B this will be the identity I have A B. Similarly I will have A^2B as a next term till A^{n-1}B ok. Now let me call this $\bar{C}$. Now what is the relation between rank of C and rank of $\bar{C}$ ok; because this is an invertible matrix that was a condition for the similarity transformation what we get is rank of C is rank of $\bar{C}$ and therefore, if this system is controllable the transform system is also controllable and vice versa right.

So, that is the proof of this at the pair A, B is controllable if and only if the pair $\bar{A}\bar{B}$ which comes as a result of this similarity transformation is also controllable ok.

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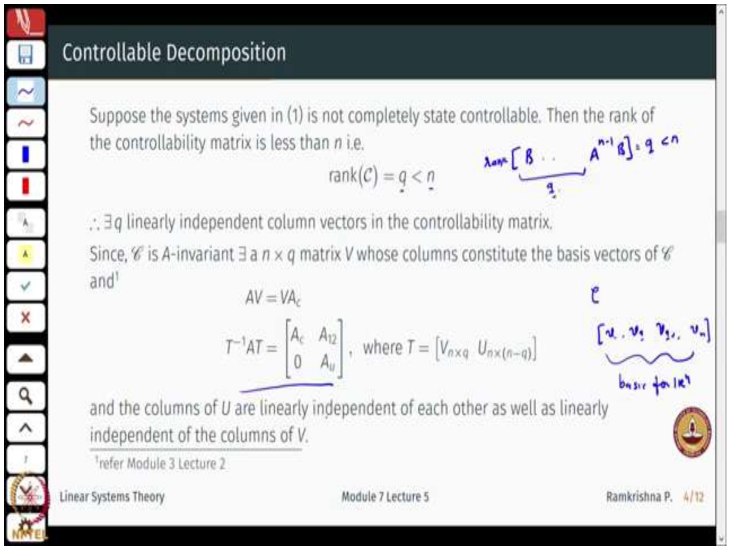

Now, this is important of we will look at what if the rank of the controllability matrix is some q which is less than n. So, the controllability matrix B to A^{n-1}B ideally you would assume that they would have n independent columns right for them to be of rank n. Now if the it is not full rank then it will be of some rank which is q and that is less than n right at least. So, which means that there will at least be q a linearly independent vectors here ok. Now this C will be A invariant ok. Now again.

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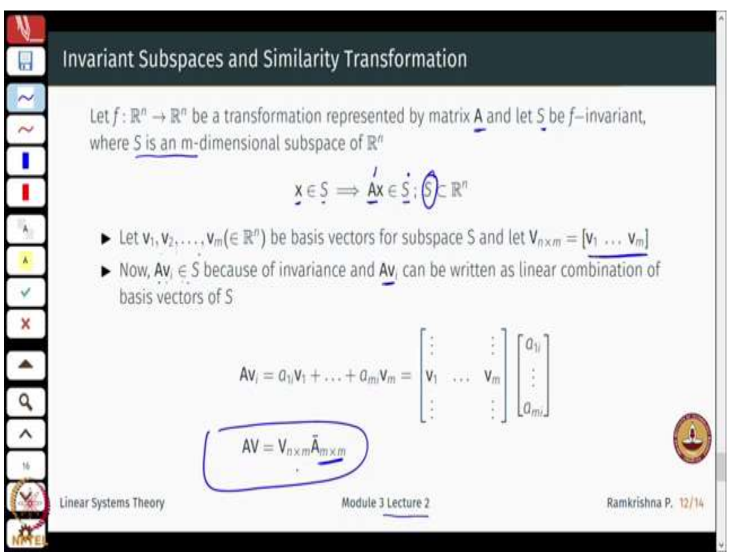

I will just recap what we did here right. So, I was in this lecture from week 3 what we had what we called as an invariant subspace. So, if we had a linear transformation represented by the matrix A S which was an m-dimensional subspace of R^n.

So, here if I look at I have some kind of a q dimensional subspace of the controllability matrix ok. So, now, this S is A invariant which means I take any element x from S I multiply it by A and I get S that I get. So, A times x is again in S which means that S is A invariant ok. Now v_1 till v_m were the basis for the subspaces and then I so, I just call them this m vectors ok.

Now I know that Av_i is in x is in S because of invariance and this Av_i can be written as a linear combination of basis vectors of f and therefore, I can have an expression like this right that AV = V$\bar{A}$ where $\bar{A}$ is a m x m matrix ok. So, this is; this is what we also what we did earlier too ok.

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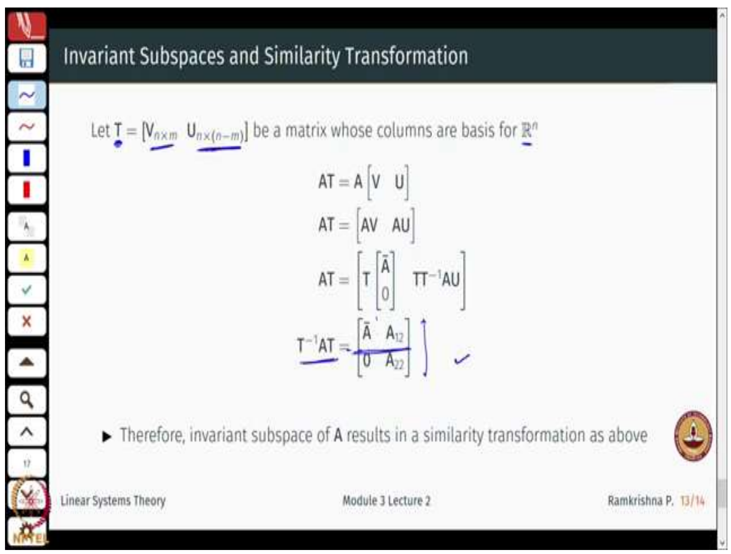

Now, in that case I take this m independent vectors which come from here m independent vectors which come from this subspace I can always add to it n minus m vectors such that this T is a invertible matrix or such that this u together with this v forms a basis for R^n. In that case my T^{-1}AT where T is constructed from here takes this form.

So, it has a nice decomposition here right. I just I can just split it very nicely in this way. Now let us see what this means in the case of controllability or the cases where I actually lose controllability ok. So, I have this q independent columns. Now to this q independent columns I add so v_1 till v_q I can add some v_{q+1} till v_n such that this will form the basis for R n and then get do the similarity transformation ok.

So, now, this T^{-1}AT will look something like this exactly similar to what we were doing here and now it will be kind of obvious of why this particular concept was taught earlier here earlier in week 3.

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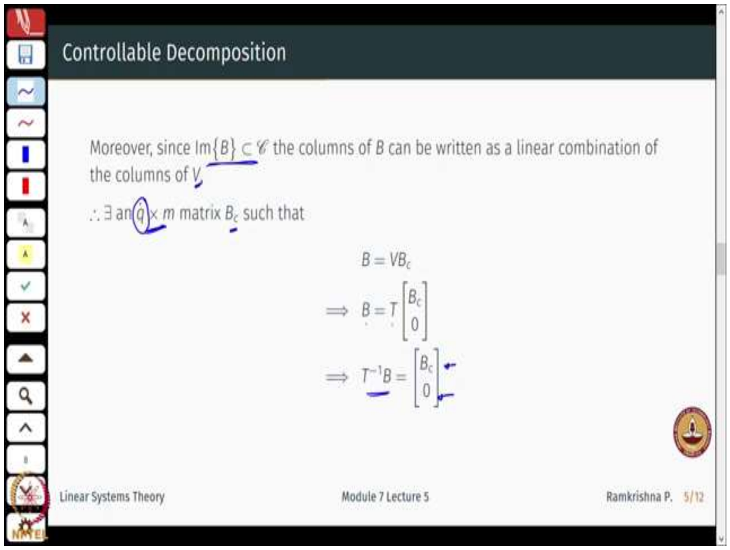

So what happens to B, now since the image of B is a subspace of C the columns can be written as now independent as linear combination of the columns of v. That is there will exist a q x m matrix again in the in the in the similar way B with a suffix c such that B= T B_c or T^{-1}B = B_c and then you have a have a set of 0s here and this will be a q cross m matrix right ok.

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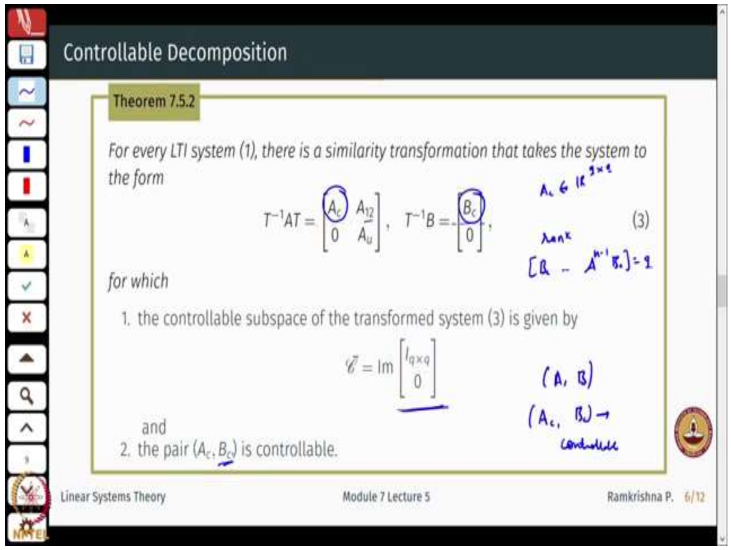

Now, for every LTI system this is what we also proved earlier. There is a similarity transformation that takes the system which is of this form T^{-1}AT looks something like this T^{-1}B looks something like this and in the transformed system the controllable subspace is also transformed like this one. So, only the q modes are controllable right.

So, here the rank(C) = q and therefore, c is now just the image of that q x q subspace and in this case the pair A_c and B_c will be controllable right and this is of A_c is of; is from R^{qxq} and the rank of this matrix B_c till $A^{n-1}B_c$ is q ok. Now this pair is controllable, not A,B is not controllable whereas, some smaller A_c and some smaller B_c is controllable which means there is at least one smaller part of the system that is controllable.

So, what happens to transfer function right? So, it is easier or it is useful now to look back at control one and say well there is something which is going wrong here and the system is not controllable; let us check what the how the transfer function looks like ok.

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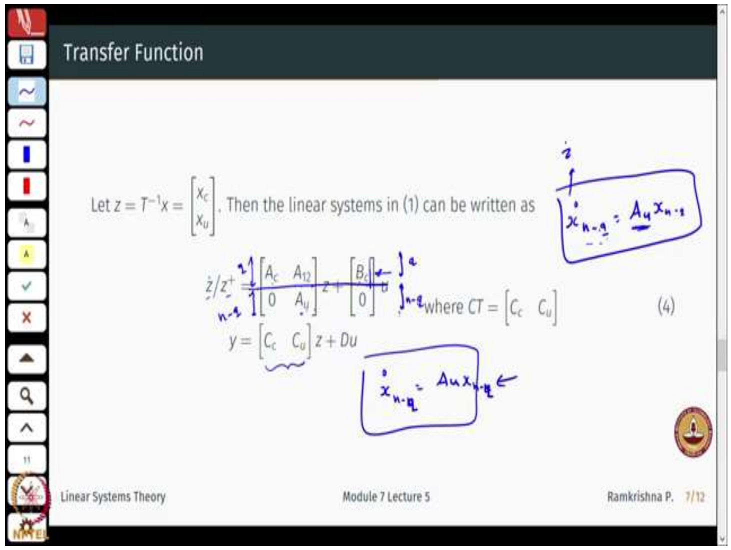

So, I have a decomposition here $\dot{z}$ is also could do the same thing in this week time ah. I have A_c, B_c and so this is with the decomposition that we did earlier. C will have some terms like this; whatever they could be.

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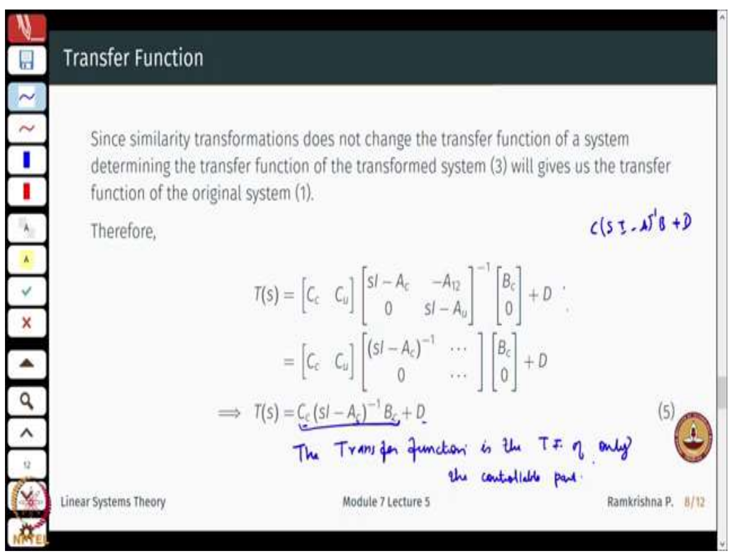

Now, if I write down the transfer function ok. How to do the transfer function? I just do C(sI - A^{-1})B + D. So, I do all those steps should be very easy to check and I just end up with the transfer function just having terms added to A_c, C_c, B_c and of course, T. So, so this is.

So, this is a little important to note here right that the transfer function notice this A_c and B_c this is the controllable part right. So, this pair A_c, B_c was controllable and therefore, I can conclude that the transfer function is the transfer function of only the controllable part. What happens to the uncontrollable part well that is disappears in those as a result of certain pole-zero cancellations.

So, sometimes when I give you a transfer function that may not really be a completely a controllable system right and if you look at also it in terms of so, this is of dimension k and of dimension n - k. So, this the; last n - k, $\dot{x}$ of n - k is just A_u. It has no control entering here right.

So, this $A_u x_{n-k}$ ok. There is no influence of the control input to this n - k states right. So, that is why we actually can nicely separate out the controllable part and the uncontrollable therefore, we call this the A_c and this we call is the A_u denoting the uncontrollable part right.

So, the B_c the control input effects only the first sorry should be q right that is what we assumed a q here and n - q right. So, q is controllable and the remaining n - q are not controllable right.

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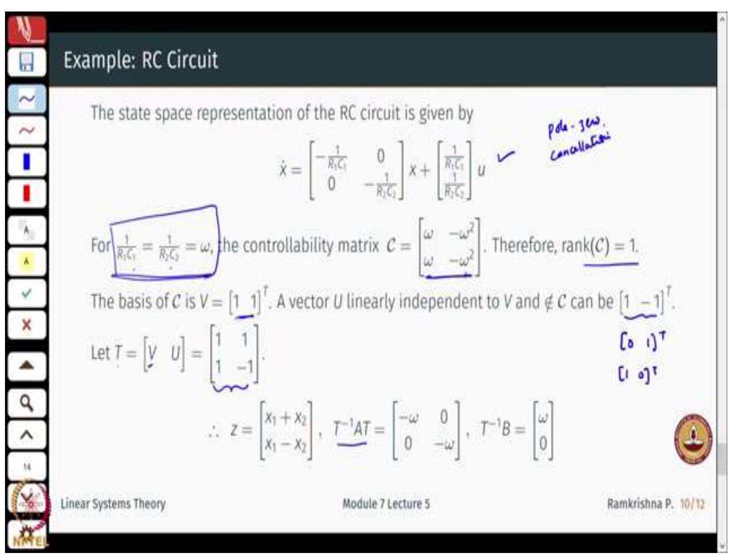

So we go back to this example of the parallel R C circuit. So, we conclude it or we could easily derive that when the time constants were equal the dimension of the subspace is just one or the system is not is not controllable that the rank of C is 1 ok. Now if I just look at the controllability matrix this is there is a linear dependency here therefore, I can say that rank of C equal to 1 if I just chose a basis $\begin{bmatrix}1\\1\end{bmatrix}$ right.

So, this vector together with the vector which is linearly independent of 1 just say $[1 \quad -1]^T$ you could also you look at $[0 \quad 1]^T$ $[1 \quad 0]^T$ and so on ok. So, I construct this transformation T from V; V which comes from the number of independent columns of the controllability matrix which is 1 in this case and U which is an additional vector we construct in such a way that this U plus V together span R^2 right.

Now I do all the transformations T^{-1}AT will look something like this T^{-1}B will be of this form ok.

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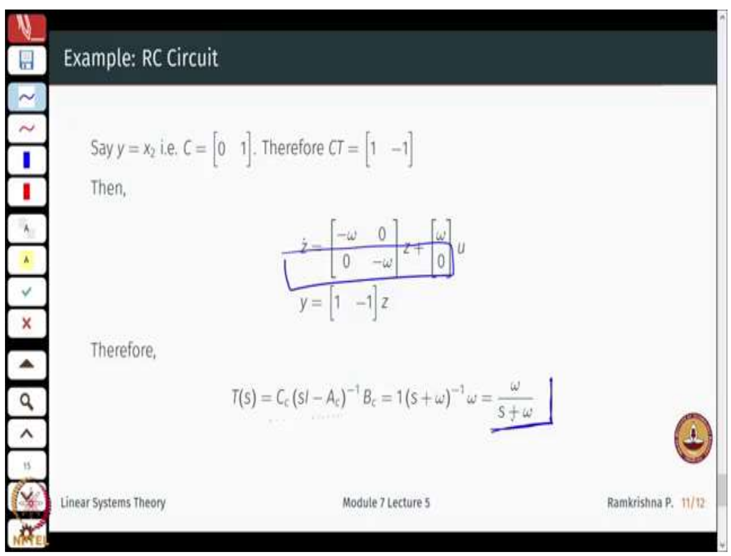

So, I have a system which now looks like this. So, if I look at here see the z_2 term does not have any control influence. It is only the z_1 that is getting effected by control by the control input. So, you would expect the transfer function to be have 2 poles, but then if I just compute the transfer function from the from this formula I just get that it is just has 1 poles sorry just a 1 pole.

Alternatively what you could also do is compute the transfer function of this system assuming a certain output and then when you plug in this equality there that the time constants are the same then you will essentially see a pole-zero cancellation ok.

So, in this case also I am dealing off us with a system which has two states or essentially 2 poles but then because of the dependence of time constants on each other or the time constants are equal there is a pole-zero cancellation and is look at it as a first order system.

Ok and this is some this is something which we miss while we do the transfer function (Refer Time: 15:27) analysis we assume that everything is nice and there is no pole-zero cancellation and so on ok.

So, now, just look at just to conclude so, if I have this modes $\dot{x_{n-q}} = A_u x_{n-q}$ ok. What do I do with this system? If I say well I can only control the first q modes what happens to this guys or it should be written in z coordinates right not in x, but does not matter. It is just a matter of notation.

So, if this A is such that the this that the eigenvalues of this are unstable then the overall system is also unstable but there is some hope when I say when this eigenvalues are stable that this x that the remaining n - q states asymptotically go to 0 then I can do something with the controllability properties of the pair A_c and B_c.

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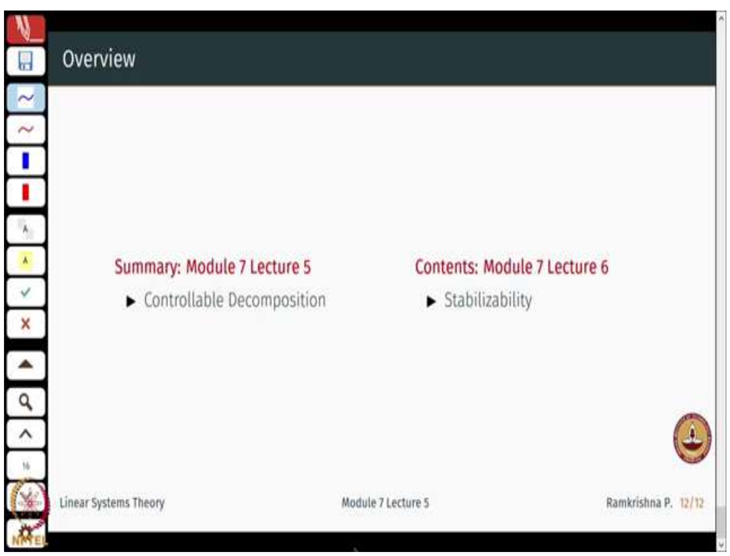

So, that is what we will look at when I, when in the next lecture where we talk of a weaker form of controllability which is essentially to do with stabilizability right. So, that; so we will focus on this controllable decomposition what to do with the uncontrollable modes and what to do with the controllable modes.

Thanks.