Linear Systems Theory
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Module - 07
Lecture - 04
Controllability for Discrete Time Systems

Hi everybody, welcome to this lecture 4 of week 7 on Linear Systems Theory. So, so far we had derived conditions for controllability. We had also checked equivalent methods starting from the PBH tests to the eigenvector test. Is there any relation between Lyapunov stability and controllability and so on?

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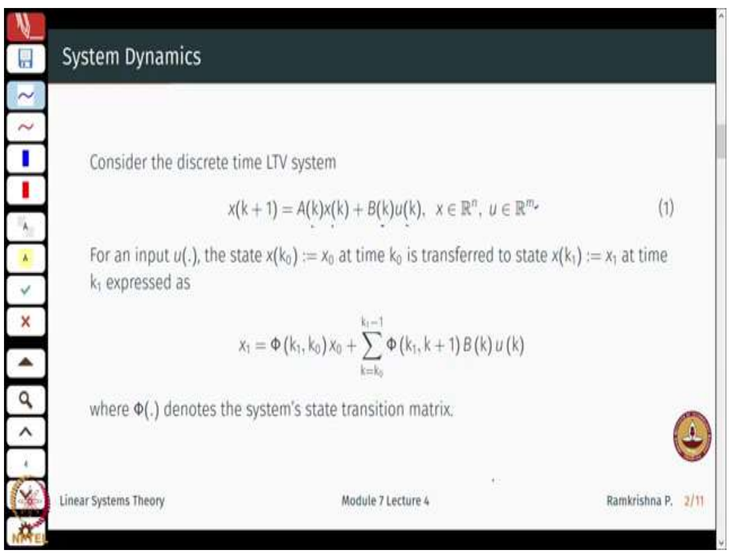

So, this lecture we will talk mostly about discrete time systems. Discrete time systems layer usually given in this form x(k+1) = A(k)x(k) + B(k)u(k) for some M dimensional input. So, how does one define for here the state transition matrix?

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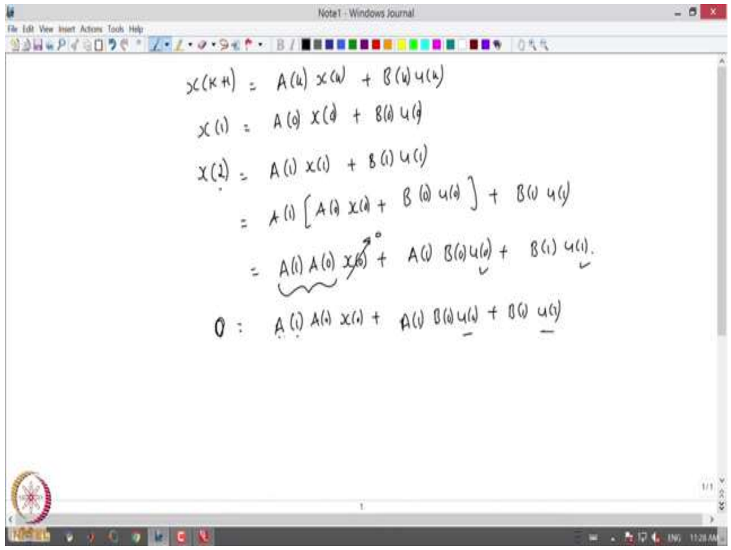

We start with systems of the form let they be depending on time like the A and B. B matrices are also depending on time ok. So, how do I go about computing the solution of this or even the straight transition matrix? Now, very similar to what we had done earlier. So, let us start with a k = 0 and check how the solution looks like, $B(0)u(0)$. Similarly I can compute x(2) as A(1)x (1) + B(1)u(1). So, this is what is x(1).

So, this is A(0)x(0) + B(0)u(0) + B(1)u(1) So, this is A(1)A(0) x(0) + A(1)B(0)u(0) + B(1) u(1) ok. So, in general, so if I call this as my state transition matrix. I can write it write a general expression for this right. So, and then x(k_1) = x_1 ok.

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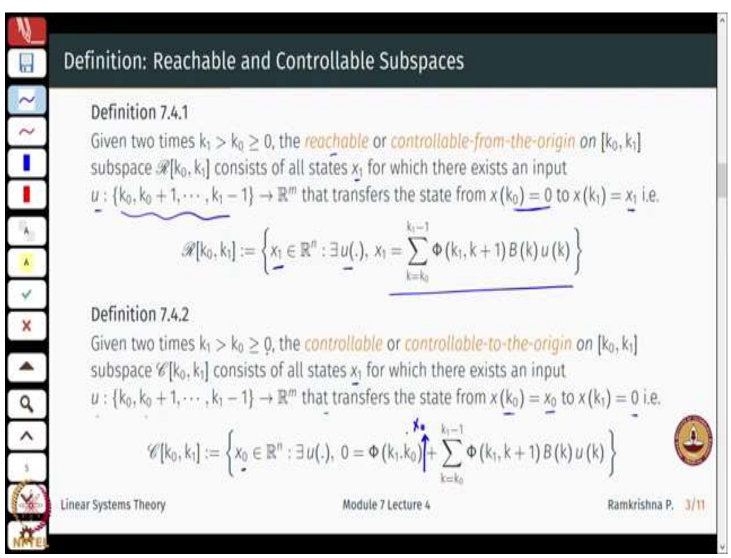

So, again the notion of the reachable and the controllable subspaces remains the same. So, we will just see about its read out the statements again. So, given two times k_1 this is greater than 0 and k_0 can be equal to 0. The reachable subspace consists of all states x_1 for which there exists an input sequence right. So, the reason why we just need this input sequence if you look at here sorry here is to define except to I need u_0 and I also need u 1 ok. So, I need this input sequence the transfers state from x(k_0) right.

This is the origin to some state x(k_1) right that is the set of all possible x_1s for which there exists a control which can solve for this equation. So, this is just by substituting x_0 = 0 in this in this expression. Similarly, for the controllable subspace so, I have given two times k_1 with the same conditions the controllable subspace consists of all states x_1 for which there exist input sequence that transfers from any initial condition to the origin. So, this is just looking at that my final state is the origin.

So, I have was 0 on the left hand side and that would just turn out to be something like this. So, all the initial states x_0 for which there exists a solution to this equation right. So, to something so, if I were to just find out in this case of 2. So, I am just say what is; so, you have 0 is A(1)A(0) x_0 + A(1) B(0)u(0) + B(1)u(1). So, I am finding a solution for this u for all possible x_0. So, that is what this expression says I think this and x_0 missing here.

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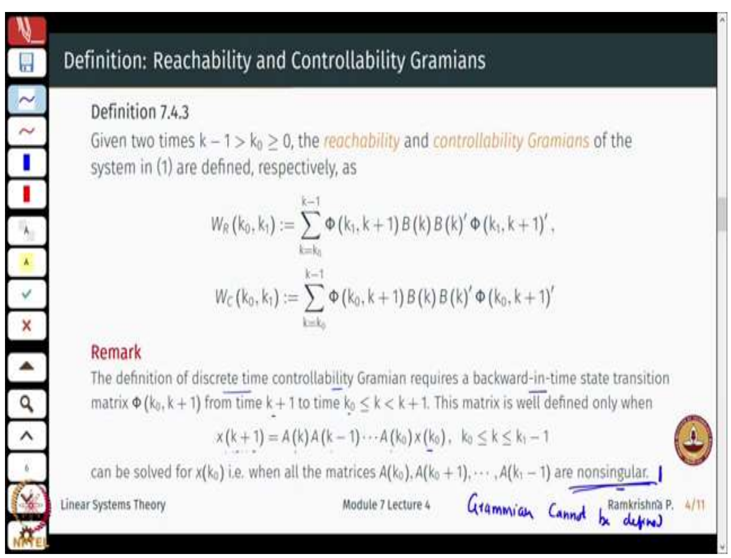

In a very similar way as that for continuous time systems we define the controllability Gramians right which will be useful for us in checking the conditions for controllability ok. So, there is a slight distinction when we translate from the control time systems to the discrete time systems ok. So, what is an important thing here to observe is that the definition of discrete time controllability it requires backward in time state transition matrix from k +1 to k_0.

So, if I just look at the first expression. So, in general x (k+1) can be written as A(k)A(k-1) all the way till A(k_0))x_0 and I can solve for this x_0 if and only if all these A's are invertible or they are non singular ok. So, we will again come back to what this means what happens to the Gramian in case these A's are singular matrices we will discuss this again when we look at LTI system. So, that we can actually fit in and compute it very nicely an interpret what this statements could be ok.

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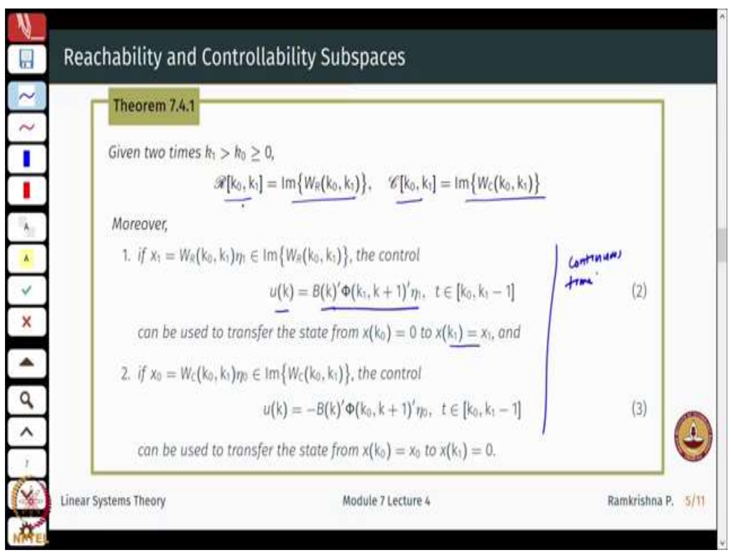

So, the first theorem is its very easy to check is that the reachable subspace is just the image of the reachable, reachability Gramian in the discrete time case and the input that steers my state from the origin to any x(k_1) is just given by this expression. Similarly I can do for the controllability test also right. So, if so the controllable subspace is the image of W_c, essentially means that I am just a point just taking all the set of points that can be reached from the origin that will be just the image of this W_r here.

Similarly for the controllable subspace the proofs are very similar to those of the continuous time case. So, I will skip the proofs over here is this exactly the same steps.

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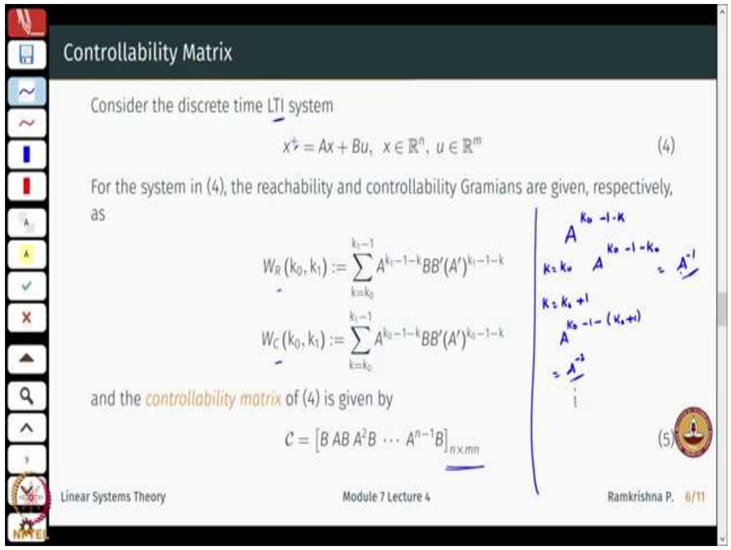

So, here well what is the notion of a of a controllability matrix. So, we start with a discrete time system just for ease of notation I just write it x+ instead of x(k+1) = A x(k) and so on. Again I just switch to linear time invariant systems, now in this case I have the Gramians defined respectively in the following way and the controllability matrix remains the same ok. So, let us reinterpret this statement here. So, what is the definition of discrete time controllability Gramian?

So, here let us say I am just interested now in this power set $A^{(k_0-1-k)}$ ok. So, when k= k_0 the first term becomes $A^{(k_0-1-k_0)}$that is A^{-1}. Similarly in the second time step I have $k_0 + 1$ is $A^{(k_0-1-k)}$. So, what is k is now? k = $k_0 + 1$. So, this is A^{-2} and so on and therefore, you see that there is some requirement for the invertibility of A right.

So, this is what it means here right. So, all these matrices are non singular and if this just not happen the Gramian cannot be defined ok. So, this is a little distinction between the control the continuous time and the discrete time evolution. No such problem though exists here. So, here we are we are kind of ok.

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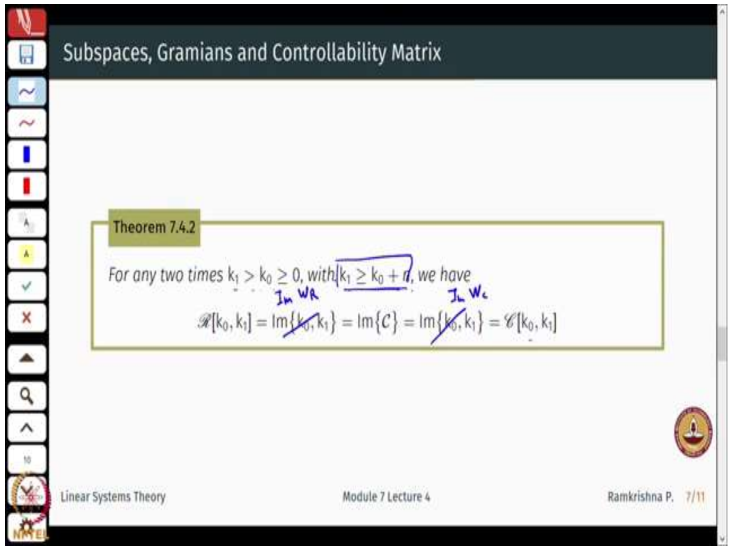

So, the next theorem says that again given any two time instances with k_1 satisfying this I will tell you why this is important. We have the same similar theorem right that the reachable subspace which is equal to this should be the image of W_R is image of C. So, this will be the image of the controllability Gramian which is equal to recontrollable subspace ok.

So, I will do a part of this proof and the one which is obvious I will just leave it as an as an exercise ok. So, and then also we also look at why this particular thing here is important ok. So, what is to be shown is the following right.

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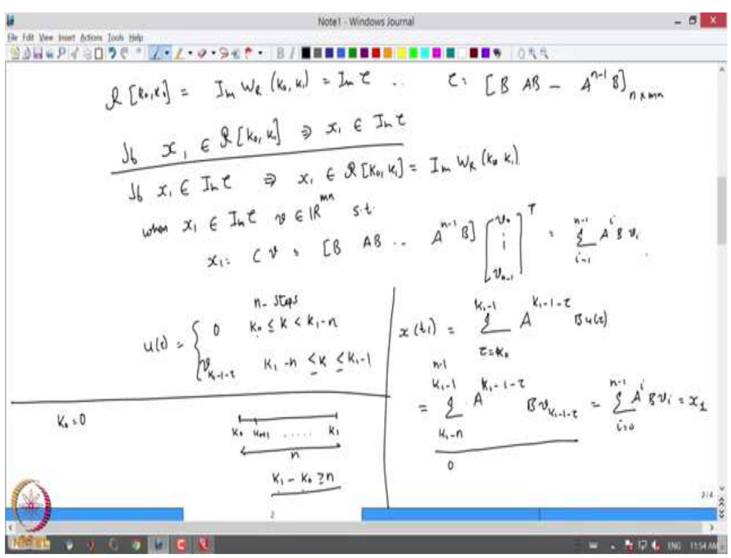

So, R (t_0, t_1) is the image of W_r sorry just do in the discrete time says k_0, k_1 this is image of C. I will leave the other the other proof. I will just do this part of the proof. So, what is easy to prove is the following. So, we start by showing the first step is if x_1 is in the reachable subspace then it is easy to show that this x_1 is also in the image of C. So, what is C is just the controllability matrix [B AB. . . . $A^{n-1}B$].

So, this part is easy. So, we will show the converse that if. So, assume that x_1 is actually in the image of C ok. Now, that we have to check if this actually means that this x_1 is also in the reachable subspace, which is nothing, but the image of the reachability Gramian from k_0 to k_1 ok. Now, when x_1 is in the image of C which means there will exist a vector sorry vector v which is in R^{mn} such that what is the dimension of this. So, this is this will be of dimension n cx mn such that x_1 is C v. So, this is [B AB. . . . $A^{n-1}B$] times this is define these vectors v_0 till v_{n-1}.

So, this is summation $\sum_{i=1}^{n-1} A_i B v_i$ ok. How do how do we understand this?

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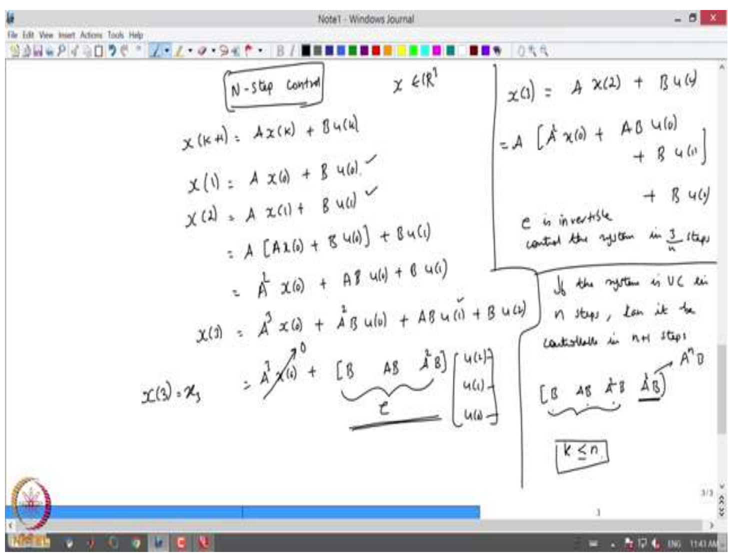

So, whenever we talk of discrete time systems we talk of the n step control ok. What does it mean by the N-step control? So, let me just write down a simple equation here x(k + 1) =Ax(k) + Bu(k) ok. So, what does this do? So, I have $x_1 = Ax_0 + Bu_0$ assume for simplicity here that u is just a scalar then $x_2 = Ax_1 + Bu_1$. So, this is A ok.

So, this is A^2x_0 + ABu_0 + Bu_1 ok. Similarly I can write x_3 = A^3x_0 plus. So, x(3) let us write this is x (3) = Ax(2) + Bu(2). So, this is what is x(2), x (2) is A^2x_0 + ABu_0 + Bu_1 + Bu(2). So, that will be A^3x_0 + A^2Bu_0 + ABu_1 + Bu(2) ok. So, let us just assume that n equal to 3 here. So, this is A^3x_0 +[B AB A^2B]$\begin{bmatrix} u(2) \\ u(1) \\ u(0) \end{bmatrix}$ ok.

So, in when we do the controllability so, this is my C matrix ok. Here I am just assuming that this u is a scalar just for ease of understanding here. So, the controllability would mean that I should be able to reach any point x_3 is say x_3 starting from the origin. So, let me assume that this is the origin then it should mean that this C is invertible right ok. Now this is like the 3-step Gramian. So, I have this step one, step two, step three. So, in generalized I talk of the n step control. So, the idea here is ok. So, if this is controllable means C is invertible and I can control the system in say 3-steps or in general this is n steps. So, if I cannot control the system in n steps. Can I control? So, I just write this down.

If the system is uncontrollable in n steps is can it be controllable in n + 1 steps ok. So, let us look at it right. So, the then my controllability the Gramian that I compute sorry the controllability matrix would be [B AB A^2B] and say A^3B or in general this would be this A^nB ok, but what do I know up from the Cayley Hamilton theorem that this A^nB or A^3B in this case can be written as a linear combination of this 3 and therefore, this does not contribute to the rank. So, if it is controllable in n in n steps it is controllable. So, and if it is not controllable in n steps it cannot be controllable in n + 1 steps.

However there are possibilities where it could be controllable in some k steps which is less than n for example. So, I will do an example with this. So, this is the idea of n step control and that is where the Gramian comes in right. So, we compute the Gramian for n steps, but if I go for n + 1 it does not really add anything to my control not to my controllability properties, but what does it also go on to show is that I must compute this for at least. So, I must let the system evolve for at least n steps right. So, this is step one, step two, step three in this case because here I am just for simplicity assuming its R^3 even though you could just generalize it for n dimensions ok.

So, back to here alright ok. So, what we have to show is now that the control at least goes around four n steps and therefore, I can write u(τ) now as being 0 for $t_0 \leq$ t < t_1 - n and this is $v_{t-1-\tau}$ for t_1 - n ≤ t_1 - 1 ok. I will quickly explain what this means, but just let us let us write down complete the proof first before we understand this. So x so I should do a k here right because I am talking in discrete times.

So, this is sorry $v_{k_1-1-\tau}$ this is $k_0 \leq$ k < k_1 - n. This will be again k_1 - n ≤ k ≤ k_1 - 1 ok. So, x (t_1) is $\sum_{\tau=k_0}^{k_1-1} A^{k_1-1-\tau}Bu(\tau)$. So, now, this now this τ is 0 from k_0 to k_1 - n. So, I just start from here right. It is this is equal to $\sum_{\tau=k_1-n}^{k_1-1} A^{k_1-1-\tau}Bv_{k_1-1-\tau}$ this is a simply $\sum_{i=0}^{n-1} A^i Bv_i$ ok. What do I know of this? What do I know of this because I know that.

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x_1 is in the image of C. So, I will get x_1 ok now let us let us quickly understand what these things mean ok. So, one way to understand this quickly is let me assume that k_0 = 0 and what do I need for control is I need to go at least n time steps. So, this is k_0 k_0 + 1 all the way till k_1 in such a way that that this should be n right that k_1 - k_0 should at least be equal to n right that is what this expression says and why do I need at least n time steps is for to compute the controllability matrix.

Now, so, this is this is one way of looking at it now if I just look at k_0 = 0 and that k_1 is such that k_1 - k_0 ≥n then let us look at how this expression translates to ok.

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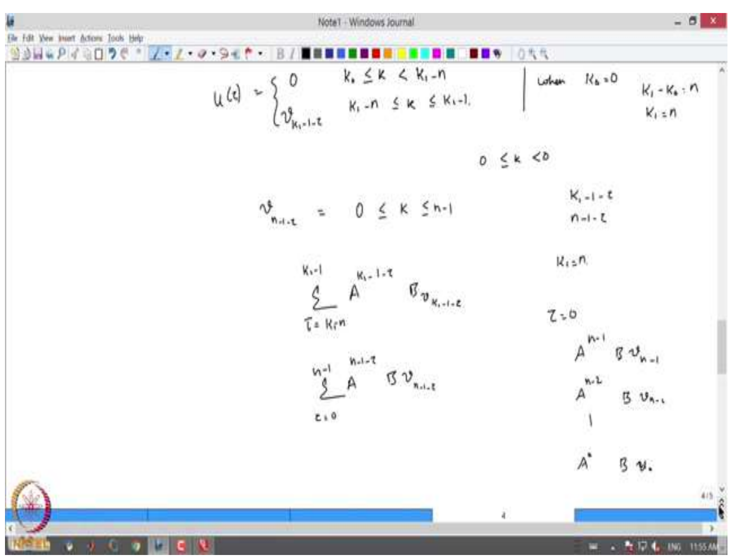

So, what is the expression that we have ok. So, I have u(τ) the way I write it as 0 for $k_0 \leq$ k < k_1 - n then I have this u is $v_{k_1-1-\tau}$ for k_1 - n ≤ k < k_1- 1 ok. So, when let us say k_0= 0 and what do I need I need. So, what do I have from this expression here is that for the n-step Gramian is k_1-k_0 should be at least equal to n. So, k_1 - k_0 let me just assumed this is n. So, I have k_1 = n ok. So, so this is 0 right the first one is anyway 0 because I have 0 less than or equal to k less than or equal. So, this is this is a strictly less than here less than 0. So, this is this is ruled out.

So, the second expression we will say that what is so far from 0 < k ≤ n - 1. I will have the input v as now I have k_1 - 1 -τ what is k_1 this is n - 1 -τ. So, I have $v_{n-1-\tau}$ ok. So, that is what I have on the right hand side here and therefore, now to look at. So, this should be now a little easier to look at because k_1 = n. So, I just look at this summation from 0 till n - 1 and then we just translates to A^i here right and then you compute backwards from n - 1. So, let us let us look at this summation part for a while.

So, what do I have here is sorry summation tau equal to sorry $\sum_{\tau=k_1-n}^{k_1-1} A^{k_1-1-\tau} B v_{k_1-1-\tau}$ ok. What do I know? I know that k_1 = n ok. So, this will be summation $\sum_{\tau=0}^{n-1} A^{n-1-\tau} B v_{n-1-\tau}$. So, when τ = 0 I will have $A^{n-1}B$ v computed at n - 1. When = 1 I will have $A^{n-2}Bv_{n-2}$ all the way till A^0Bv_0 that is what was here right this one ok.

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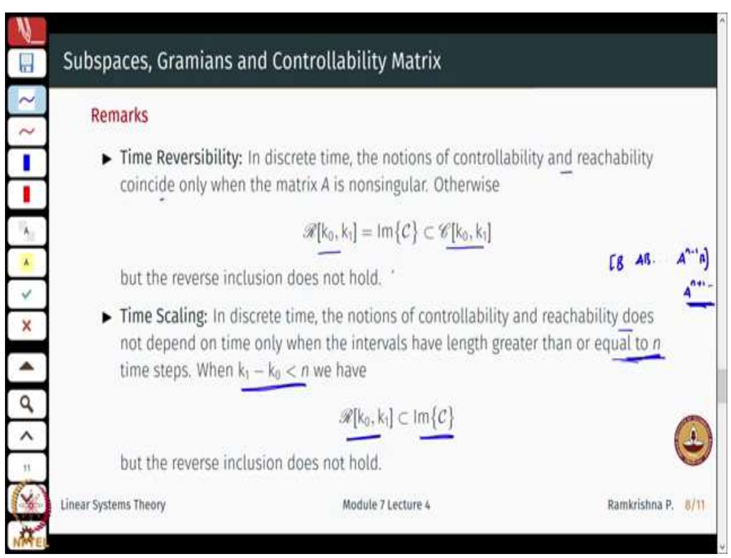

Now, what is this lead us to observe or to conclude that in discrete time the notions of control ability and reachability coincide only if the if the matrix is singular right otherwise well that the reachable subspace is or the reachable set is a subset of the controllable substrate, but the reverse inclusion does not always hold ok. So, the second thing which we will derive with the help of an example we said in the discrete time the notion of controllability and reachability do not depend on time do not depend on time only when the intervals have length larger than n ok.

So, what does it mean that if I again look at the n-step Gramian I have B sorry n step controllability matrix I have n - 1 B right. So, what happens when I look at a time n or n + 1 or so on. This does not really contribute right. So, this is what it means right. It does not depend on time only when the intervals have length greater than or equal to n. So, after n I think any of those computations will not lead us to anything, but whereas, I if what happens if this interval is less than n if $k_1 - k_0$ is less than n. So, let us just check with the help of an example ok.

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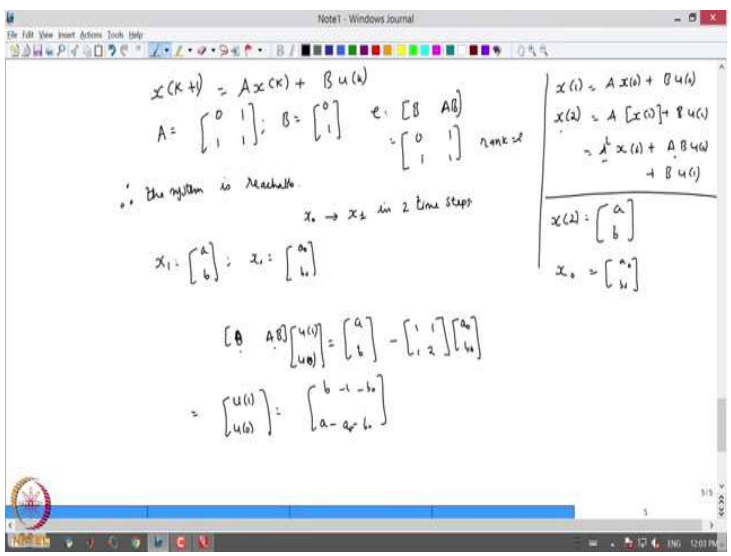

So, let us say I have a system in R^2 x(k + 1) = Ax(k) + B u(k) where A is $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ B is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ then the controllability matrix which is [B AB] is $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ and the rank is two here you can check that easily. And therefore, the system is reachable which means that I can transfer from any state x_0 to any other state x_1 in 2 time steps ok.

Now, how do I let us say I have x_1 has some vector $\begin{bmatrix} a \\ b \end{bmatrix}$ and x_0 is such that this is $\begin{bmatrix} a_0 \\ b_0 \end{bmatrix}$ ok. Now, I want to compute what are the control inputs that will steer the system to any other point in the state space ok. So, again we just look at x (1) = Ax(0) + Bu(0). x (2) = Ax(1) + Bu(1) this is A^2 x(0) + Abu(0) + Bu(1) ok. So, I can substitute. So, what do I know I know what is x(2) here is (a,b) this is a final state which I want to reach and x_0 = $\begin{bmatrix} a_0 \\ b_0 \end{bmatrix}$ ok. I know the matrices a b and so on. So, I just substitute it over there to get the following.

So, I can also write this as [B AB] that is with u_1 u_2 is x(2), what is x(2) that is the final state that I want to reach a b - a square x naught ok. What is A square? A square is 1 1 1 2. What is x_0 that is a_0 and b_0 ok. So, I know these two therefore, I can compute my u to be u 1 sorry this should be u_0 u_1 $u_0$is simply b - 1 - b_0 a minus a_0 minus b_0 the minus here ok.

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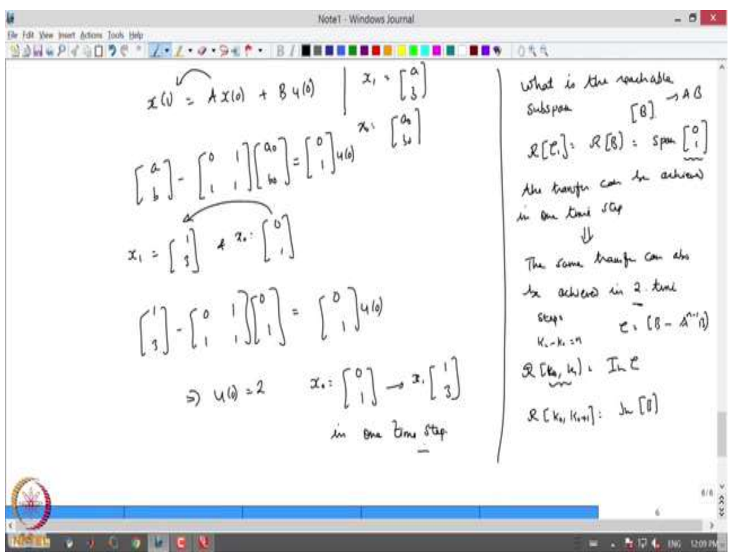

Now, on the other hand let me choose ok. So, let me just check the set of point I say I can actually reach in one time step which means x (1) = Ax_0 + Bu_0 and let me again just say what are the final states x_1 is a,b and x_0 is a_0 and b_0 ok. In this case I just do this rearrangement x_1 is a b minus a what is my A A is I just give this here a is $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ x_0 what is x_0 is $\begin{bmatrix} a_0 \\ b_0 \end{bmatrix}$ plus the B matrix here is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and u_0 ok. Here I can actually solve for u_0 for example, if x_1 is (1,3) and x_0 is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ if I substitute this here what I have x 1 is $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ - $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ok.

So, this should be equal to sign here this equal this equals to $\begin{bmatrix} 0 \\ 1 \end{bmatrix} u_0$ ok. So, this implies that with u_0 = 2 I can transfer the state from x_0 which is (0 1) to x_1 which is $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ in 1 time step ok. So, in this case what is the reachable subspace because when I do the when I compute see I only look at B right ok. So, the reachable subspace here is if I can look at just R I just compute the first term of the controllability matrix which I just say that is C_1 what is the first term of the controllability matrix that is B because I do not compute A^2B anymore right sorry. I do not compute A B anymore right.

So, this is the span of all vectors span of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ right and in this case the transfer can be achieved in one time step ok. The same transfer can also be achieved in 2 time steps and therefore, what do I have is we just go back to what we were saying here right. So, in the discrete time case the notion of controllability does not depend on time only when it is greater than n right, but what if it is what if the intervals are of length lesser and I still have a reachable subspace, but these reachable subsets will just be a just be a subspace of this image of c ok.

But the reverse inclusion will not hold that is that is easy to check right. So, so image of C will not be as a subset of R right. So, which means that I can there are some states which I can reach in less than n time steps. In this example there are states for example, from $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$ which I can actually reach in 1 time step not I do not exactly need 2 time steps. So, there is some reachable subspace and that subspace is just classified by the span of C because also how do we look at R the reachable subspace reachable subspace say k_0, k_1 was equal to the image of C and if this was usually because the way we compute C was [B AB ...$A^{n-1}B$].

So, I am looking at k_1 - k_0 be equal to n at least could also be greater than 1, but what if it is less than 1 I am just looking at 1 time step; so, R[k_0, k_1] where k_1 or I will just do this k_0, k_0 +1. So, I am just looking at just this oh this is image of B ok. That is what exactly I am I am doing over here right ok.

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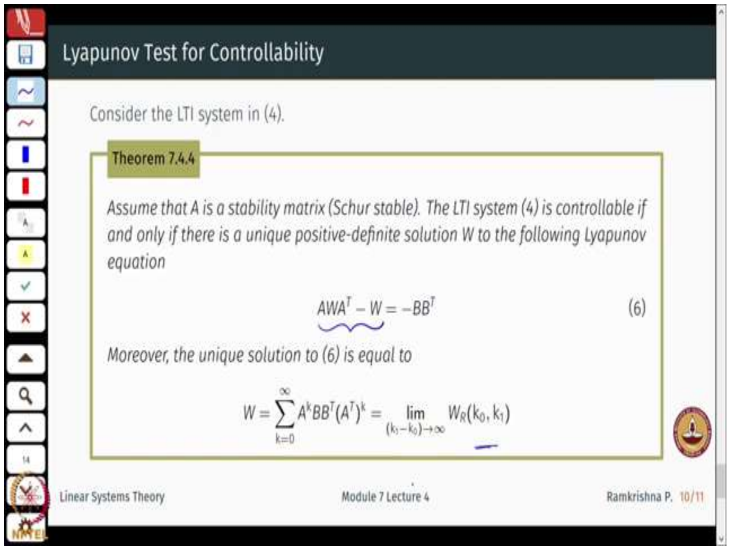

So, the controllability tests and all are very similar in the discrete time case that the rank of C should be equal to n ok. I will skip this proof; I also skip this proof of the Lyapunov test for controllability right. So, this is again exactly the same and we know now the reason why this matrix looks in this form A^TWA - W = BB^T. I will skip these steps, but it is just useful to know to know the results.

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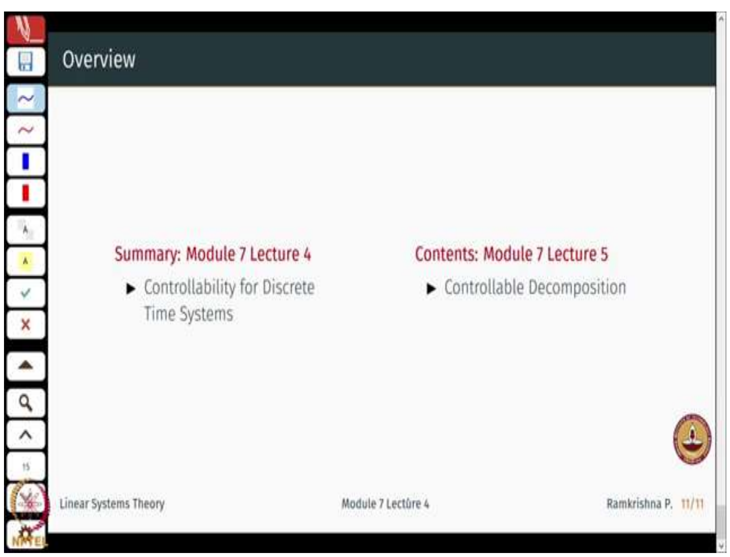

So, I just conclude this lecture over here and in the next lecture what we will see is ok. So, far we have asked ourselves of when is the system controllable how do I test whether or not the system is controllable?

An important question that we will ask is, what happens if the system is not a controllable. How do I even check? So, system being not controllable might mean that there are some modes which I cannot control or some states which I cannot control. How do I even identify those? So, we will do that with the help of what we call as a controllable decomposition. So, that is coming up in the in the next lecture.

Thank you.