**Linear Systems Theory**
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**Module 01**
**Lecture - 03**
**Part 01**
**System Models**

Hello everybody in this 3rd lecture where we still continue to learn to develop models, we will learn a very special class of systems of course, this will not be a major part of the course starting from week 2, but it is good to know a model. So, far we have seen linear models, we have seen non-linear models, we have seen how linear models can be used to analyze network systems for example, in the wireless sensor networks.

We also saw examples about hybrid systems where you have continuous dynamics which are also governed by certain discrete events which happened ok. So, this class of systems are called infinite dimension systems or infinite dimension models, a more engineering term for this is also called as lumped sorry distributed parameter models.

So, far what we saw is that if I take a resistor it is just a little element of what I call as a lump, that it is just if it is just a small element which does not depend on any other parameter ok. So, what I mean by this distributed parameter models.

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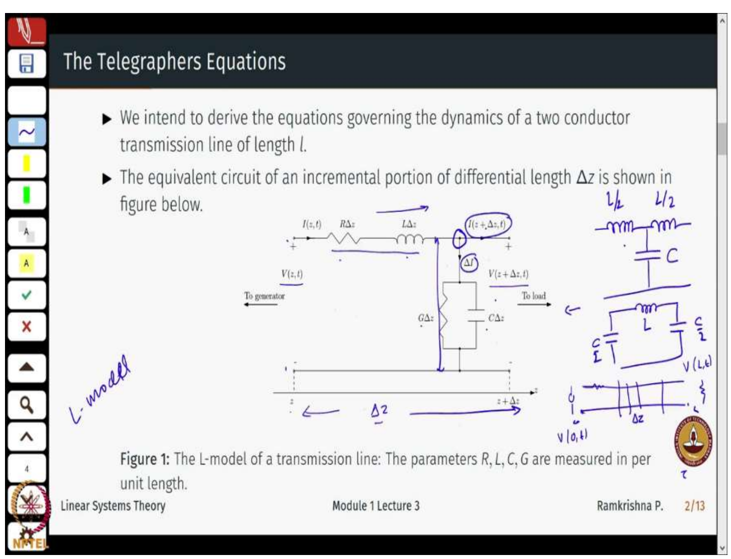

So, we will start from examples which we know from say electromagnetic text ok. So, now, if we remember models of transmission line well it is even if you do not remember. So, we had several models right we had something called the T model approximation which was something like this for I just ignore the dissipative elements here we also had something called the pie model right which was be something like this where the entire capacitance was clubbed into just one single capacitor, the inductors were distributed into two, $\frac{L}{2}$ this side, $\frac{L}{2}$ this side.

Similarly, if I have what I call as a pie model I have to entire inductance of the transmission line, lumped into just a single inductor and capacitors just $\frac{C}{2}, \frac{C}{2}$ like that into two different elements ok. So, a simpler version of this is also the L model which we consider here right.

So, I just have R L C G and ok, what is the difference between having this model and what actually happens is that the resistance is distributed over the entire length of the transmission line, say if I take say if it is a 5 meter lineType equation here. the resistance from 0 to 2.5 might be different than the entire line right. So, you have these parameters R L and C which are like distributed over the entire line. So, in this and I think so, we start with what is a simpler version of these models called the L model. So, it is called the L model of the line ok.

So, if I take a small element starting from z to $z + \Delta z$ which is that this entire thing is of length $\Delta z$ ok. So, if R L C and G are measured per unit length in a little element of the transmission line. So, this is my entire transmission line, starting from 0 to L and is interested in a little element here $\Delta z$ and this little element in this length $\Delta z$ looks like this L model ok. So, what will be the R for this element $\Delta z$ will be R *$\Delta z$ similarly with L, G the conductance and C the capacitance right ok.

What do I know about this things I already know the circuit laws right. So, if I apply KVL to the outer loop so, here at z I denote my voltage at as V(z,t), the voltage at z + $\Delta z$ as V( z + $\Delta z$ ,t). So, that I have to do this, because if I say am I have say generator here which is a typical setting going to some load here. So, the voltage at length 0 at any time t will definitely be different than voltage at length L at the same time in instant t essentially because there might be a drop in the line right.

So, the line has it is own resistances so, the voltage here might be different than voltage here, then here ultimately until it reaches the it reaches the load and there are several methods of how to preserve, how to make this drop to be lesser and so on that is a part of a of a power system course. If you are an electrical engineer and if you have done that course, if you have not done it does not really matter because we are not really analyze power systems here ok. So, if I apply the basic voltage laws V equal to so, that if I call this current through this; through this branch as I( z,t).

So, I just have the simple equation here, that V (z, t) is R for the little element $\Delta z$ the current times $L\frac{di}{dt}$ right. So, the voltage here from here till here is a voltage across R L and the voltage here which is what which is simply V( z + $\Delta z$, t) again the time is fixed here right am, just looking at how the things vary with space. So, far we were interested in say $\frac{dx}{dt} = Ax + Bu$. So, there is another dimension here in terms of space ok.

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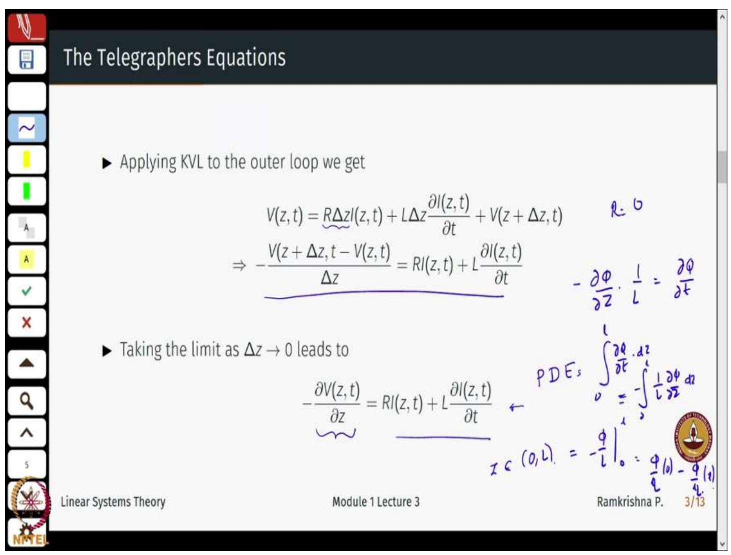


So, I just rearrange terms and write my equations like this right and if I will take the limit as $\Delta z$ tends to 0 this looks familiar to what I learned in calculus as definitions of derivatives right. Now for a small elements also this turns out to be the derivative or the partial derivative with respect to z is on the right hand side I have things like this.

So, this is the first equation now we start from ah lumped model of length $\Delta z$ to which I write down the equations for the entire line it would seem something like this. So, this is valid for all z between 0 and L and these are essentially systems which are now governed by Partial Differential Equations also called PDEs, but infinite dimensional systems are the systems governed by partial differential equations. The another transfer dimensional systems could just be like time delay system. So, one way to look at things here would be that if I send voltage here at time t = 0, it might just reach the load which is at length L with some delay top right. So, transmission lines can also in some cases be modeled as pure time delay systems right.

So, there is possibly no loss maybe in some ideal case just that there is a time difference by the time I transmit my voltage from z = 0 till z = L again there is a lot of literature on that. So, the reason of also doing this is doing this model explicitly is that the same system can be represented by different models is what we claimed in one of our earlier lectures and this is one you know indication or one example of that ok. So, how do I write the second equation so, if I look at if I apply kcl to this node here. So, I have I(z,t) coming from here, I have the $\Delta I$ here and I have I( z $+\Delta z, t$ )as again all at the same time same time t. So, I just applied the current laws ok.

So, I just write down I (t , z)= $\Delta I$ + I(z+ $\Delta z$, t) and I just write down this delta I in terms of V (z + $\Delta z$,t) and this 2 elements here ok, because that the potential at this point of the voltage at this point is just V( z $+\Delta z$, t) ok.

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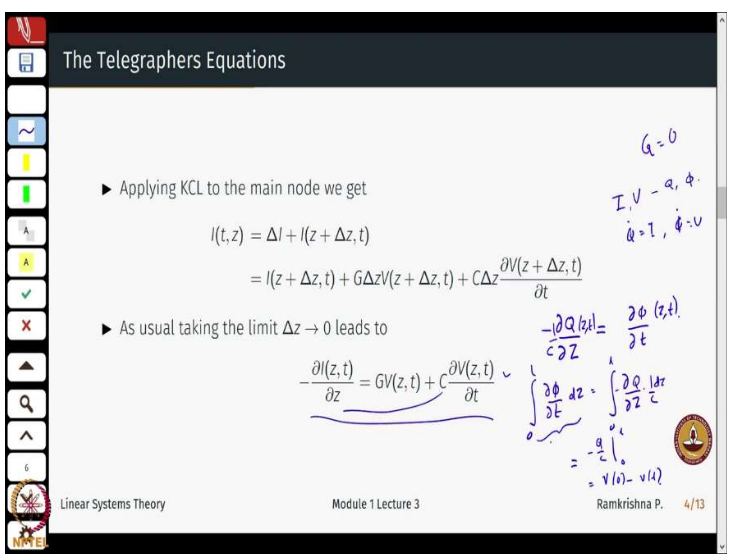

So, and then I just rearrange terms take the limit as $\Delta z$ goes to 0 and then I have other partial differential equation here right ok. So, these two equations together constitute what are also called the telegraphers equations or the equations wave equations for a transmission line and this are also similar if you come from the mechanical domain of a vibrating string there is nothing. So, there is a traveling wave there is a traveling wave also here, the structure almost looks similar and both come from set of conservation laws right.

So, what I said here is like you know we are essentially looking at systems of conservation laws ok, what are the things or what are the physical quantities that are conserved here ok. So, let me just take this equation for a while, now for simplicity let me assume G =0 what I would have and let me write down I and V in one of the fundamental terms right this is called the charge and the flux.

So, we know that $\dot{Q}$ is $I, \dot{\Phi}$ is $V$ and after all the necessary things have been taken care of I can write this also as $\frac{dQ}{dZ}$ is or what the partial of Q which again depending on z,t till dΦ by ok. Now if I integrate this over the spatial domain what I have is from say 0 to L say again d z and this would be this will have a $\frac{1}{C}$. So, this guy goes here so, times $\frac{1}{C}$ so this so, the rate of the flux or the rate of the flux around the entire or in the entire spatial domain would just be so, this has dimensions of say $\frac{Q}{C}$ from 0 to L with a minus sign.

So, this Q this has just the dimensions of voltage this is V at 0 minus V at L ok. So, whatever is the rate of change in the spatial domain is just what is being exchanged through the boundary ok, similarly I can write also for the other equation right over here. So, if I again right it in terms of Q and Φ as R = 0 what I will have is $\frac{d\Phi}{dz}$ where Φis the flux say $\frac{1}{L}$ is $\frac{dQ}{dt}$ and similarly to what I had here I can derive the conservation laws. So, if I integrate from 0 to L $\frac{dQ}{dt}$ dz is from 0 to L. So, this will be the rate of current or the rate of the charge or the increase or decrease of charge in the spatial domain will just be what is being exchanged through the boundary.

So, this will be $\frac{\Phi}{L}$ with a negative sign from 0 to L this would be $\frac{\Phi}{L}$ at z equal to 0 $\frac{\Phi}{L}$ at z capital L ,z =L ok. So, the conservation laws so, whatever is happening in the special domain is what is being exchanged with the boundary against in right same expression for the energy conservation at the rate of change of energy in the spatial domain which is from between 0 to L it just what is being exchanged through the boundary ok.

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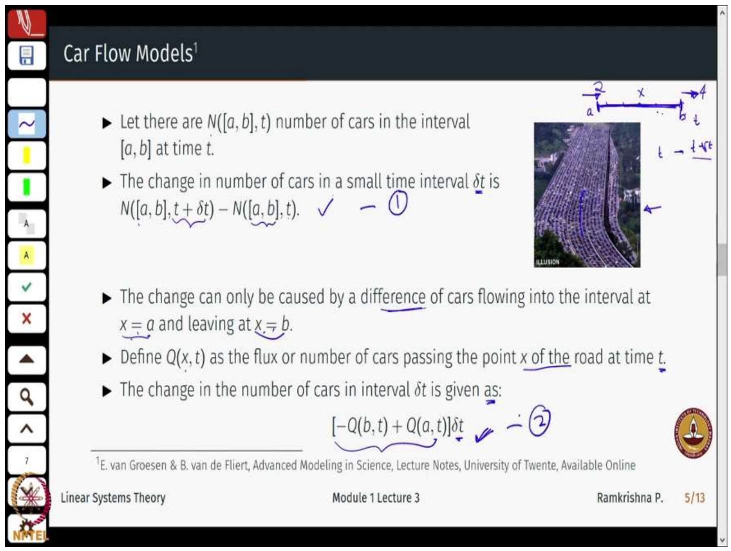


So, keeping this in mind can I derive expressions or equations which say for example, model a traffic flow right. So, this comes picture from one of the busiest highway in the US. So, I will just look at these as one dimensional flow right. So, because everybody there follows lane so, I can just say model through a one through one lane and then it just like one dimension traffic flow right you do not really go like zigzag as we do in an Indian condition in a lane less situations.

How do I model this right. So, for example, if I say what does it mean by even modeling this. So, can I have an expression that relates the flow of the cars through a special domain also the entire length of the highway ok, this will clear a little slowly right. So, let me take a stretch of the lane right and you know these points as a to b ok, at any time instant t let N denote the number of cars in this interval from a to b.

So, this is the number of cars which (Refer Time: 15:14) say there are like may be 1 2 3 4 5 cars ok. Now what happens from t to Δt well it is a moving traffic so, the number of cars could possibly have changed right. So, the change in number of cars in a small interval Δ t is what it is it just a difference right.

So, I look at the number of cars in the interval a b at the time t+ $\Delta$ t minus the number of cars in the interval (a ,b) at time t. So, am just writing down the change in the number of cars in a small interval at $\Delta$t. So, am just interested in this interval a to b how does the number of cars change from t to $\Delta$t ok.

Now if say that the traffic is just still right it is not moving what will be the change, the change will be 0 right, because no cars enter and no cars leave. On the other hand if say that the traffic is moving in a very steady state right say it is like one car enters from here and one car leaves from here still the number of cars would be just 5 for example, in this case right. So, one is entering other is leaving so, the change from t to t +$\Delta$t will again be 0.

So, the change is caused in this the change will be caused only when there is a difference of cars flowing into the entire flowing into the interval at x = a and leaving at x =b, say there are 2 cars leaving this interview entering a and say 4 cars leaving b then there is a change right kind of obvious ok. So, now, if I am just standing at this point say at a and if someone else is standing at this point b, I can measure the number of cars passing this point a for example, at time t and I call it the flux Q (x,t) and I can do it for point b I can do it for any point between a and b this is generally call it x right so, the number of cars passing the point x at time t.

Now, so, the change as I said is caused only by difference of what happens at point a and point b ok. Now the change in total number of cars in this interval will just be given by this one assuming that cars leaving are treated as negative at point b and then cars entering are treated as positive right in the interval $\Delta$t ok.

Now this change is equal to this change, because here am just count counting what is happening between points a and b and I know that the change that happens between points a and b is only governed by what happens at x =a and x = b ok, that I quantify as this number. So, this one is equal to this 2 right so this what I write here.

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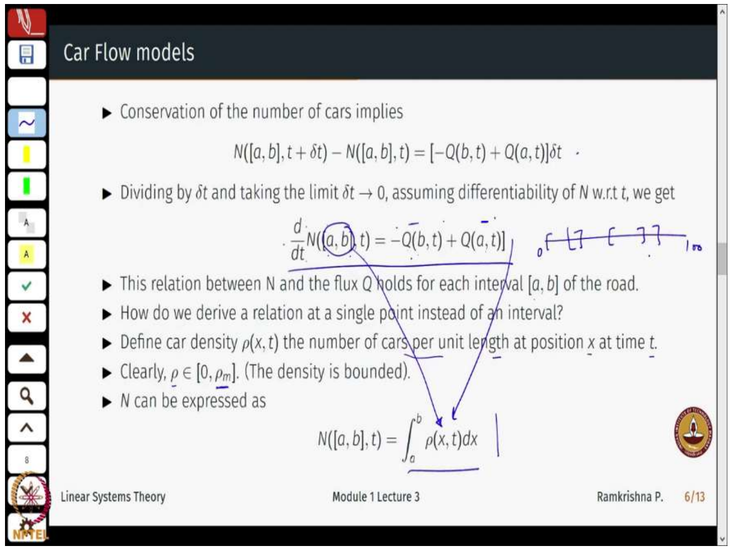


Now, I divide by Δt here and then take the limit and have a nice looking ordinary differential equations in terms of an the rate or the rate of change of number of cars in the interval (a,b) is just defined by what is the number of cars that are leaving point b and the number of cars that are entering point a. So, what I have done is, I have just written down a model for this statement here and the change can only be caused by a difference of cars flowing into the interval at x = a and leaving at x =b right. So, if these two are equal then this is 0, $\frac{d}{dt}$ if you this and this are equal then $\frac{dN}{dt}$ is 0.

So, I just have a nice looking model for the statement, this statement I know is always true right this is basic observation and this relation holds for any point just say this is like starting from length 0 to 100 kilometers it holds for maybe a b here, a b here, here or any interval so ok. I am not really interested now in defining this interval, can I just write a general expression right for any point right ok, to do this I define x another quantity which again is kind of obvious right there is nothing really to be surprised about this.

So, given a scenario like this right I can write like this I can always define the car density the number of cars per unit length at position x and time t physics teaches us about density and what I know about density or from what I know from physics would be that, there will be I cannot have infinite amount of cars per unit length right. So, the density is upper bounded I can only hold maybe 4 cars per meter or per 5 meters or something ok. Now if

I define density I can then write down the number of cars in the interval (a, b) at time t simply as the function of density, like this is again from physics.

So, now, I can on I can right relate this two expressions also now to the density ok. So, this I so, what do I eliminate here itself this (a ,b) here is actually can be written more generally here right and this (a , b) can be arbitrary anything ok.

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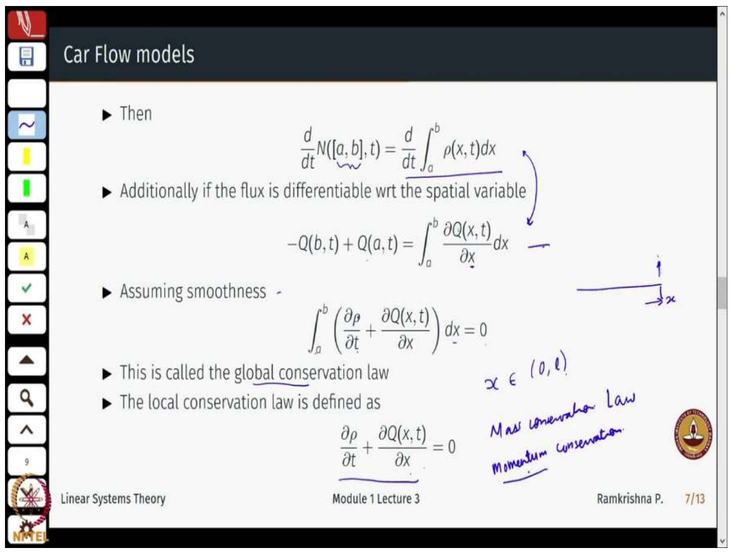


So, this the rate of change of cars in the interval a b at time t is this can be rewritten in terms of the density ok. This also well, then this is also related to this that is what the next expression says N - Q (b,t) + Q (a ,t) is just now written this I can right. So, assuming that this Q is differentiable is this Q is differentiable with respect to x I can write minus Q (b ,t) + Q (a,t) as expression and now we just combine these two ok, these two why are this expression and I get something like this right.

From the integral a to b $\frac{d\rho}{dt} + \frac{dQ}{dx}$ dx= 0 right I assuming smoothness and all that nice properties ok. This is what I call as a global conservation law right the number of cars are conserved that whatever is going from point a is actually coming out at point b and that there is no all of a sudden the car does not disappear or appear from nowhere. A little proof will also show we will skip the proof that I can just write this down as a local conservation law and this for any point x and this point is valid for any say any point between say 0 and L, if L is the length of the entire hybrid ok.

So, this also resembles the mass conservation law influence in fluid dynamics, what is the relation right. So, if I look at so, what I do here is, I just look at car says really small infinitesimal points moving through a straight line that is essentially how I model fluid also a very small particles moving in a straight line and it is just a one dimensional motion it just go x am not really worried about this direction or this direction at the moment ok.

So, even though well physics or the fluid dynamics is completed by another conservation law called law so the momentum conservation ok, will not go into the details of these things. So, just to show you an illustration of how starting from a physical observation of traffic flow we can actually write them down as partial differential equations ok.

Of course the aim of the course will not be to look at solutions of these equations, which will depend on initial plus boundary conditions and so on, but just to give you an idea of how to start from basic physical observations to arrive at mathematical expression.

Now of course, these are very simple models and you can derive more complicated models on is the flux depending on the density, is a flux depending on the velocity of the car upper bound on the velocity and there are more and more things on there. So, for those details I can just refer you to this notes which are available online ok. We pause here before we go to the next set of examples which will be on computing systems.