Linear Systems Theory
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Module - 07
Lecture - 03
Controllability Tests

Hello everybody; welcome to this lecture 3 of week 7, on the course on linear systems theory. Until now, we had a good characterization of the reachable and controllable subspaces which gave us a nice build up to the idea of controllability. What we possibly would have learnt in our undergrad course on control usually, we were dealing with SISO systems and then we had that if the matrix C is invertible and the system is controllable and so on.

So, now have a bit of a generalization of that to MIMO systems of systems which have more than one inputs and we slowly build up to the controllability condition of the controllability matrix C, being of rank n or having full rank and we had some nice proofs also to the build up in the in terms of the reachable subspace, in terms of the controllable subspace and so on and what we also saw is in the case of LTI continuous time systems that the concept of reachability and controllability are just similar.

So, if a system is reachable it is also controllable and vice versa and that the conditions x are exactly the same to verify those in terms of the controllability matrix or even the dimension of the reachable or the controllable subspace ok.

So, this lecture we will do a little more proofs on controllability, what are the other methods to prove? Sometimes the this might actually be easy to verify, because it might be computationally inexpensive and so on. So, we will see some of these proofs which are quite elegant to begin with and will also make towards the end of this lecture a nice connection between Lyapunov stability conditions and controllability conditions ok.

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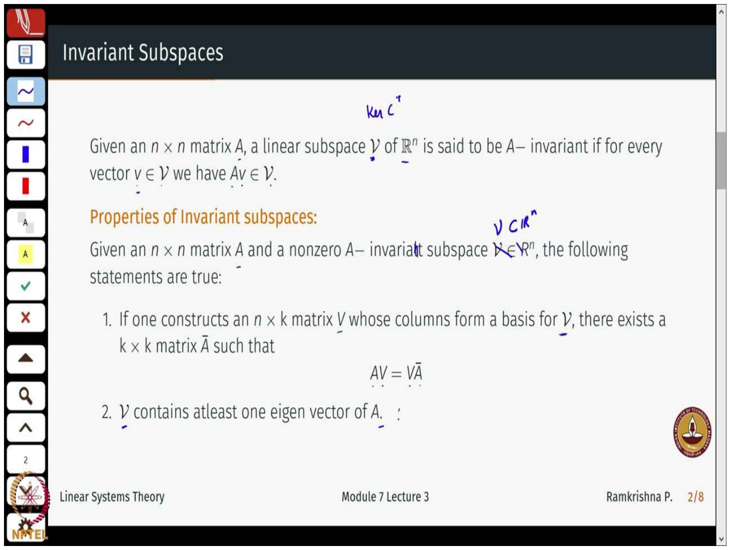

So, the first so, let us begin with a brief recap of invariant subspaces. Some of it we had done in over here right in our module 3, lecture 2 right. So, where we had a brief introduction to linear to invariant subspaces ok. So, let us let us do a bit of a recap of that.

So, given an n x n matrix A; A linear subspace V of R^n is said to be A invariant. If for every vector in V we have that A times V is also in V and this comes with a couple of nice beautiful properties, first one is if I have our n x n matrix A invariant subspace ok. It should be n a non zero invariant subspace V of R^n of V so, this should be V will be V in R^n then the following statements are true.

So, if one constructs an n x n matrix V, whose columns form a basis for this curly V, then there exists a k x k matrix such that A V = V$\bar{A}$ exactly, what we derived in this lecture of module 3, lecture 2 ok.

In addition, it also has another very useful property that V so, this curly space V which is which is the subspace that is of interest this subspace V has at least one eigen vector of A ok. So, we will begin by proving this ok.

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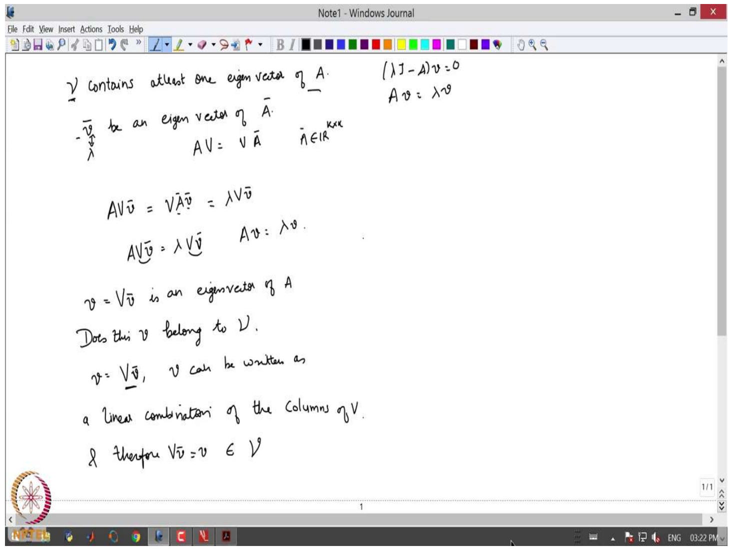

So, let us say so, what we want to prove is that this V contains at least one eigenvector of A ok. So, let us say let us V, let us begin with let be $\bar{v}$ be an eigenvector of $\bar{A}$ and what we know about $\bar{A}$ is that AV = V$\bar{A}$.

Now, so this $\bar{A}$ is of dimension k x k ok. now, let λ be this corresponding eigenvector sorry, eigenvalue to this eigenvector $\bar{v}$ ok. What we know then is well AV$\bar{v}$ = V$\bar{A}\bar{v}$ ok. Now, what do I know that $\bar{v}$ is an eigen vector of $\bar{A}$.

So, in general, what is how do I write eigen vectors (λI – A)v = 0 or in other words Av =λv. So, this will then turn out to be λ, because $\bar{v}$ is a vector eigenvector of $\bar{A}$. So, the V capital will be as it is and I have this one ok.

So, now if I look at this expression closely AV$\bar{v}$ = λV$\bar{v}$ ok. So, this has a has a has a vector form right. So, this is all the same V's therefore, this vector V, which can be written as this one is an eigenvector of A because I can write this as Av = λv. Now, the next thing now, the next step to proof is does this v, this v belong to the subspace the curly V ok. Now, how is V.

So, V is of this form V times $\bar{v}$ the small v bar, which means that this vector v can be written look at this expression carefully can be written as a linear combination of the columns of V and therefore, this V $\bar{v}$ right or which is equal to V must naturally belong to

V and what we now, proved is the statement that V contains at least one eigenvector of A. So, we will use that that statement shortly in one of our proofs ok.

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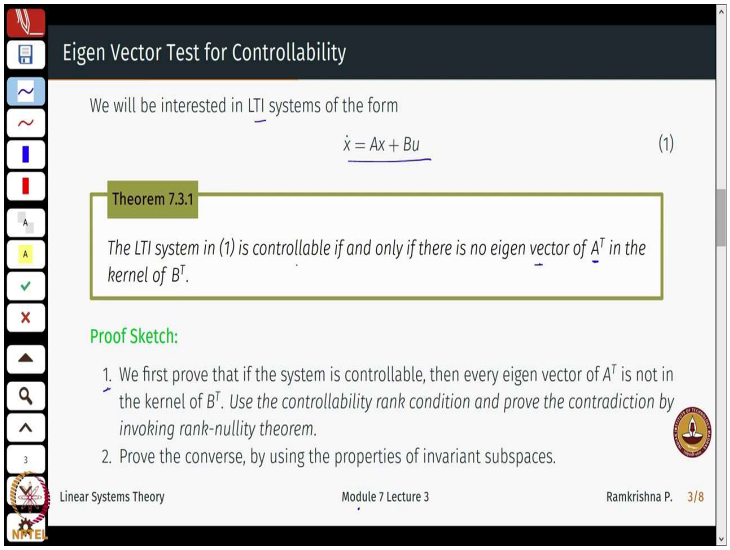

So, the first theorem for today's lecture would be the following. So, we will be as usually interested in LTI systems, the two of continuous time. So, the LTI system one is controllable if and only if there is no eigenvector of A^T in the kernel(B^T) ok.

So, what does this mean so we have to prove both ways right. So, first we assume that it is controllable and then prove that there is no eigenvector of A transpose in the kernel of B transpose and then we will do the reverse right we will do the converse then ok.

So, let us start by with the first step so.

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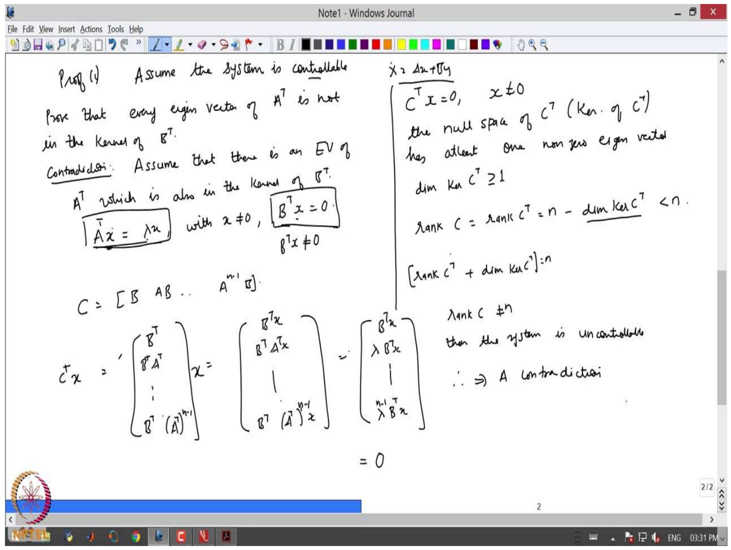

So, the proof ok, so the first step of the proof will be assume, the system is controllable ok. So, once the system is controllable, then we will check what is then will have to prove that every eigenvector of A^T is not in the kernel(B^T). Now, what do we do is well as usual right much of the proof techniques rely on proof by contradiction ok.

So, first what I will do is; I will assume that there is an eigenvector of A^T, which is also in the kernel of B^T ok, which means that ok, let me call this eigenvector to be to be x is λx and please do not confuse this x with that $\dot{x}$=Ax + Bu. So, this is just the eigenvector nothing to do with the state with x ≠ 0. Now, if this eigenvector of A is in the kernel of B^T, then this will satisfy this expression, B^Tx = 0.

Now, what we will show is that if this is true that there is one eigenvector of A^T which is in the kernel(B^T), then the system will lose controllability ok. What is the controllability matrix; C was [B AB ....A^{n-1}B].

C^T I can write as B^T then here, I will have $B^T A^T$ sorry, these are all transposes all the way till $B^T {A^T}^{n-1}$ ok. Now, I use this expression A^Tx = λ x.

So, first what I just I just multiply here by x and the x will show up here. So, I will have here B^Tx, $B^T A^T$x all the way till $B^T {A^T}^{n-1}$x and this will be nothing, but B^Tx, λB^Tx and so on until $\lambda^{n-1} B^T$x. So, I just do nothing, I just start with the controllability matrix take it is transpose multiplied by x and make use of the fact that λ is the eigenvalue of B^T with the corresponding eigenvector x ok.

So, what do I know well this B^T is x that is what I assumed right. I assume that let there be an eigenvector of A^T which is in the kernel of B^T, which means this will go to 0, if this is 0 then it actually means that C^Tx = 0 when x ≠ 0. So, which means that the null space of C^T is also the kernel of C^T has at least one non zero eigenvector and then which means that the dim(kernel(C^T))>1.

So, it is at least one which means that the rank of C which is also equal to the rank of C^T is n minus the dimension of kernel of C^T ok, where did we get this from? We get this from the rank nullity theorem, that rank of C^T plus the dimension of the kernel of C^T should be n ok.

Now, if the dim(kernel(C^T)) > 1, which would mean that the rank of C^T would be definitely less than n n - 1, n - 2 or whatever, but it will it will always be less than n, because the dim(kernel(C)) ≥ 1 ok, which means that if rank(C) ≠ to n, then this renders the system uncontrollable.

This is a contradiction so have contradiction. What is the contradiction that if there exists eigen value an sorry an eigenvector of A which is also in the kernel of B transpose. So, sorry if there exists an eigenvector of A^T which is also in the kernel of B^T and the system becomes uncontrollable right; and therefore, if the system is controllable, then there is no eigenvector. So, this actually will not hold true.

So, B^Tx will never be 0 right. So, that is what is the first part of the proof right. So, what does the statement say a system is controllable, if and only if there is no eigenvector in the kernel of a B^T. So, we started by again proving or assuming that a system is controllable and then show that, if this condition is satisfied in the system becomes uncontrollable which was the proof by contradiction ok.

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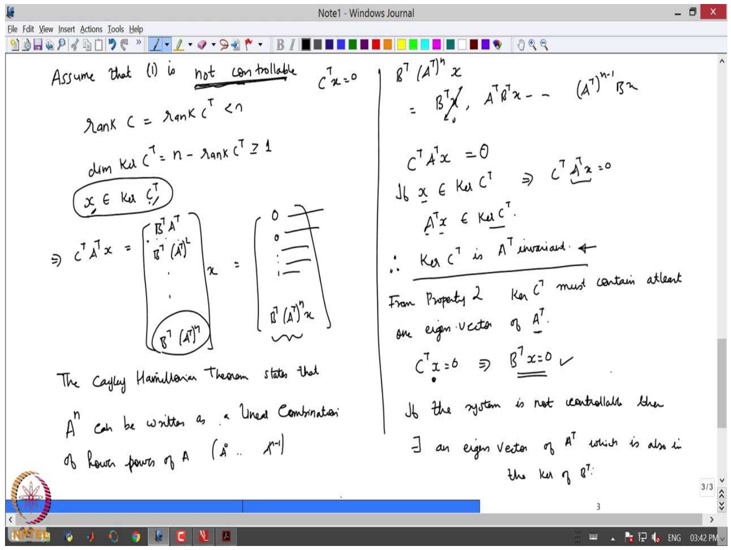

In the next part we will do the converse, we will assume that ok, assume that the system is not controllable ok. If the system is not controllable, then rank(C) = rank(C^T) < n. So, what will I show here is that if the system is not controllable, I just want to see what happens with this eigenvector of A^T does it lie in the kernel of B transpose and so on.

So, so again I invoke the rank nullity theorem to rewrite this as dimension of C of the kernel of Cc^T is n – rank(C^T) ok. Now, because the rank is always less than n the kernel will always have a dimension greater than or equal to 1 ok.

Now, let us say there is an element x that belongs to the kernel of C^T, which means that $C^T A^T$x can be written as C is again the controllability matrix. So, this is $B^T A^T$ $B^T {A^T}^2$, all the way till $B^T {A^T}^{n-1}$x ok.

So, if x is in the kernel of C^T, which mean C^Tx = 0. What is; what is C^Tx look like? Well, C^Tx looks like this one here, the first term here and C^Tx is 0 means C^Tx = 0, $B^T A^T$x = 0, all the way until the n - 1 ok.

So, we write that here. So, the first is 0, second term is 0, all the way till B^TA sorry, $B^T {A^T}^n$x ok. Why does this term remain that terms remains, because there is here the terms only go till n minus 1 ok. What happens to this term now? Is this also equal to 0?

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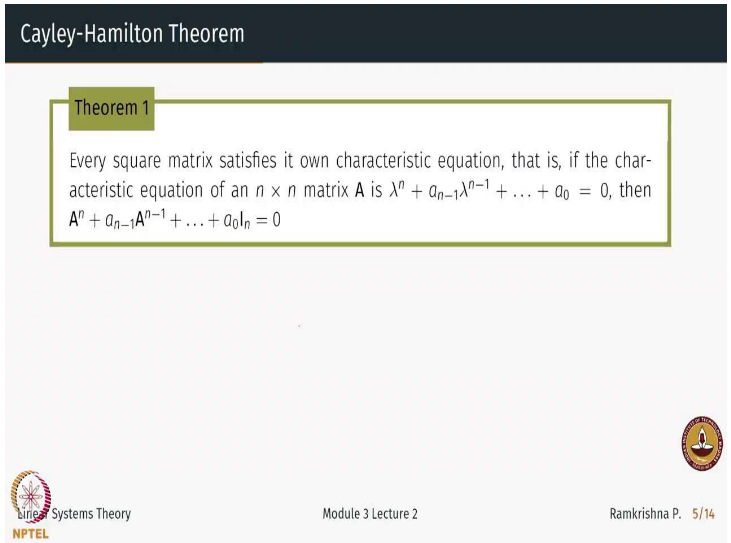

Ok the answer to that comes from the Cayley Hamilton theorem, which if I look at this the last equation of this that ok, the statement says that each matrix satisfies, it is own characteristic equation and therefore, A power n can be written as a linear combination of it is lower powers, all everything from here right.

So, the Cayley Hamilton theorem states that A^n. So, here my A^n can be written as a linear combination of lower powers of A that would be from A^0till A^{n-1} that is obvious from this expression right. So, I do not need to write down it all over again. Ok therefore, what is the problematic term is this term times x.

So, can be written as $B^T {A^T}^n$x can be written as linear combination of B^Tx, then I have $B^T A^T$x and so on until ${A^T}^{n-1}$B x and all of these terms go to 0, because of this, because of the because the first term here right because of because x belongs to the kernel of C^T all of these terms go to 0.

C in the same way, as all these terms here go to 0 and therefore, what do I have now is $C^T A^T$x = 0 or the 0 vector which means that if x belongs to kernel of C^T, it also means that $C^T A^T$x = 0, look at this carefully and this also means that this A^Tx belongs to kernel(C^T) and therefore, this kernel of C is kernel of C^T is A^T invariant ok.

Look at what was the definition of invariant subspaces; that linear again, I have a n x n matrix A which is a linear subspace of V, this V is said to be a invariant if whenever I take a vector V belongs to V, this curly V then AV also belongs to V.

So, in my case here this subspace V is the kernel of C^T right. What do I do? I take an element of the kernel x which belongs to C^T and what I see is that A^Tx also lies in that kernel of C^T and therefore, kernel of C^T is A invariant ok.

So, when kernel of C^T is A invariant then this kernel of C^T, where the second condition contains at least 1 eigenvector of A^T. So, from the property 2, this kernel of C^T must contain at least one eigenvector of A^T ok.

Now, since C^Tx = 0, we also have that B^Tx = 0 ok, from the first line of the first entry of the matrix C^Tx = 0 ok. So, what we what does this mean? So, where did we start with? We started by the assumption that the system is not controllable and then we ended up proving that this vector x ok, which was in the kernel of C^T ok.

So, this vector which was in the kernel outs of C^T via this property of kernel of C^T being A invariant, it contains one eigenvector of a transpose. Let me assume that this x is some axis is eigenvector.

So, this eigenvector which is also in the kernel of C^T now, also satisfies B^Tx = 0 ok. So, this means that if the system is not controllable, then there exists an eigenvector of A^T, which is also in the kernel of B^T ok.

So, that is that that concludes the proof. Again, we may just want to write these steps carefully for yourself and each of the properties that we are using are the one which we learned in week 2 and week 3 ok.

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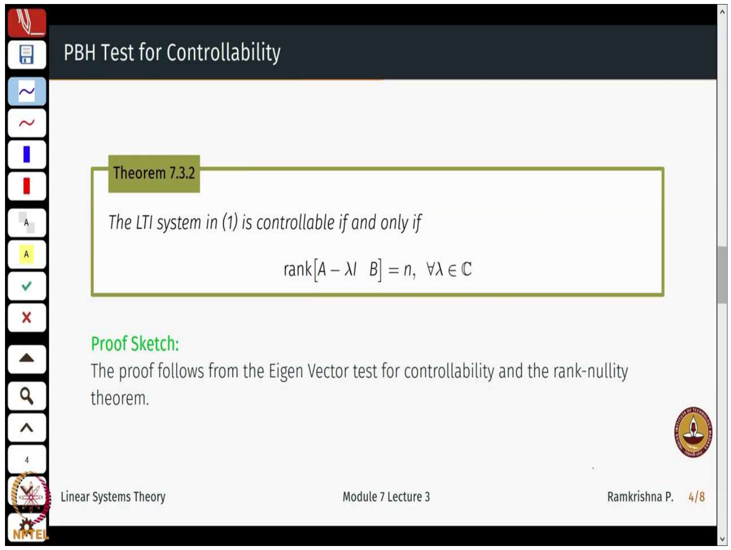

Another interesting test; so, this system again the same system is controllable, if and only if this condition holds A - λI would be the rank of this is equal to n for all λ belonging to the set of complex numbers ok. So, it is, it turns out to be pretty simple this proof again we start. So, what do we have to show right.

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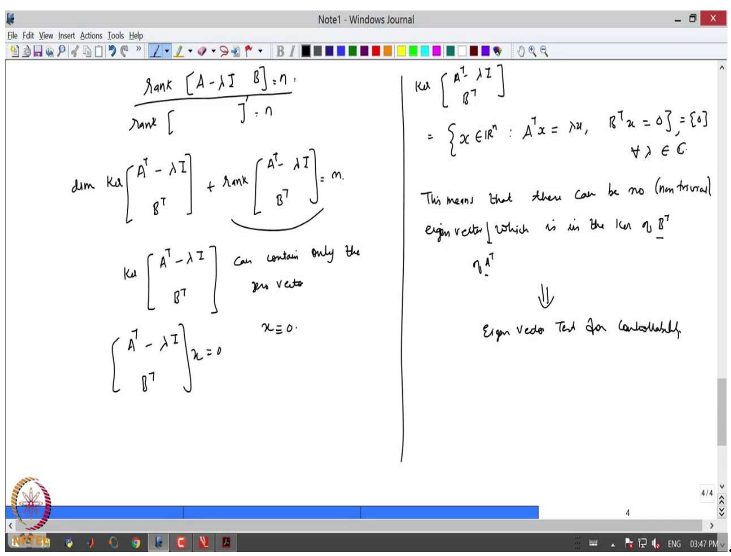

So, the condition that we want to verify is this one or what happens when this particular condition is satisfied A - λI with B is equal to n. This also means that ok, rank of the same thing with the transpose will also be equal to n.

Now, if I again look at the at the rank nullity theorem, I can write this as dimension of kernel of the transpose right. So, it is an obvious reason why I take this transpose; (A-λI)B^T plus the rank of the same thing. Rank((A^T-λI)B^T)= n ok. now, this is this is a pretty right, if the rank of this is n then it is easy then ok. So, then what am I left with dimension of kernel of (A^T-λI)B^T ok, which means that ok, just remove this dimension that the kernel of this guy can contain only the 0 vector right.

So, so, if there exists an x, which satisfies this relation then x should only be 0. This is also equivalent to saying the following therefore, kernel of (A^T-λI)B^T this is the set of all x in R^n, such that A^Tx = λx and B^Tx = 0 again, for all. So, this set is only the null set for all λ in C ok.

So, what does this mean? This means that there can be no eigenvector which is in the; so now, no eigenvector of A^T which is in the kernel of B^T or no non trivial eigenvector ok. This is precisely the eigenvector test and therefore, we conclude that if the rank ((A- λI)B) = n, this actually means that there cannot be an eigenvector of A^T which is in the kernel of B^T and therefore, the eigenvector tests for controllability says that this system is actually controllable ok. So, that is a simple trick here ok.

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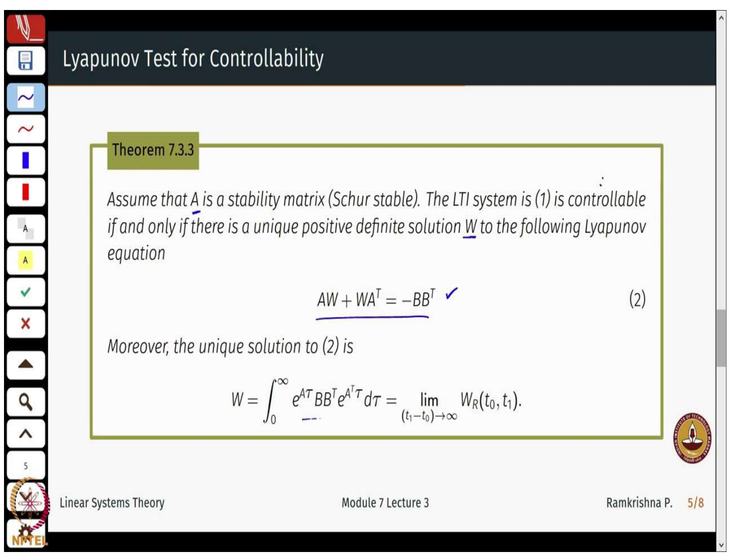

So, the next theorem will try to find a relation between the Lyapunov test or the Lyapunov stability proofs that we did and is there a relation between controllability and stability. So, the first result in the direction is the following. So, we assume that A is stable that all its eigen values are less than 0, then the LTI system is controllable if and only if there is a unique positive definite solution to the following Lyapunov equation right and moreover, this unique solution is equal to of course, this is the Gramian integrated from 0 to infinity right.

So, we will do a proof of this. So, what is the assumption that ok, we know that A is stable ok, then if a stable we show that controllability is equivalent so, as finding a solution for this or if I can find a positive definite solution W for this equation this amounts to controllability. So, these are the two things that we need to show.

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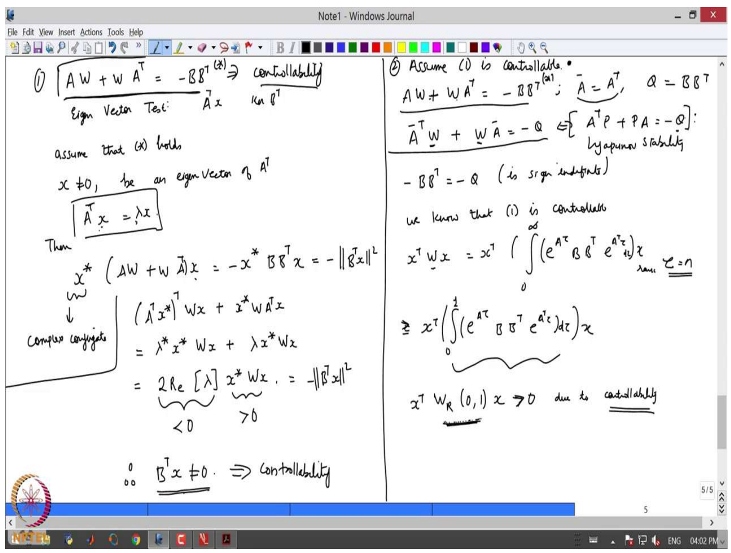

First, we start with this equation which is that AW + W^TA = -BB^T implies controllability. Again, we use the eigenvector test ok. What does the eigenvector test do? The eigenvector test again has to do with the eigenvectors of A^T and then you relate that to the kernel of A^T right ok.

So, assume that this equation ok, let me call it star for here assume that this equation holds right and then we have to write down steps which will show us that this will actually lead us to controllability ok. So, now, let x ≠ 0 be an eigenvector of A^T, which is associated with eigenvalue λ, which means something like this A^T x = λx ok, then I just do this little pre and post multiplication.

So, x^*(A W + W^TA)x = -x^*BB^Tx. So, this x^* is the complex conjugate, because this λ can actually be complex numbers also.

Now, if I write the left hand side carefully that would look something like this at so, I have $(A^T x^*)^T$ Wx + x^*WA^T sorry, this is a little missing key. So, this is WA^T, this is also WA^T ok. So, x^*WA^Tx this, because of this relation, we will take the following form.

So, I will have $\lambda^* x^*$Wx + λx^*W x. So, this we will just be so, I am just like it is almost like adding up a complex number and it is conjugate right. So, what will I left a left, what will I will be left is just the twice the real part of λx^*W x ok.

So, again we will see, where did we start with. We started with assuming that this is true right in such a way that W is positive definite and then we were to show that it actually means the controllability. Now, if this is true ok, I just do x^* pre and post multiply by x^* where x is an eigenvector of A^T associated with the eigenvalue λ, then I end up with this, with this expression.

Now, what do I know about this; that because A is stable this must be negative not only that W is positive. So, this should be positive what do I have on the right hand side? On the right hand side I have so, this x star B this can equivalently B written as something like this $||B^T x||^2$. So, this is - $||B^T x||^2$.

On the left hand side what do I have; this is strictly less than 0, this is greater than 0, this is strictly less than 0, because of the of the assumption that A < 0 or A is a stability matrix and therefore, B^Tx cannot be equal to 0, which means that this eigenvector of A^T is not in the kernel of B^T which actually means controllability ok.

I started again by assuming that this is true right and then showed that if this is true that there is no eigenvector of A^T which is in the kernel of B^T, which actually means a controllability ok. So, that concludes the first part of the proof ok.

Now, the second part assume that the LTI system 1 is controllable ok. If this is controllable, then we will show that equation that this equation mark by star is actually true. So, so, this is how the equation looks like W A plus W A transpose is -BB^T and by letting $\bar{A}$ equal to A^TQ is BB^T. I can rewrite this equation in the following way. Now, this is $\bar{A}^T$W + W$\bar{A}$ =-Q right.

This is very similar to saying A^TP + P A = -Q right as in the Lyapunov stability of proof. So, now we will just use some of those techniques over there to prove that, if the system is controllable, then there exists a positive definite solution to this to this equation ok. So, this expression tells us that if A is a stability matrix, $\bar{A}$ is also a stability matrix.So, to conclude that W is the is a unique solution, we just make use of the proofs which we had in Lyapunov stability of the previous weeks lectures ok. So, we will, what we need to show now; is that this W is actually positive definite and that if this is also a solution to this equation ok.

So, the only problem that we will have here is that this -BB^T which we called as -Q is sign indefinite or we can actually conclude if this is positive definite or not ok but what do we know? We know that the system 1 is controllable ok.

Now, let us see if we can make use of this. So, let me just start with some arbitrary vector x^TWx. Now, what is this W? Well, what I want to prove is that the W which is given by this expression is a unique solution. So, uniqueness I can prove, but now we I will prove that this is actually the solution here and is also that W is a positive definite solution ok.

So, this will be x^T. So, in the inside I have $x^T(\int_0^\infty e^{A\tau}BB^Te^{A^T\tau}d\tau)$x. Now, this is can be a check that I can easily write this as this is greater than or equal to $x^T(\int_0^1 e^{A\tau}BB^Te^{A^T\tau}d\tau\,)x$ here ok.

Now, this guy is just the controllability Gramian defined from the initial time 0 and a final time 1 ok. Now, based because the system is controllable what will I have is x^T W_R R x is always greater than 0 right, because of you can look at the definitions of the, image of W_R being if so, this all I have to do with the rank of C being equal to n and then we had relations between the rank of C with the image of the Gramian and so on. So, this is actually true, because of controllability ok.

So, this actually means that this W is positive definite ok, what did we use to show the positive definiteness? That positive definiteness follows from the controllability and therefore, what again? Let us do a quick recap system was can we assume that the system was controllable and we showed that this equation (Refer Time: 44:01)by star had a positive definite solution W. So, I just so this expression looks similar to what I had in the

Lyapunov stability theorem just that there Q for every Q > 0, I found out a P, which was greater than 0, which satisfied the Lyapunov equation ok.

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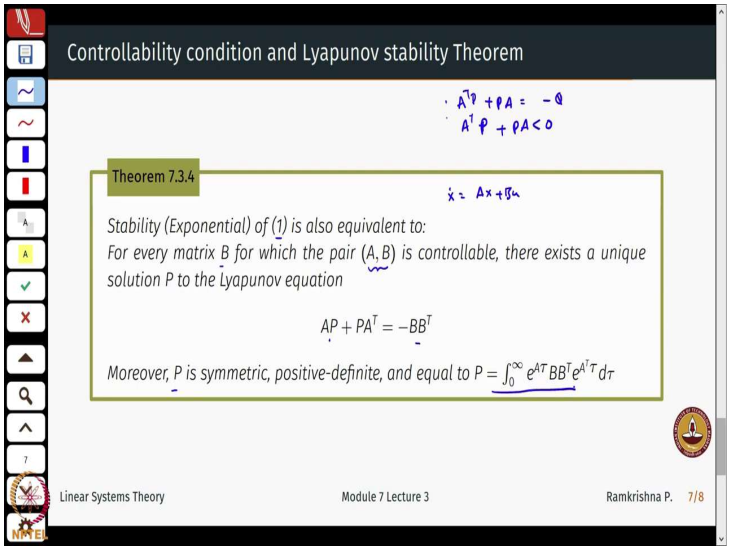

So, the uniqueness and all I can establish from there what I just show here is that this W indeed is positive definite, which follows directly from the assumption that one is controllable and which concludes the proof here.

So, the last result of today, we will quickly wind up with the proof shows that so, we had earlier, five conditions of stability. It started first with the eigenvalues being strictly less than 0, this was equal to asymptotic stability, this is also equal to exponential stability, then we had two equivalent conditions to prove that for any positive matrix Q there existed a solution to this expression.

So, given a Q, A^TP + PA = -Q this also meant asymptotic stability or exponential stability similarly, with this A^TP + PA < 0, all these were equivalent starting from eigen values being less than 0 to asymptotic stability exponential stability and these two things. So, what we have that in addition that we have this one.

So, for every matrix B for which the system or the pair A B is controllable, which also means that ok, the system written by $\dot{x}$ = Ax + Bu is controllable, then there exists a unique solution P to the Lyapunov equation given by this. Moreover, this P is symmetric positive definite and equivalent to this expression ok. So, the proof will just you make use of all the little tricks that we used in today's lectures of the earlier proofs.

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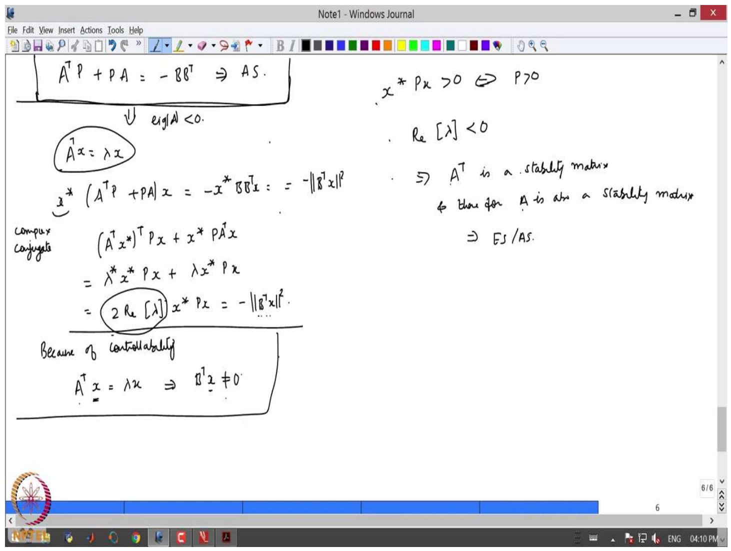

So, we need to show that this condition A^TP + PA = -BB^T actually means asymptotic stability or even exponential stability. So, in the previous expression what we had in the previous theorem, this condition ok, we started with when A is stable then such a situation, then such a equation is satisfied with W being positive definite and taking this expression ok.

So, now, we have to show a bit of the some something like the converse of it that if A^TP + PA = -BB^T, then it is this will also mean that the eigenvalues of A are strictly less than 0 or the real parts. Let us assume now, that that this expression holds and as usual let x be an eigenvector associated to the eigenvalue λ ok. The same tricks again, x^*A^T P + PAx = -x^*BB^T x = -$||B^Tx||^2$ ok, again this is the complex conjugate ok.

Now, very similar tricks as we did earlier. So, this the left hand side can be written as A^Tx^{*T}Px + x^*PA^Tx is again, making use of this expression here, I have this to be λ^*x^*P x + λx^*P x again, this will give me 2Re(λ)x^*P x and on the right hand side, I have -$||B^Tx||^2$.

Now, what do I what do I assume. So, so this condition I know is equivalent to controllability from this theorem right, from this theorem here. So, this theorem just tells me about controllability right. So, if system is controllable if and only, if I can solve for this expression where W is a positive matrix given by this expression here or W is actually a positive definite solution.

Now, because of controllability ok, because of controllability what does controllability tell me, if the system is controllable which means that there is no eigenvector of A^T, which is in the kernel of B^T and this should definitely not be equal to 0 ok. Now, P look at this expressions here now, this is not allowed to be 0, because the system is controllable so, this should definitely be less than 0, x^*P x > 0, because P > 0.

Now, I am just left with this term and what should the overall term be? Overall term should be less than 0 because this is not allowed to be 0 and then I have the norm with the square.

So, this number should always be less than 0, because of the negative sign here and therefore, the only solution left is for the real part of the λ< 0. This shows that A^T is a stable matrix or a stability matrix and therefore A is also stable matrix ok.

So, starting from this assuming that this expression is true, we followed these steps made use of the fact that this, because the system is controllable. There is no eigenvector of A^T which is allowed to be in the kernel of B^T, which led us to conclude that A^T is the stability matrix and therefore, A is also stability matrix right and this means exponential stability and also asymptotic stability right. So, this concludes the proof of this result.

A bunch of things we derived today, starting from the definition of controllability and then to do with some eigenvalue eigenvector properties of $A^T$and B^T. We also found out a beautiful relation between controllability and Lyapunov stability ok. So, these are details which you may it may help for you to work out work it out by yourself and check for those proofs again.

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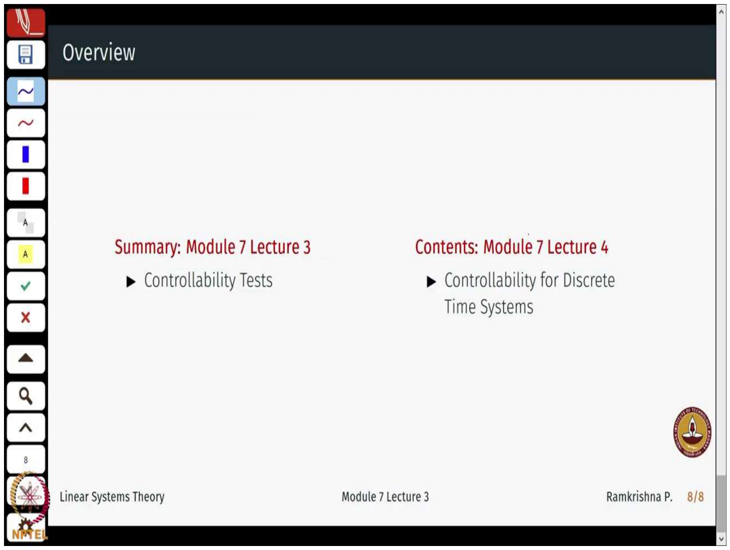

So, in the next lecture we will talk about controllability for discrete time systems. So, our the properties can we just lightly take this and put it there and everything will fall in place ok, that we will see if the discrete time systems throw up some surprises or everything is just copy paste appropriately ok. So, that is coming up in the next lecture.

Thanks for listening.