Linear Systems Theory
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Module – 07
Lecture – 02
Controllability Matrix and Controllable Systems

Hello everybody, welcome to this second lecture of week 7, where we deal with controllability and reachability analysis. So, so far what we have seen is to characterize reachability and the concept of controllability in terms of subspaces of R^n where my states evolved in r n. We also had a characterization of what was the minimum energy required when I say I can control a system from any state to the origin or I can do the reverse that I can go from the origin to any point x_1 in finite time with application of some arbitrary control input as long as it is bounded.

So, this lecture we will see or here also. So, this will also be familiar to what we would have done earlier called the concept of the controllability matrix. So, whenever we talk of controllability the first course on state space will tell us about the controllability matrix or the Kalman rank conditions ok.

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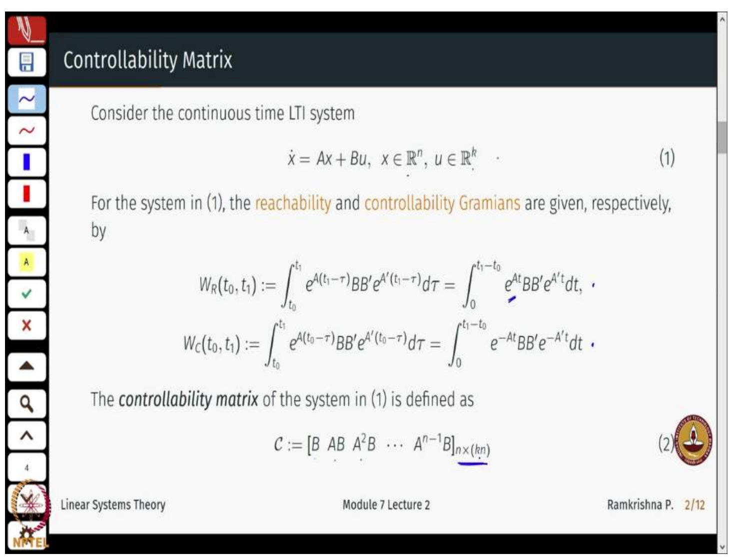

So, let us focus here on LTI systems where the equations look a little nicer than that for LTV systems ok. So, I start with $\dot{x}$ = Ax + Bu with x in R^n, u in R^k. So, we have k inputs for the system and the reach ability and the controllability Gramians take this form right. So, I can just dis compute them to be something like this and so, e power A t is my state transition matrix for the LTI case ok.

So, the controllability matrix is defined as the following. So, you have C is [B AB A^2B $A^{n-1}B$] and this is easy to check that this is an n x kn a matrix ok.

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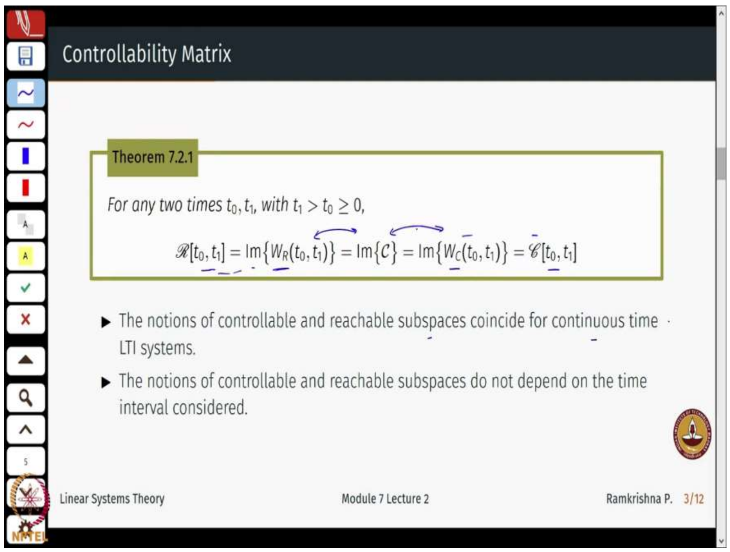

Now, what is the relation of this control ability matrix defined here to what we learnt in the previous lecture on the reachable subspaces and its relation to the image of the reachable Gramian and the controllable subspaces with this appropriate relation to the controllability Gramian ok. So, before we do this, I will we can I know do a proof of this, but this is a little intuitive. So, the notions of controllable and reachable subspace coincide for continuous time LTI systems. So, if I say a system is if I say a system is controllable, it also means it is also reachable in the LTI case continuous time.

So, there the so, that is what I will emphasize on whenever they are not equal I will tell you why they are not equal and they also not do not really depend on the time interval considered ok. So, let us let us check how to prove this. So, I know these two are equal I also know that these two are equal. So, let me just prove may be either this or this. So, if I do a proof for this will follow immediately and vice versa ok.

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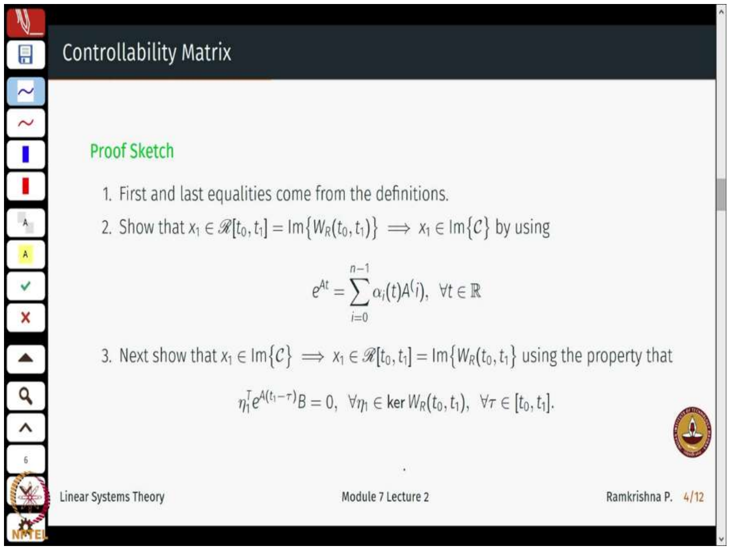

So, let us start by showing that if $x_1$is in the reachable subspace which is also equal to this one then x_1 also belongs to image of C. So, what is given to me ok?

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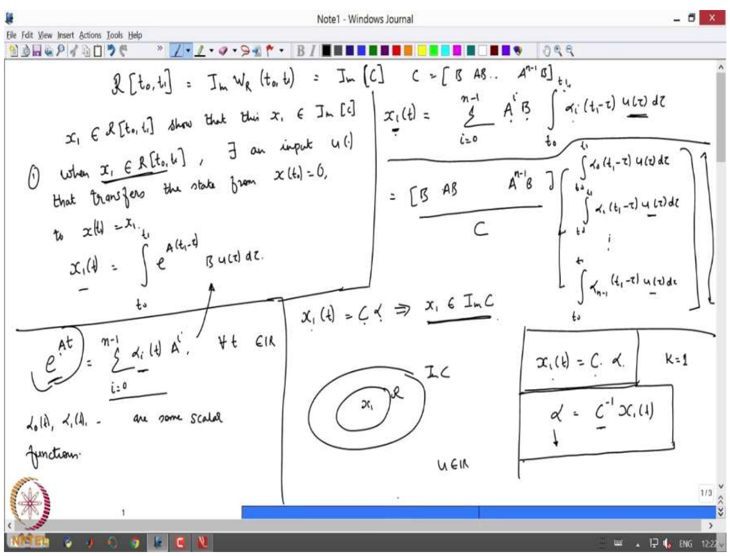

So, I need to prove this that the reachable subspace or the reachable set is the image of $W_R(t_0, t_1)$ this we proved in the previous lecture, now we also proved that this is actually also the image of C, but C was defined as [B AB,,, $A^{n-1}B$] ok. So, let us we follow the same philosophy for the proof right let us first begin by assuming that there is an x_1 in R and we show that this x_1 also belongs to the image of C and then we do a do the reverse ok.

So, we will do this first ok; so, first to check this or to verify this. So, when x_1 is in the reachable subspace, there exists an input u that transfers the state from x at t_0 which is the origin to x at t_1 which I call as x_1 which means in the LTI case it looks something like this, $\int_{t_0}^{t_1} e^{A(t_1-\tau)}Bu(\tau)d\tau$.

So, I just mentioned this briefly here, but we have done this earlier. So, if I compute e^{At} from the Cayley Hamilton theorem what I know is the following that I can write e^{At} as the following $\sum_{i=0}^{n-1} \alpha_i(t)\, A^i$ for all t in R where this α_0(t) α_1(t) etc are some scalar functions. So, there is a way to define this, I will skip this step, but these are to do very well with what we did in one of our earlier lectures especially I think in week 3 where we were computing the state transition matrix or we were also discussing about the Cayley Hamilton theorem ok.

So, once I know this expansion of e^{At} where things become a little easier for me now ok. So, I can write $x_1(t)$ as $\sum_{i=0}^{n-1} A^i B \int_{t_0}^{t_1} \alpha_i(\mathrm{t}-\mathrm{\tau})\mathrm{u}(\tau)d\tau$. So, this is just by simply substituting this expansion for e^{At} into this the solution of the equation; this can also be written how will this look for i = 0? I will have B here which is multiplied by i =0 will be $\int_{t_0}^{t_1} \alpha_0(\mathrm{t}-\mathrm{\tau})\mathrm{u}(\tau)d\tau$.

Similarly when alpha; so, when i = 1 I will have AB and similarly here I will have $\int_{t_0}^{t_1} \alpha_1(\mathrm{t}-\mathrm{\tau})\mathrm{u}(\tau)d\tau$ and so on ok. So, until I go to e^{n-1}B where I will have $\int_{t_0}^{t_1} \alpha_{n-1}(\mathrm{t}-\mathrm{\tau})\mathrm{u}(\tau)d\tau$ ok. Now what does this mean? So, this is my controllability matrix C which I defined here this one. Now what does this expression tell me? If I just write it. So, this is x_1(t) is C let me just call this some vector say α, C times some α right all these are numbers and then these constitute vector ok.

So, this means that this x_1 comes as a result of this vector α transforming where this map C_1 or this linear transformation which means that x_1 is also now in the image of C. So, this x_1 which was in the reachable set we now proved at this x_1 is also in the image of C ok. So, if I again just write down x_1 is in R is also in image of C. So, the first step of the proof well is not completed we showed this one by making use of this expansion of e^{At}. Next we will show that if x_1 is in the image of C this will actually mean that x_1 is also in the reachable subject or the reachable set of the system or this r is also equal to the image of W_R, again we use a property like this we will derive this what this mean ok.

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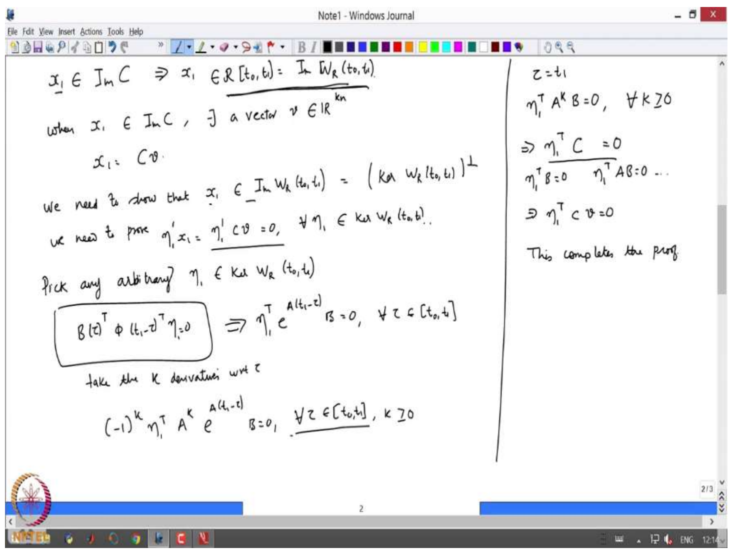

So, the second part of the proof we start from x_1 belonging to the image of C and show that this actually implies that this x_1 also belongs to R(t_0, t_1) is also equal to the image of $W_R(t_0\ t_1)$ ok. So, how do we do this? When x_1 is in the image of C there exists a vector v which belongs to R^{kn}. So, that is how the controllability matrix dimensions look like. So, this is the n x k n matrix. So, there will exist a vector of from R^{kn} such that x_1 is this Cv ok.

Now, we need to show that this x_1 is also now in the image of W_R which is my reachable subset ok. Now, I know again that this is true that the image of W_R is the Kernel of $W_R(t_0, t_1)^{\perp}$ ok. So, very similar to what we did to prove this equality we will use exactly the same steps. So, when I say that the x_1 is in the image of C and I need to show that x_1 is in the image of R which means I we need to prove something like this that $\grave{\eta}_1\ x_1$ is what is x_1? $\grave{\eta}_1\ x_1$ is Cv = 0 for all η_1 which come from the kernel of $W_R(t_0, t_1)$ ok.

Now, to verify this, just pick any arbitrary η_1 from kernel o W_R. So, from if we if you recollect our earlier proof we had in 1 of the steps we had something like this, that $B(\tau)^T \Phi(t_1 - \tau)^T \eta_1 = 0$ this is in in one of the last steps of proving this equality right. So, that translates to saying something like this here that $\eta_1{}^T$ this in the LTI case is $\eta_1{}^T e^{A(t_1-\tau)}B$= 0 and this is valid for all τ belonging to $[t_0, t_1]$ ok.

Now, I take the k derivatives with respect to τ and what I get is that $(-1)^k\eta_1{}^T A^k$ $e^{A(t_1-\tau)}$B=0 for all τ in $[t_0, t_1]$ and k ≥ 0 ok. So, now so, I just say that this is valid for all τ between t_0 and t_1 this properly t_0 and t_1. So, in particular when τ = t_1 what I have is $\eta_1{}^T A^k B = 0$ for all k ≥ 0.

So, this means that $\eta_1{}^T C = 0$ right because if say k = 0 I have $\eta_1{}^T$B = 0 for k = 1 and so on ok. So, when $\eta_1{}^T$C = 0 this also means $\eta_1{}^T$Cv = 0 right that is what we wanted to show here right and therefore, this completes the proof ok. So, just to understand this I think you can just write down each step by yourself and then check what they have been and once you get a get a grab of 1 or 2 proofs I think the remaining ones would be much easier to to understand and interpret.

So, I will spend a lot of time in teaching you proof techniques that will also help you when you read some research papers or you want to pursue some projects in this direction and so on ok. So, since your advice is to write down all the proofs for yourself and check them step by step. So, what we prove now is this one that this is true and the other part of the proof that image of C is actually equal to the image of W_c is a very obvious consequences the proof goes in exactly the same lines ok.

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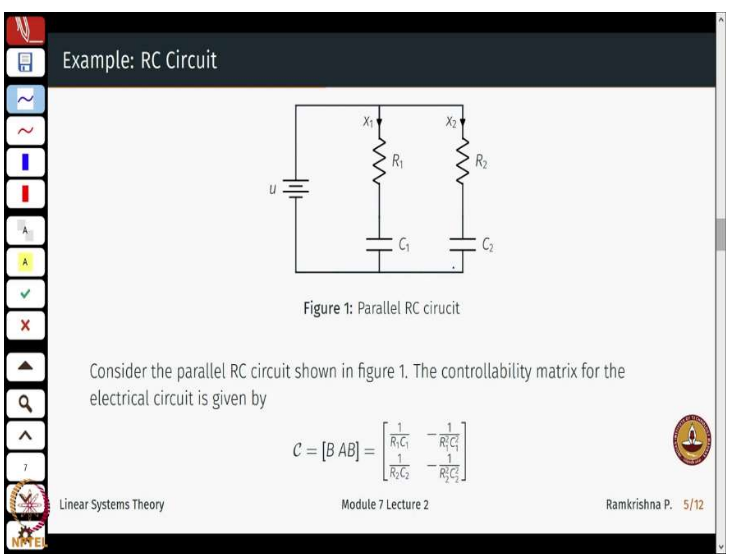

Now, we come back to our examples. So, we again start with that parallel RC circuit and check what is the rank of the controllability matrix the rank of the accountability matrix or the or the C takes this form $\frac{1}{R_1C_1}, \frac{1}{R_2C_2}$ and so, on ok.

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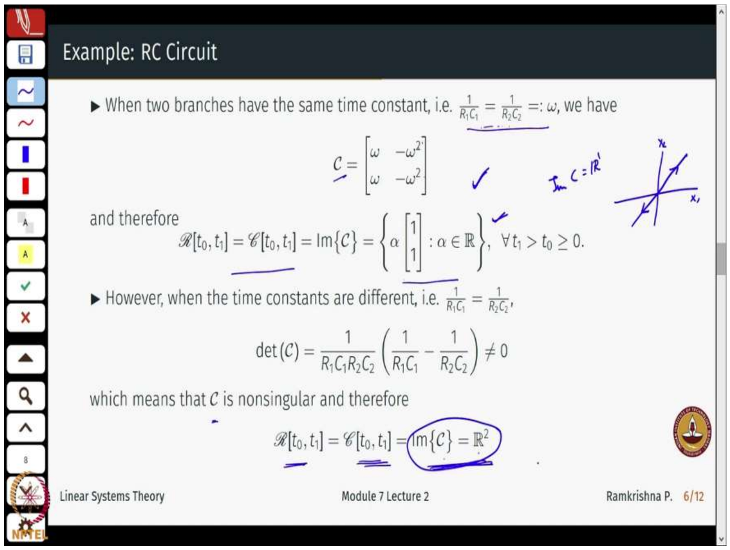

So, what happens when both the branches have the same time constant then C takes some form like this. And therefore, R of C which is equal to the image of C will be something like this or this of dimension 1; however, when the time constants are different. So, this case we already inferred earlier now we are just computing it via the controllability matrix.

So, when the time constants are different C is non singular and therefore, image of C is R^2, in this case the image of C is just you just R^1 that in the first case right because C is non-singular. And therefore, the image of C will be R^2 image of C is also equal to the controllable of space and also equal to the reachable subspace. So, what does this now have to do with say for example, this quantity here or this thing here where the image of C was only equal to R what does this have to do with controllable systems ok.

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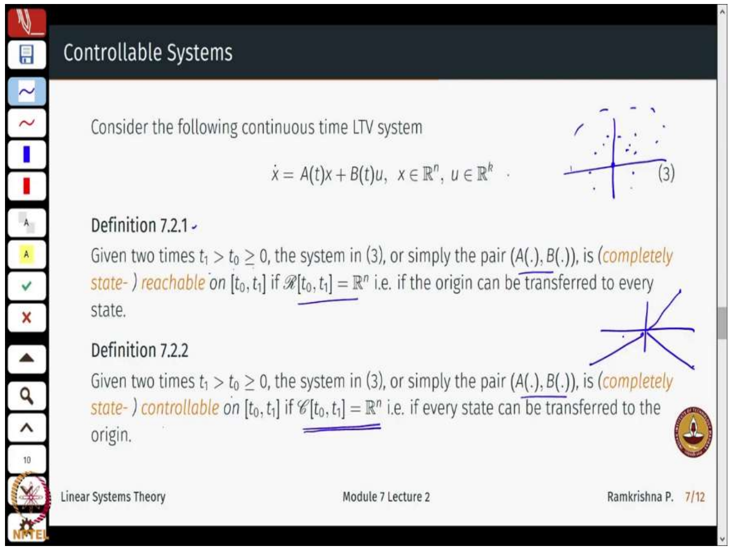

Now, let us formally define the notion of controllability ok. I start with a LTV system. So, given two times $t_1 > t_0 \geq 0$, we can say that the pair AB is controllable or completely sorry we start with reachable completely state reachable if $R(t_{0,}t_1)$ is R^n or if the origin can be transferred to every state right; so, in any point in the state space.

Similarly, I can say about controllability given two times t_1 $t_0 \geq 0$ or they can say that the pair AB is completely controllable if the dimension of controllable of C is R^n that is if every state can be transferred to the origin. So, the first statement or this definition means that ok. So, can I go from this point to any point in the state space any arbitrary point any bounded point in the state space? In the second case it means that can I transfer any point starting from the state space to the origin here starting here to the origin starting from here to the origin.

So, what does this what is the interpretability of this well in case of sorry in case of f of this whether time constants were equal what we could check is that the only possible points that I could reach were on this line and this was x_1 and x_2. Whereas, if the system is completely controllable then I should be able to reach any point or I should be able to come back to the origin from any point in R^2 in this case ok.

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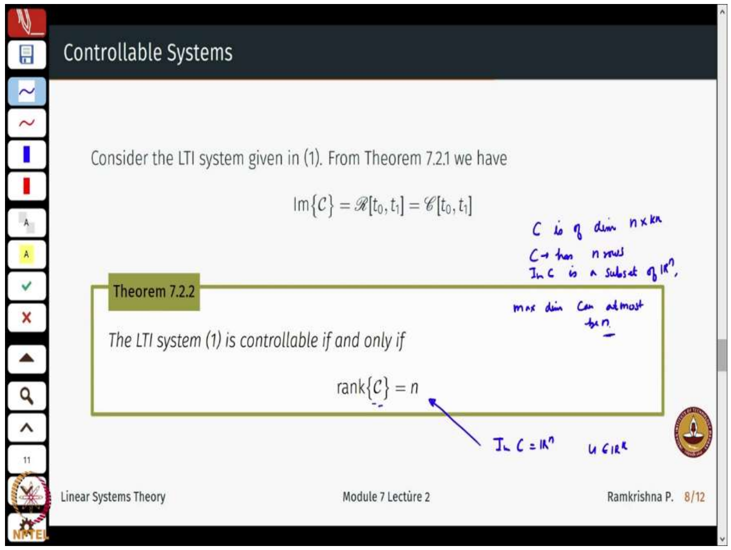

Now, how do we check this well the linear system is controllable if and only if the rank of the controllability matrix is n ok. So, 1 one way to interpret this is what is C, C is of dimension n x k n ok. So, C has n rows an image of C is therefore, a subset of R^n and the maximum dimension therefore, can at most be n ok. And now for controllability what we know is that image of C should be equal to R^n and therefore, we write the statement does a system is controllable if and only if the rank of the controllability matrix is n is there well can I look at it in a slightly different way ok.

So, let us let us take this definition here for example so, in this expression what I have is x_1(t) is C is some vector α ok. Now what is given to me in this if I look at this entire expression closely, the initial condition is given to me which is 0 I know what is x_1 at the final state the final time is this also possibly given this is also a finite number A is given to be B is given to me ok. So, I can compute these α because this curve α directly comes from e^{At}. So, once I know e^{At} once I know A I can compute what are these α.

So, what is unknown here? The unknown here is does there exist an input u which will steer my system to x_1 in some finite time t_1. Essentially I am solving for an equation like this. So, when is this equation solver is will this alpha is $C^{-1}x_1$ and this alpha has the information of my unknowns which is my input. So, this C which is a controllable matrix; so, for a solution to hold here the C matrix should be invertible right ok.

This is a little interpretation when case when k = 1 even though for a higher case it may have a different interpretation, but all the undergraduate takes on control we will start with deriving the Kalman rank condition of this form right. When you want that the inputs are the scalars this is u is in R, then I can interpret this as just solution to a linear equation which is possible if and only if C is invertible right. So, and which n with now generalizes to the case when u is in some R^k to saying that the C must be of full rank. So, what is the full rank? The full rank is as a maximum possibility of being n ok.

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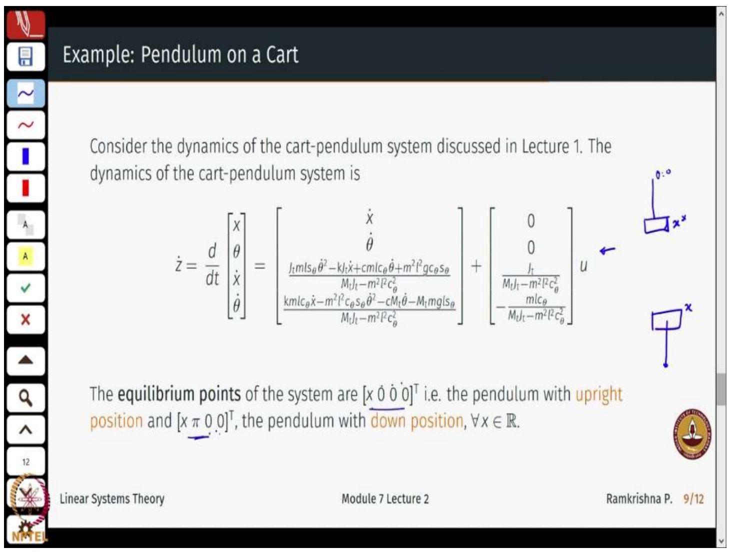

So, I just start again with the dynamics of the cart pendulum system which was discussed in the first lecture the well at we derived the dynamics to be of this form it will have certain equilibrium points right. So, with that is which means that the pendulum is in the upright position and also with the pendulum being in the down position. So, this is an equilibrium position for any x^*. So, the theta equal to 0 and at steady state the velocities will go to 0. So, this will be the theta equal to 0 the velocities both being equal to 0. Another possibility is just a downward position theta is π some value of x and the velocity is going to 0 ok.

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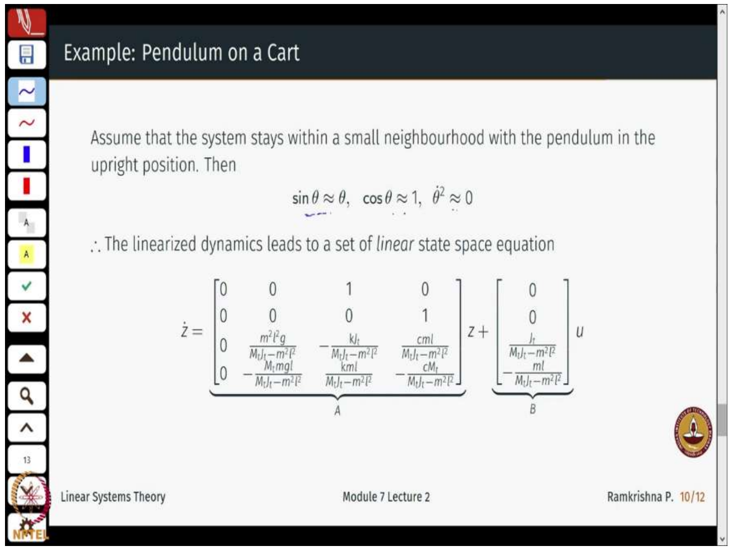

So, just to do a little approximation I just approximate sin(θ) as θ for and cos(θ) as 1 for the θ and $\dot{\theta}$being close to 0 ok.

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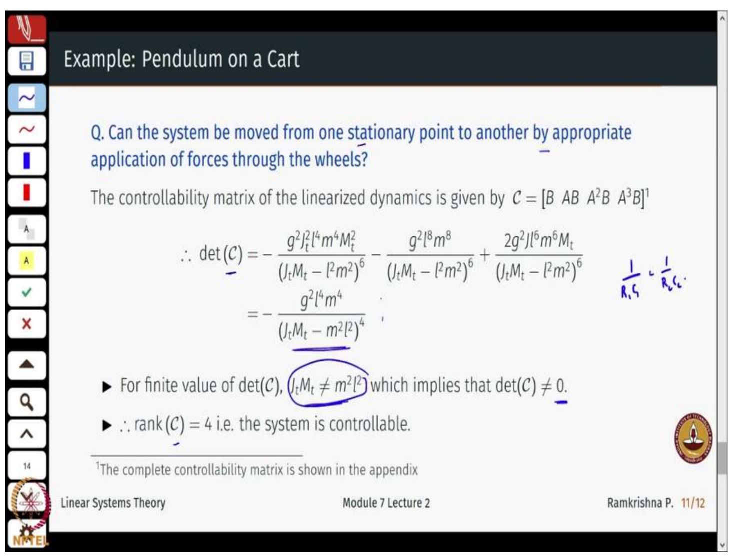

Now, I can ask a question, can the system be moved from one stationary point to any other stationary point by application of forces through the wheels? Then I compute the determinant of this matrix which turns out that for finite values. So, if as long as this does not hold which implies that the det(C) ≠ 0 which turns out then that the rank(C) = 4 and therefore, the system is controllable ok.

So, just some steps to for how to check whether or not the system is controllable and if it loses control at what point does it lose? But in general for all these systems we call them to be generically controllable or even structurally controllable because, for almost all values of parameters of J, M, m, l they are controllable except for a very small set ok.

Similarly even the RLC circuit that was controllable for all values of R and C except when this was with this was true. So, these are systems which are structurally controllable which is when a property directly related to controllability, but a weaker notion of controllability also, where the system is controllable for all parameters except some very small values of parameters. As long as we can avoid those, we can steer the system from any point in the state space to any other point in the state space.

So, we not deal with the literature on structural controllability, but you can just a dig out something from literature it also turns out to be a very interesting ideas right and then they do a bunch of stuff starting from graph theory and all. We will not really touch up on that, but if anyone is interested I can point you to the to the appropriate literature ok.

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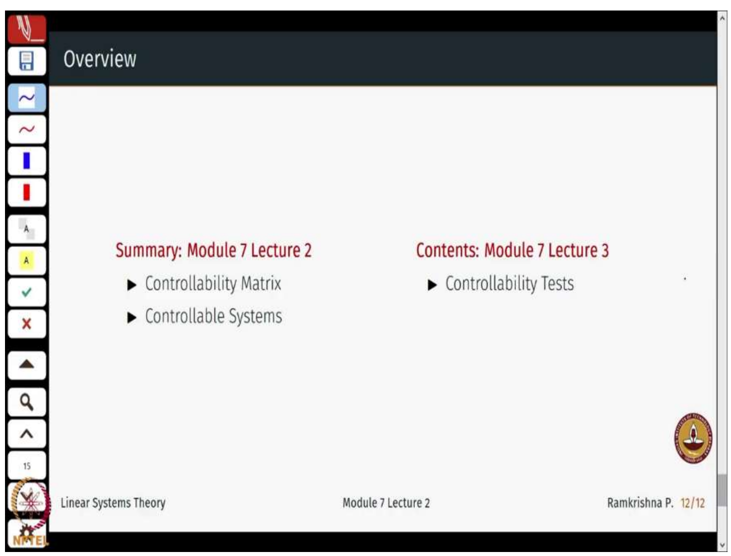

So, that concludes our discussion on controllability matrix and what are controllable systems; in the next lecture we will see ok. So, one test that we did today is about the rank of C being n or C being full rank right or in the one dimensional case where this whether was scalar inputs, the problem translated to C being an invertible matrix. Now, are there any other tests of controllability? Right. Are there any other tests relating to the Lyapunov stability properties, that we studied in the previous lecture? So, that all will be coming up later in lecture number 3.

See you then thanks for listening.