Linear Systems Theory

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Module - 07
Lecture - 01
Controllability and Reachability

Hello, welcome to this lectures of week 7 on the course Linear Systems Theory. So, so far we did a bunch of stuff building up true concepts in the last 6 weeks starting from modelling to some concepts in linear algebra, we had studied notions of solutions to equations equilibrium points, stability and also how to verify stability in the LTI case. So, we had some kind of functions, we had an energy base of physical interpretation of those.

So, we will today we will talk a bit about control. So much of the undergraduate course are focused on say compensated designs using say PID or the Lead Lag Compensators or at some point of time we also talked a bit about pole placement, right. And in much of the cases we never ask ourselves a question is a system controllable, right. But I just start with the assumption that the system is controllable and then go about designing a lead like or PID controllers or whatever, right. So, we can either through the root locus method or through the frequency domain methods why are the plots.

So, today we will do a little more qualitative analysis of systems or what does it mean by controllability of a system is controllability a binary, yes or no answer or there is a little more information to it. What happens if a system loses controllability, what happens if the system loses controllability, what happens to those uncontrollable modes, are they stable, are they not stable and so on.

So, we will answer a few of those questions. We will also look at then some tools to test whether the system is controllable or not, some of them you would have read a bit earlier like the Kalman rank condition and so on. So, we will do a little more detailed analysis on those, because we are now equipped with some tools on linear algebra in the form of looking at systems as evolving over vector spaces, subspaces and so on, ok.

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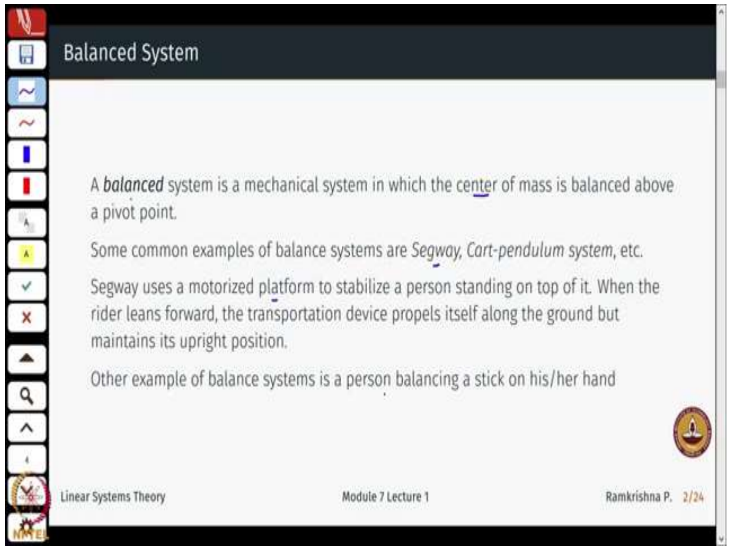

So, we begin with a little example of I will start with a balanced mechanical system. Well, you would have seen a lot of these in real life. So, balance system is a mechanical system where the centre of mass is balanced about a pivot point. Example is a Segway scooter where it is just a motorized platform to stabilize a person with who just stands on top of it, and when the person leans forward the device propels itself along the ground, but still maintaining the upright position.

So, the person standing on top does not fall, but he can still move forward. So, you could have seen this in many places on TV or also in real life. Another day-to-day example that is that we do is just of balancing a stick on a person's hand, we would have done this a lot ok.

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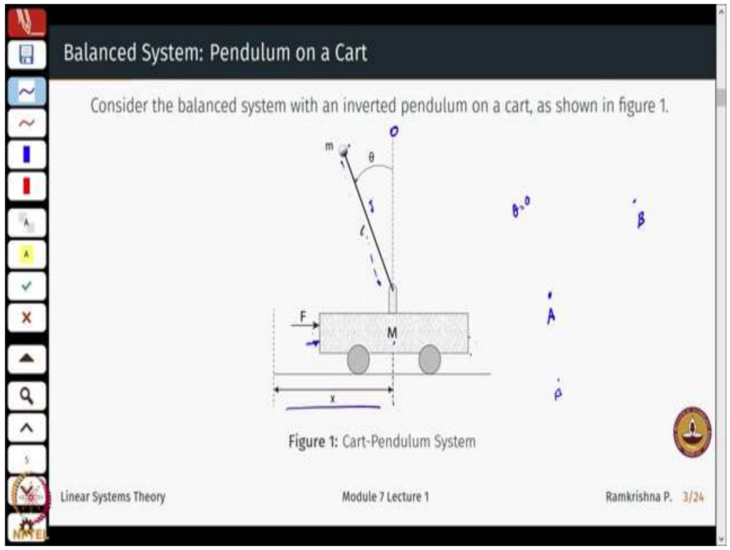

So, if I were to model that or look at of a simpler version of that, that would just look like or a segway scooter can approximately be looked upon also as a pendulum on a cart wheel, ok. So, I can move this cart where a force here not only can the cart move from say a point A here to some point B here, this force also helps in stabilizing the position of this mass to the upright position or as to the upright position as possible especially when we encounter a certain disturbances, ok.

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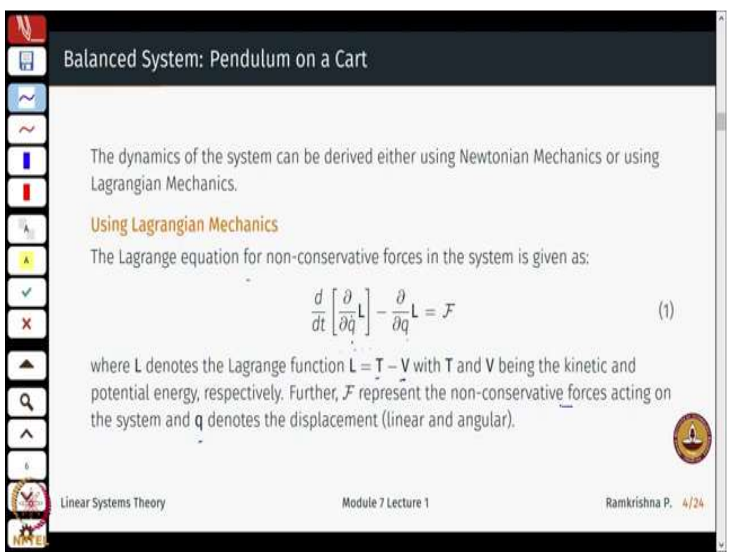

First is how do we model this? So much of in one of our earlier lectures we talked of Newtonian mechanics or even the Lagrangian mechanics, ok; I am not going to teach you Lagrangian mechanics. But just to teach you of how to write down the equations of motion through Lagrangian or the Euler lagrangian equation right digressive a bit, but that that should be, ok.

So, if I look at a system, so the Lagrange equation for non-conservative forces is given by this equation where in this case a q would define would denote the generalized positions, $\dot{q}$ would denote my generalized velocities could be both a linear and angular, L denotes the difference of the kinetic and the potential energy, and F represents some kind of non conservative forces acting on the system ok. So, what does this mean in the context of the cart pendulum system? So, I will have kinetic energy corresponding to the mass M here, I will have kinetic energy corresponding to the mass M here and also this little the thing here which would have its own moment of inertia and therefore, have it is own kinetic energy.

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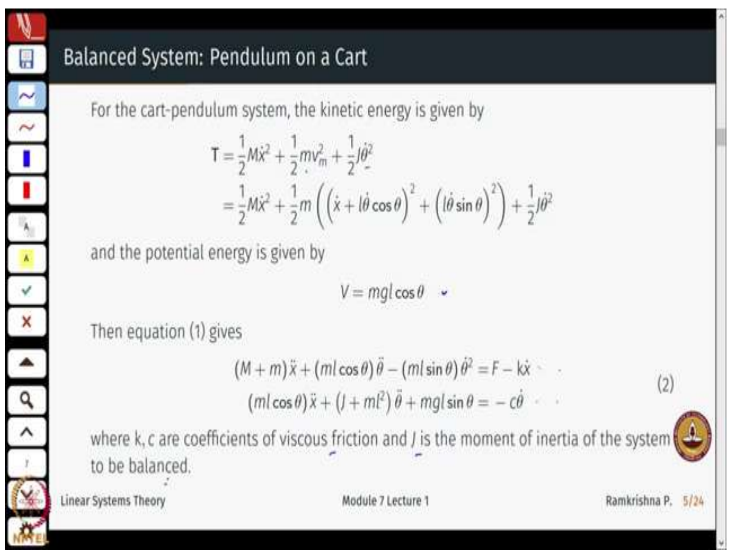

So, the total kinetic energy would be $\frac{1}{2}$M $\dot{x}^2$, $\frac{1}{2}$m for the first small mass and then to do with the θ and similarly have a certain potential energy of the system, that I just take T - V plug it in here and yeah I will get the equations for the system which looks like this, ok. If we just work this out, these are simpler techniques to work.

So, these are the equations of the system with which is coefficient corresponding to viscous friction, the moment of inertia and so on and this is the some what we would like to the balanced, ok.

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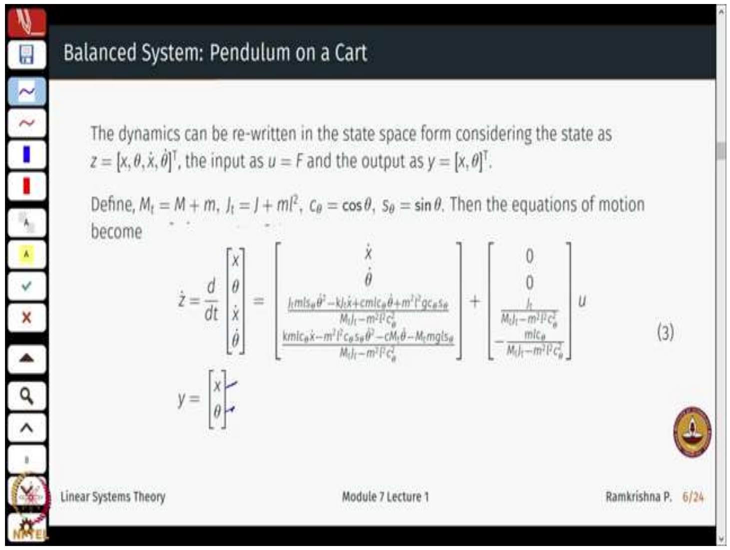

So, I can write down the dynamics now in the state space form which essentially turns out to be a non-linear equation. I just do some little simplifications and then end up with equations like this. And I can assume that I can measure the horizontal position and the angular displacement of the pendulum also which is sitting on the cart. So, this x is the; x will denote the motion of this mass here on which, so this is the x which I can measure, ok. So, what are we interested here right so much of the time, the problems were spent on can I stabilize this to $\theta = 0$ without really worrying about this thing.

Now, if I talk in terms of a segway what I really want is to move from a certain point A to certain point B while keeping theta equal to 0 or as close to 0 as possible, so that the person who is standing on the scooter does not fall over, ok.

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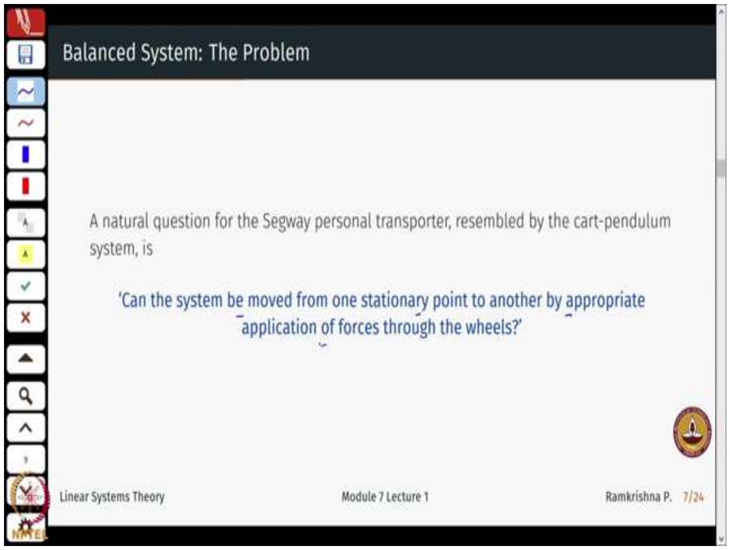

So, that is a little control problem that I can ask instead of; so, I have a stabilization problem of keeping the mass or the pendulum at the upright position or the person standing on the scooter at the upright position. In addition can I ask a question, can the system be moved from one stationary point to another by appropriate application of forces right, through so, this is my input here, I do no have any control on I do not have another input here, right. So, there is another set of input here, then my problem would have been much mush easier, but unfortunately we do not have any control over this part of the system, ok.

Now, this gets us to defining the notion of a controllable and reachable subspaces, ok. Why I am a using two different terms for controllable and reachable, ok; that I will shortly tell you that, but we will look at each of these definitions or these properties in terms of some general subspaces of R^n, if n could be the a vector space where my system dynamics evolves on.

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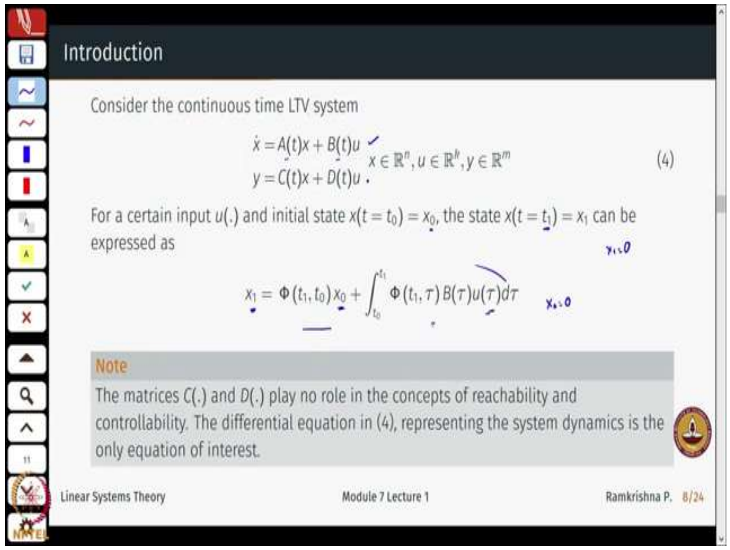

So, what is the problem? So, I have a system $\dot{x} =$ A x + B u could in general be LTV or even LTI; I know the solution looks something like this, right. So, I start with an initial state at t = 0, what I want to reach is a final state x_1 at t = t_1 right. And then so what does the solution tell me, well, can I resolve starting from an initial state, can I reach x_1 by some application of control and this is how the solution looks like, ok. I do not really look at what at how the outputs look like, because they are not really important in looking at the evolution of states, right. So, we are just looking at the first equation here.

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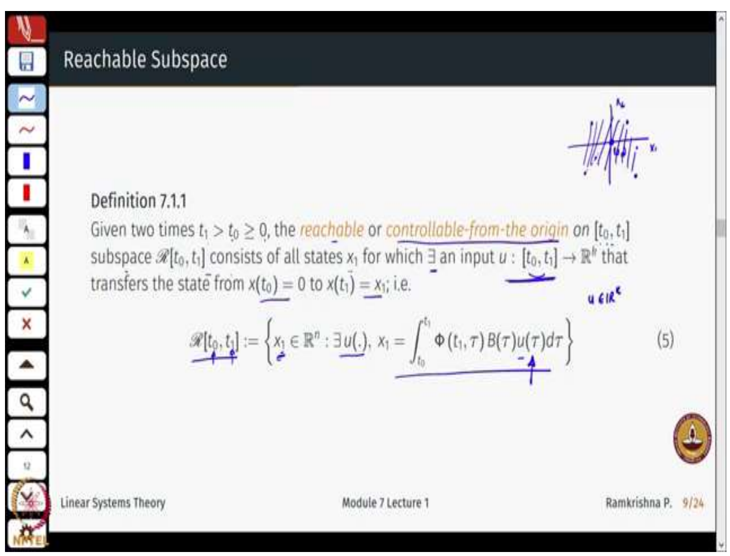

So, I do not reachability or controllability yet, but I will just define first in terms of subspaces. So, if I am given two times t_1 and t_0 > 0, the reachable right or it is also called controllable from the origin. So, what do I mean ok, I am say from the origin. So, this is my origin (0, 0) if I am in R^2 on (t_0, t_1), right. So, given two times, the reachable subspace consists of all state x_1 for which there exists an input u right, defined over this interval such that, so k is my number of number of inputs.

So, u belongs to certain R^K, that transfers the state from this origin to some other state x_1. So, what does it mean? So, starting from here can I what are the points I can reach? Can I reach this point by application of some control in finite time t_1? Can I reach this point, can I reach this point, this point, this point, this point and so on and what I do is, so there is no restriction on the input as long as it is bounded. So, I just take the set of all points and then that set of all points that I can reach from the origin to the point at time t_1 with application of some control u is called the reachable subspace, right. So, that is the mathematical expression of this that R(t_0, t_1) so, what are the points I can reach starting from t 0 at the origin to x_1 which is at t_1?

So, set of all x_1 for which there exists an input u such that x_1 satisfies this, ok. Where does this come from? This just comes from substituting x_0 = 0 here right. I am just talking of reachable from the origin. It can also be generalized to some other non-zero point; non-zero initial condition, but that does not really change the analysis. Because I can just shift my origin accordingly or just for use off use of notation and computation here I am just doing that, but in general we can start from any other non-zero point.

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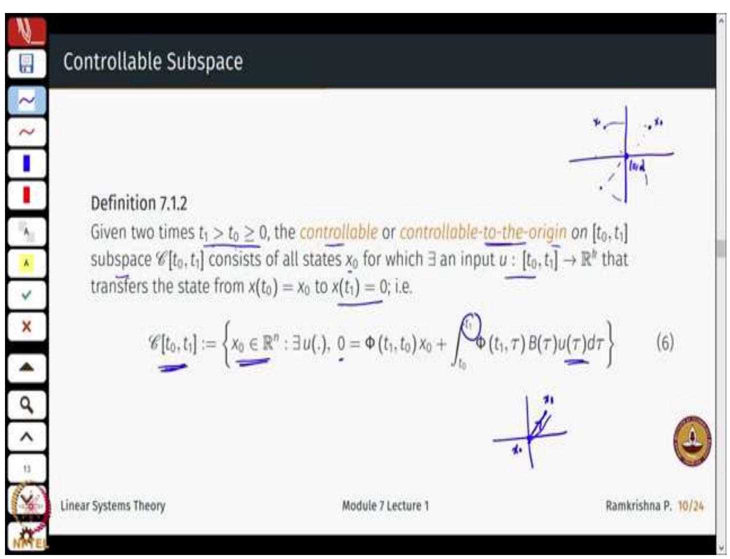

Now, similarly given two times t_1, t_0 the controllable to the origin. So, here I look at a different point, right. So, let us say I want to reach this point, I want to reach the origin and I take all the points. So, I take this point x_1 and can I ask the question, can I reach the origin starting from this point. I take this point can I reach the origin starting from this point, this point and so on and then I do a collection of all these points, right.

So, that is what it means; so this statement says at given two times t_0 and t_1, the controllable subspace consist of all states x_0. So, these are all my initial states right x_0 for which there exists an input u that transfers the state from any of these x_0s to the origin, right. So, at x(t_1); at x(t_0), I start from any of this point v and x(t_1), I should be able to reach the origin.

So, the collection of all these x_0s which transfers me to the origin in finite time t_1 with some application of control is called the controllable subspace, ok. Again this comes from the equation I guess. So, what is now the final state? Final state x_1 is 0, right. I start and then I just look at all the initial conditions for which there exists a solution to this equation. We will see what the solution to this equation means in terms of controllability or reachability and their appropriate subspaces.

So, the reachability; so, the reason I call it controllable from the origin is can I control starting from x_0, can I go to this point right. So, it is like so here I am controlling to the origin or controllable to the origin, I am looking at all points where I want to go to the origin. And then I am looking at when I say reachability I am looking at all the points which I can reach starting from the origin.

So, usually it is like reachable from the origin or controllable to the origin and we will quickly see if this subspace, so, is there any relation between the controllable subspace and the reachable subspace right, which intuitively you could say that if I can starting from this point, I can reach. So, this is my x_0 at the origin. If I can reach x_1, I should be able to again start at its same x_1 and then come back to the origin, intuitively; we will see if that is really true or not. If true for what kind of systems that is true if at all if it is so, and if it is not true, why it is actually not true.

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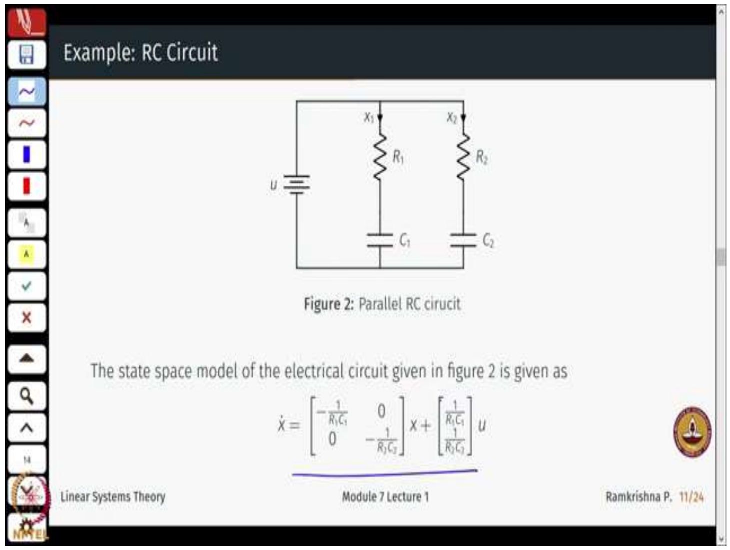

So, let us begin with an example. So, I have this parallel RC circuit whose dynamics I can write in this way straight forward to compute just apply the voltage in current loss that we know, ok.

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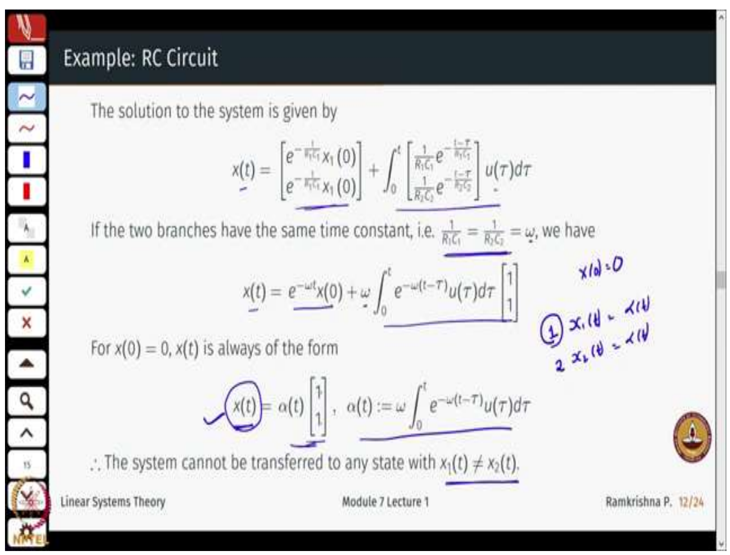

So, I am looking at the solution of this right for some input. So, x(t) we have some initial conditions and then you have this term it is on writing down this term over here, ok. Now, assume it the Rs and the Cs; the resistances and the capacitance are such that they have the same time constants that $\frac{1}{R_1C_1} = \frac{1}{R_2C_2}$ and let me call that ω right. In that case my solution x(t) will be something like this, $e^{-\omega t}x_0 + \omega$ times this one, ok.

In the reachability analysis I am interested in x_0 is 0 and therefore, what I am left with here is the x(t). The set of all x(t) that I can reach starting from the origin take some form like this right, some α(t_1) here and a 1 here, if I just write it down. So, x_1 (t) is α(t) x_2, is also α(t) right where α(t) is just this one, ok. So, what does this mean if I were to plot the reachable subspace so, if I just say here $x_1 = x_2$.

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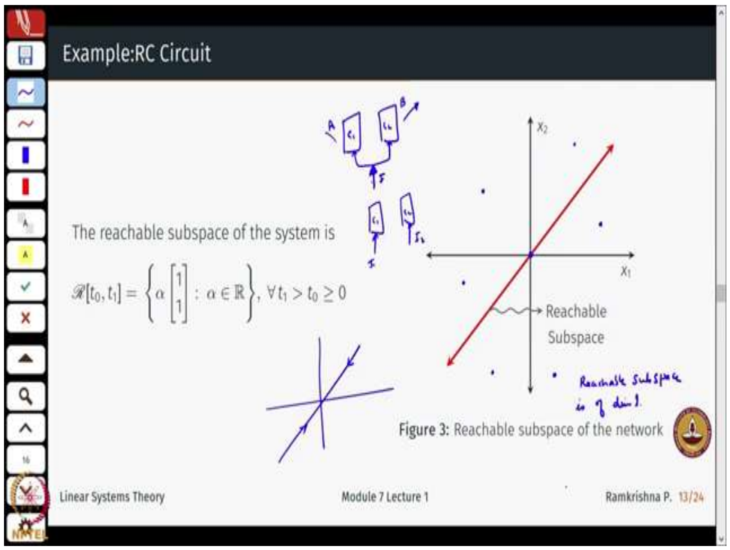

So, the points I can move or I can reach starting from the origin is only on this red line which also means that I cannot go to this point or this point or this point ok. So, I am restricted or my movement is restricted only to this red line or it also means the system cannot be transferred to any state with $x_1(t) \neq x_2(t)$. So, I cannot have that I want to go to a point where $x_1(t) = 1$ and $x_2(t) = 2$, whenever I say $x_1 = 1$, I am immediately imposed the condition that $x_1 = x_2$ right from this equation.

So, I cannot; the system cannot be transferred to any state where $x_1 \neq x_2$ and therefore, if I look at this reachable subspace, the reachable subspace is of dimension one, right. So, the reachable subspace is of dimension one, right which is again a subspace of R^2 and my state space is R^2 ok.

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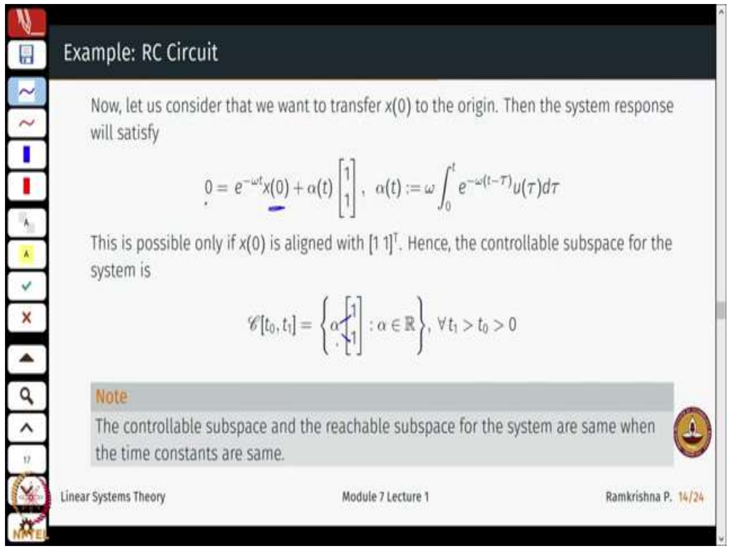

Similarly, I can do the controllable part, right; so, what are the set of points starting from some initial conditions which can reach the origin, ok. So, it is again, it again turns out to be the same that the that the controllable subspace well, it will again be the same right because if I look at this one, so again I will have a x_1 is α, x_2 is also equal to α, ok. Therefore, the reach the controllable subspace also reaches like they also looks like this. So, all the points that can be controlled to the origin will have some form like this.

So, what does this mean? So, let us say I have two cars say this is car 1, this is car 2 and then I have the same input force which is being applied to both the cars, ok. Now, the point is can I take this car to a destination A and can I take this car to a destination B? The answer is no right because resuming all the properties are identical and so on.

So, this force acts equally on these two cars and therefore, I cannot take this independently to some other location and this independently to some other locations or I cannot control both of them individually. Whereas, if this is the case c_1, c_2, I have a f_1 here I have a f_2 here. I can actually take each cars wherever I want. So, this is a little interpretation of this ok. What happens if the time constants are not the same, then we will have a better looking solution that I can actually reach all the points in the state space here, here, here and so on.

So, we will eventually come up with conditions of what is a dimension of the subspace or what are the points I can reach starting from the origin if at all I can reach, ok. So, that will be the objective or the larger objective of this lecture, ok. So, now I define what are called the controllability and reachability gramians, ok. The definitions will be clear as we do the proofs and understand them, ok.

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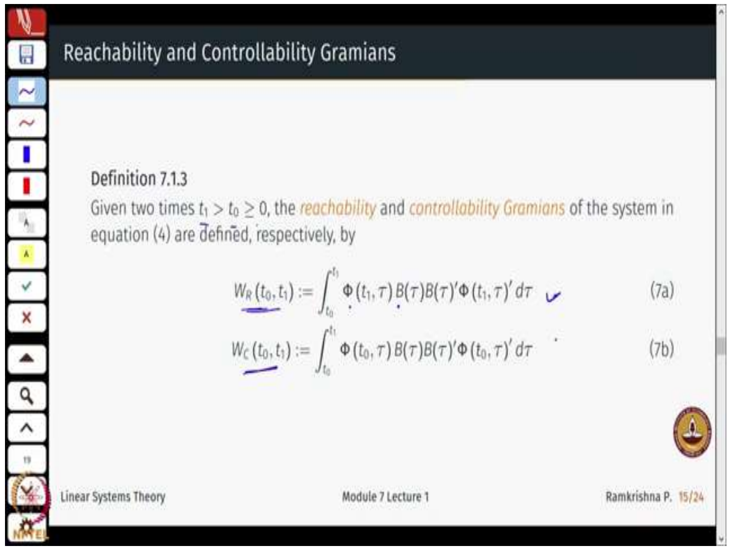

So, given two times again t_1 and $t_0 \geq 0$, the reachability gramian and these are they are defined something like this. I know Φ is the state transition matrix, B is the input matrix and so on. So, this is how I define. Well let us assume that this is that, I know that this is true what is the motivation for defining it; this way we will slowly come to that while we do a bit of bit of the proofs.

So once I have this reachability and the controllability gramians, ok, they will have some properties of their own. So, just check for yourself from the properties that we know will the reachability gramians if I compute this? Will they be sign definite in the sense of will they be positive definite, negatives definite or this cannot say anything about the sign of them? Just a little trick question that you can infer from maybe your lectures from week 3 right, ok.

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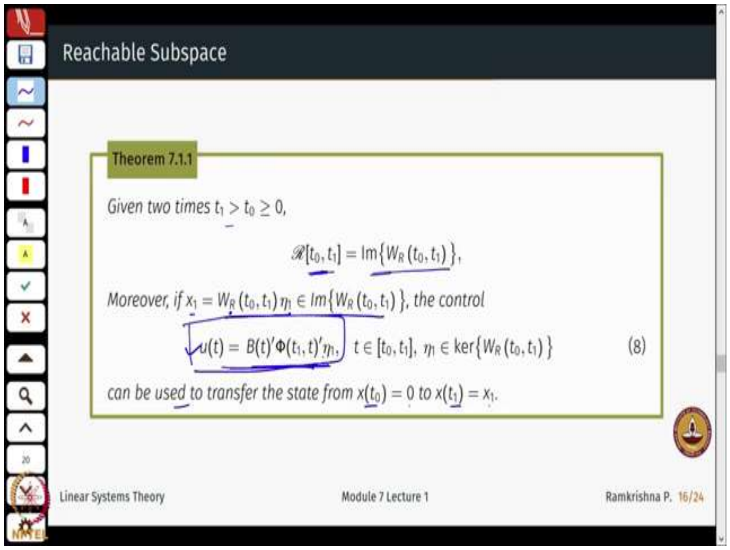

So, what am I interested now right, so given a system $\dot{x} = $ A(t)x(t) + B(t)u(t), how do I compute the reachable subspace? So, this question answers, this theorem answers that question. So, again given two times, the reachable subspace is just the image of the reachability gramian, this one sorry, this one right. So, if I just take the image of this map that will be my reachability gramian or that will be my reachability subspace ok.

Moreover, if x_1 belongs to this W_R, so the times η_1 with this x_1 belonging also to the image of W_R1, then the control u(t) given by $B^T \Phi^T \eta_1$ can be used to transfer the state from x(t_0) to x(t_1) right, ok. So, what is given to me now what I want to prove is that the reachable subspace or the reachable space is the image of W_R, right. So, when x is in the image of W_R, there will be a vector η_1 which will transfer η_1 via W_R to x_1, and then this and what is the control? This is the control that actually transfers my state from origin to some x_1.

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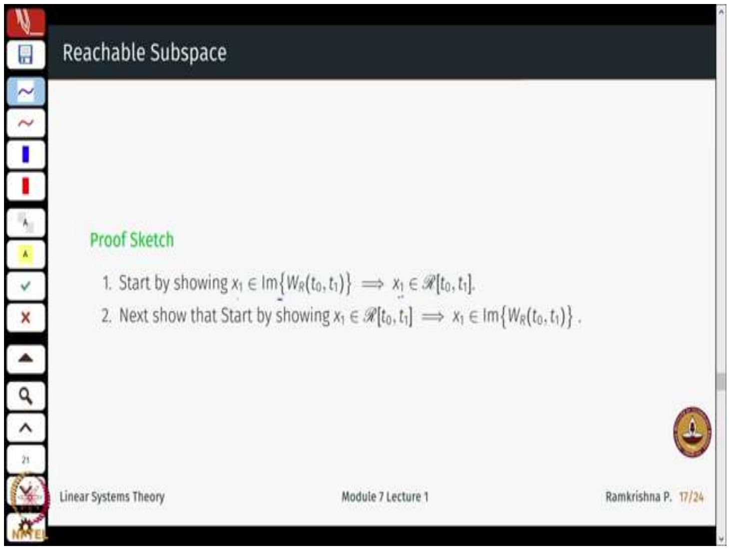

So, the proof sketch well we shall first start by showing that if x_1 is in the image of W_R then this same x_1 is also in R(t_0 t_1), ok. So, what does that mean; so, you just write down the proof for ourselves, ok.

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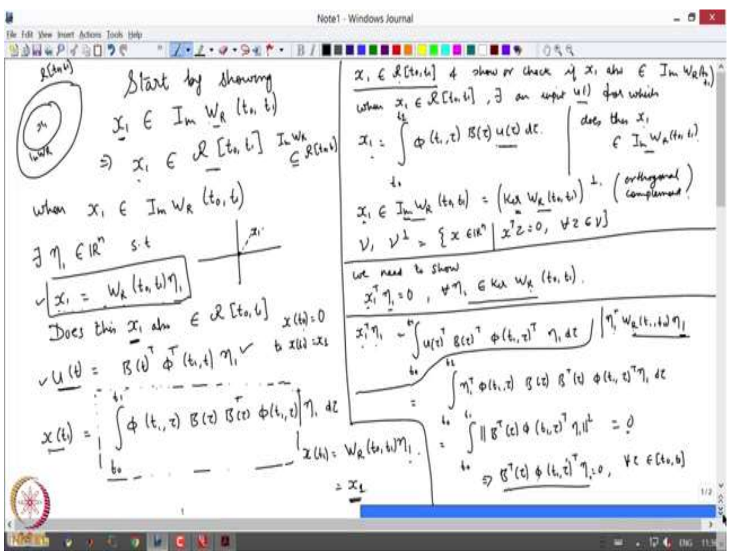

First we start by showing that if x_1 is in the image of $W_R(t_0, t_1)$, this would also mean that the same x_1 will belong to the reachable subspace, ok. So, what does it mean, when x_1 is in the image of W_R (t_0, t_1); this means that there exists some vector η_1 in R^n such that this x_1 = $W_R(t_0, t_1)\eta_1$ like the definition of the image, now we need to show that or just check does this x_1 also belong to R(t_0, t_1) ok. So, which means that if x this x_1 belongs to R(t_0, t_1) it should mean that starting from this point, from the origin I should be able to reach this point via some application of control. Now, what is that control that control? The control is given in the theorem statement this control this guy. So, this control of u(t) $=B^T(t)\Phi^T(t_1, t)\eta_1$. So, it should be a transpose here also ok.

So, now, I will just check it does this control or transfer my system from the origin to this point x_1 that should be easy to check from the solution. So, x(t_1) ok; so, what do I just substitute all of these into the solutions here, right. So, I know what is the input u, I know that x(t_0) = 0 and I compute what is x_1, what does that give me, x(t_1) is the initial condition x_0 = 0. So, the first term goes away is $\int_{t_0}^{t_1} \Phi(t_1, \tau)$B($\tau B^T(\tau)\Phi^T(t_1, \tau)\eta_1d\tau$. Now, look at this carefully this expression in the dotted box this expression in the dotted box is essentially the reachability gramian, so this x_1 =$W_R x_1$ ok.

So, again let us go through the steps again. So, what I know is that x_1 is in the image of W_R, which means something like this happens. So, this is given to me, now what I want to show is that with the application of this input u does this transfer my system from an initial condition which is the origin to x_1, sorry x(t_1) is η_1 ok.

So, in the solution I just substitute the value of u and what i get at this x (t_1) is nothing, but this should be an η_1 here is sorry this x(t_1) = $W_R\eta_1$. Now, what is $W_R\eta_1$? $W_R\eta_1$ is precisely my x_1, and this x_1 is in the image of W_R. So, what did I prove now that if x_1 is in the image of W_R the same x_1 is also in the reachable space because I can reach this point. So, by application of this control starting from the origin where do I end up with, I end up at precisely this x_1.

So the first thing is done, right. So, the x_1 which belongs to the image of R also belong; so, the image of W_R also belongs to the reachable space. So, if I just say this is my x_1 which belongs to the image of W_R, it also belongs to the reachable space. So, this picture also means that the image of W_R is a subspace of R(t_0, t_1), it could be something like this.

Now the second thing that we need to show is let us assume now that x_1 is actually in R(t_0, t_1) and show or check if this x_1 also belongs to the image of $W_R(t_0, t_1)$; now, what does it mean when x_1 belongs to R(t_0 t_1). So, when x_1 is in R(t_0, t_1) this means then that their exists an input u could be whatever right, for which x_1 is $\int_{t_0}^{t_1} \Phi(t_1, \tau)\, B(\tau)u(\tau)d\tau$ (Refer Time: 30:32). Now, what I would want to show is that does this mean or does this x_1 belong to the image of W_R lets show this, ok.

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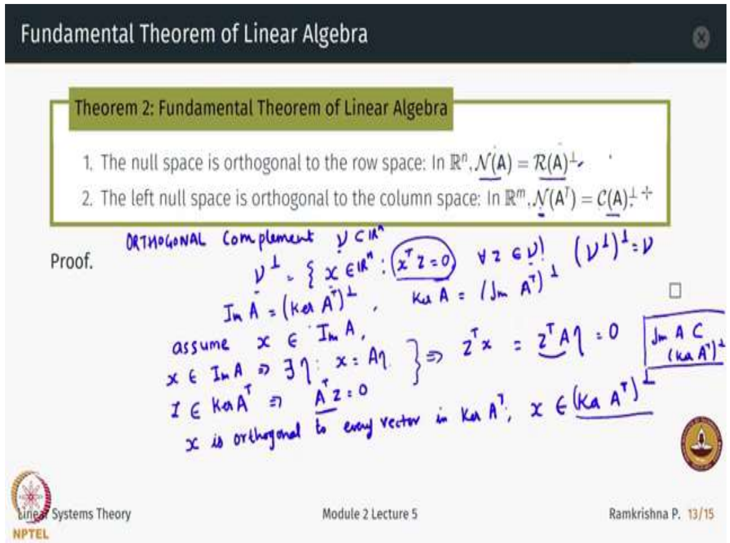

Now, some things that we know from our previous lectures is something like this right. So, this these two things here this and this, right. So, I will just quickly write down those here in with respect to the theorems that we will do here.

So, if I were to show that x_1 is in the image of $W_R(t_0, t_1)$, this image of W_R is also equal to the kernel of $W_R(t_0, t_1)^{\perp}$ with this orthogonal complement. I assume, we can recollect what is the definition of this (Refer Time: 31:33), ok. If not then I can just write down. So, this is the orthogonal complement ok; so, if I take a subspace v, it is orthogonal complement $v^{\perp}$ is the set of all vectors. So, this $v^{\perp}$ is the set of all vectors x in R^n, assuming that v comes from R^n, such that x^Tz = 0 for all z in v, we just recollecting the definition that we had earlier of the orthogonal complement.

Now, what do we need to show? So, again we know that x_1 belongs to the reachable subspace which means something like this, holds what I want to show is that this x_1 which is in image of W_R which is also equal to the kernel of $W_R(t\_0, t\_1)^{\perp}$. So, sorry I need to show that this x_1 is also in image of W_R which is equal to the kernel of $W_R(t\_0, t\_1)^{\perp}$, which means that we need to show whether something like this is true, ${x_1}^T\eta_1$ = 0 for all η_1 which come from the kernel of $W_R(t_0, t_1)$. Now, so, this is what we need to prove ok. How does this ${x_1}^T\eta_1$ look like? ${x_1}^T\eta_1$ is what is x_1? x_1 comes from here and if I do the transpose, it will simply be $\int_{t_0}^{t_1} u^T B^T \Phi^T \eta_1 d\tau$.

Now, this η_1 is in the kernel of W_R which means that if I define a quantity like this ${\eta_1}^T W_R(t_1,\ t_0)\ \eta_1$, this will be nothing, but $\int_{t_0}^{t_1} {\eta_1}^T \Phi(t_1, \tau) B(\tau) B^T(\tau) \Phi^T(t_{1,}\tau)\eta_1 d\tau$ ok. That comes from the definition of the W_R, the reachability gramian ok. So, if η_1 is in the kernel of W_R just to simplify this I can also write this way right $\int_{t_0}^{t_1} ||B^T(\tau)\Phi^T(t_{1,}\tau)\eta_1||^2 d\tau$. Now, this is equal to 0 because η_1 is in the kernel of W_R, ok. This also means that the quantity inside this which is $B^T(\tau)\Phi^T(t_{1,}\tau)\eta_1$ = 0; now this is valid for all τ in t_0, t_1, ok.

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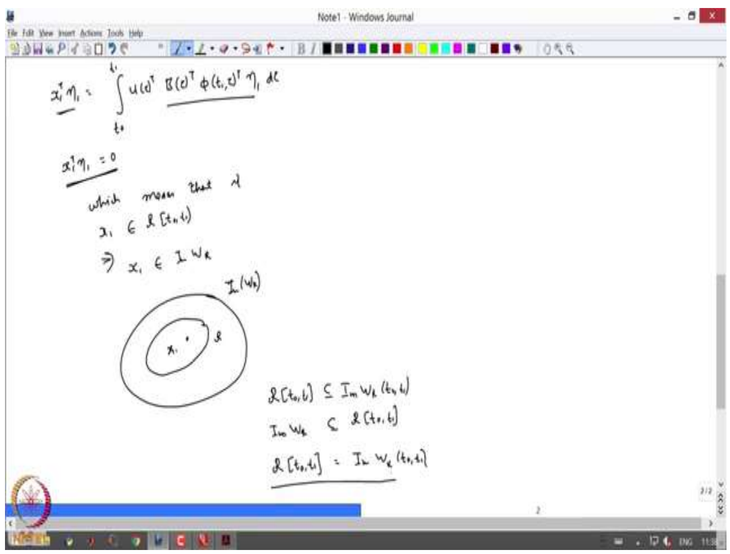

Now, again go back to this expression what was ${x_1}^T\eta_1$, ${x_1}^T\eta_1$ was $\int_{t_0}^{t_1} u(\tau)^T\ B(\tau)^T \Phi(t_{1,}\tau)^T \eta_1 d\tau$. Now look at this expression closely. So, this thing here BΦη so, 1 is equal to 0, sorry and this is equal to 0 and therefore, ${x_1}^T\eta_1$ = 0 which is what we wanted to show right, ok.

Again let us go through the proof again, what we wanted to show is that if x_1 is in the reachable space which means there exists an input u such that this equation is satisfied, then we also we need to show that this x_1 is in the image of W_R and the image of W_R has a form like this, ok. So, equivalently this means that we must show that if x_1 which comes from the reachable space if it has to be in the image of W_R, it should satisfy something like this where η_1 comes from the kernel of W_R of the reachability grammian, ok.

So, now $x_1{}^T\eta_1$ expands in this way right this one, ok. Now making use of this fact that η_1 comes from the kernel of W_R, I compute $\eta_1{}^T W_R \eta_1$ which is 0. This being 0 also means that $B^T\Phi^T\eta_1 = 0$ right which actually sits in very well in the computation of $x_1{}^T\eta_1$.

And therefore, $x_1{}^T\eta_1 = 0$, which means that if x_1 is in R(t_0, t_1), it implies that the same x_1 is also in the image of W_R, ok; what does this mean? So, I have x_1 here, this is in R and this x_1 is also here in the image of W_R which means R(t_0, t_1) is something like this. What is the first expression tells me? Well it tells me the opposite right, that the image of W_r is a subset of R(t_0, t_1) and this is possible if and only if R(t_0, t_1) is the image of $W_R(t_0, t_1)$ and this is what we wanted to prove, and the reachable subspace is the image of the reachability gramian right. That was what the statement was that R(t_0, t_1) is image of $W_r(t_0, t_1)$.

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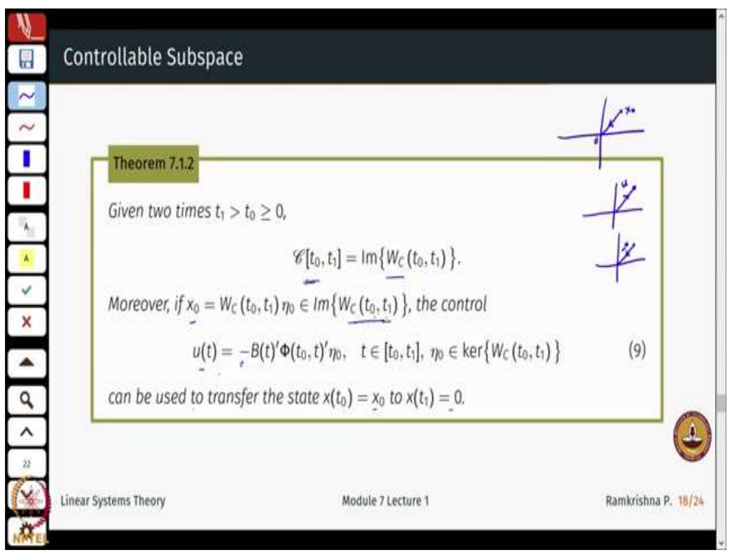

So, similarly I can do for the controllable subspace that the controllable subspace is the image of the controllability gramian which is defined this way. The second expression here, nothing much changes in the proof steps. So, I would leave the details to you. So, the again the theorem statement says that that C the controllable subspace is the image of W C and moreover if x 0 is in the image of $W_c(t_0, t_1)$, the controllability gramian the control u(t) which is now, ok. What was it earlier? It was $B^T\Phi^T\eta_1$.

Now it is just with a negative sign u(t) = -B$\Phi\eta_0$, this can be used to transfer the state from x at any initial condition to x(t_1) = 0, ok. What does this mean? So, I am just looking a control log which will take all points x_0 starting from here till the origin; loosely speaking if I can go from here till here with say some force or some application of input u, it can be thought of that if I want to come back I just apply a force of -u, right. So, this way I apply u this way, I apply -u and that is a loose interpretation of why this minus sign is sitting here, and the rest of the proof follows exactly the same steps. So, I will skip that over here.

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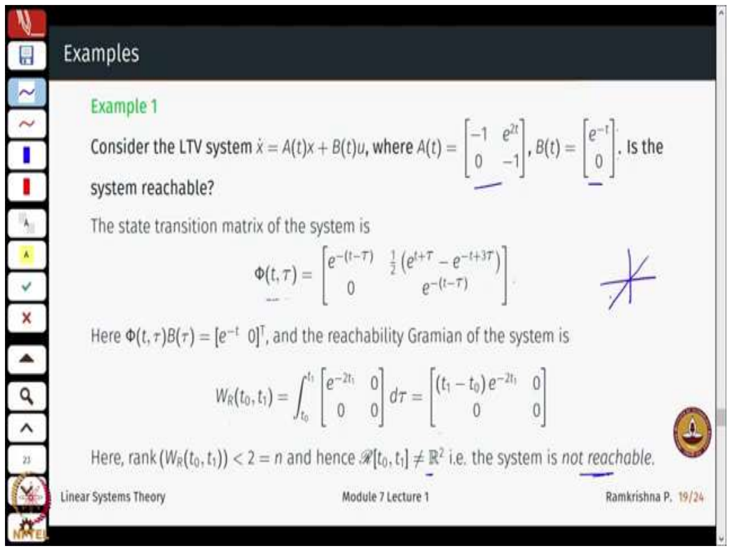

So let us do a couple of examples on how to actually compute this. So, I have a LTV system where A(t) looks like this B is like this. So, I am just computing ok; I will ask a different question not is the system reachable, I will just compute what is the reachable subspace before computing is the system reachable. So, based on week 4s lecture I can compute the state transmission matrix at to be of this form.

Now, I know that B is given in this form, I compute the reachability gramian. The reachability gramian is a 2 x 2 matrix of which looks like this, ok. The rank of it is well you can just say that it has a maximum ok, have a rank of a maximum of 1, right. So, and therefore, the reachable subspace is at best 1 and therefore, the system is not I would say completely reachable.

So, the reachable subspace could be as in the starting example we did right some subspace of one dimensional subspace of R^2. So, we call the system completely reachable if it can; if it reachable set or the reachable subspace is of the dimension of the state space which is 2 in this case, right. So, we can conclude this system is not reachable, ok.

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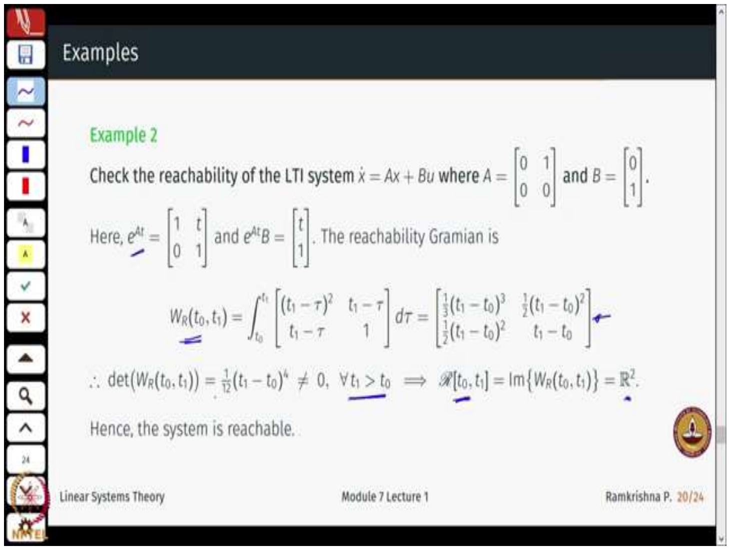

Now, I do a little simpler case of an LTI system where again I compute what is e^{At}. This is my state transmission matrix in the case of an LTI system. The reachability gramian takes this form and well I can look at what is the rank of this the rank of it is non-zero for all $t_1 > t_0$. This is what we wanted. We do not want to start at t_0 and end up at t_0, that does not make sense. And therefore, the reachable subspace is of dimension two and therefore, we can say that the system is reachable

You can do this computations, they are pretty straight forward to do. If I were to go from point A to point B, I would want to go with um a minimum possible energy right. I do not really want to over use my energy resources. So, what is the minimum energy that transfers my system from point A to point B, ok.

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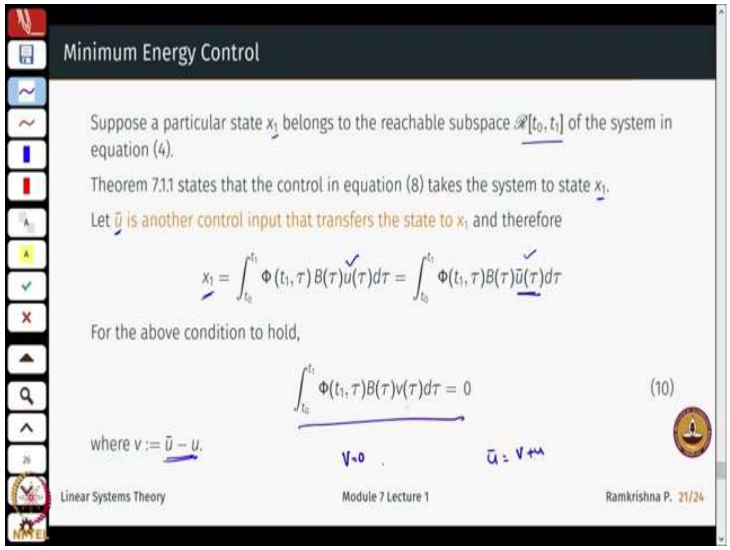

So, we will just look at look at the theorem statement and say well what was the input that we used earlier? The input that we used earlier was this one, u = $B^T\Phi^T\eta_1$ ok. Now, let me check if there is some other input, right. So, let us start with the reachable subspace. So, let us say a particular state x_1 belongs to the reachable subspace of the system we know that if it is in the reachable subspace, there certain control input takes it to this state x_1.

Let me assume now that there exist another $\bar{u}$ which also transfers the system to the state x_1, this one, right. I can reach the state x_1 with u or also $\bar{u}$ ok. Now if this is true well I can just do subtract and say that well this condition is also true that $\Phi B v^T$ = v(τ)dτ= 0 where v is just a difference between $\bar{u}$ and u.

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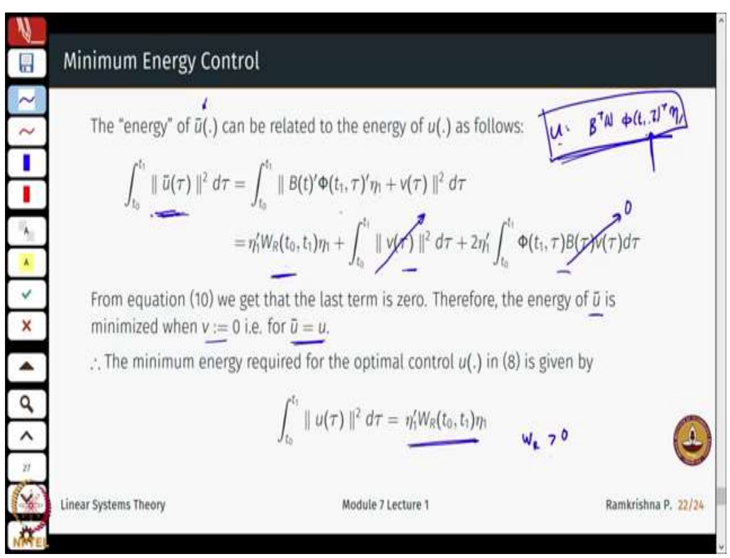

Now if I just look at the energy which just can be interpreted as a two norm, let me calculate the energy of this guy $\bar{u}$; $\bar{u}$ can take this form. So, I know what is u, u from my previous equations is this one if I am talking about reachable subspace this one. So, I will just write it down here u = $B^T \Phi^T \eta_1$. So, u = $B^T \Phi^T \eta_1$ ok. Now I am just looking at, ok; so, what is so from this expression $\bar{u}$ = v + u, ok. So, this is what I get, right. So, the energy of $\bar{u}$ is can be expanded this way. Now I know that ok; so, I just use the standard A plus B whole square formula and then I get something like this, ok.

Now, what we know from here is that this term is actually 0, ok; now I am left with this term and this term here, ok. Now when the question I would like to ask here is when does u bar take a minimum value? So, the new control input that I use which is slightly different from what I defined here, what is its minimum or when does it take a minimum value? Well look at this term here. This term W_R is always to be positive for the system to be reachable. That is if I look at the example closely here if W_R is of full rank, then the system is completely reachable, ok.

So, therefore the W_R should be a positive matrix, ok. Now, this I cannot get this to 0. What is possible is to minimize this term by setting this to 0. So this, the energy of u bar is minimized when v equal to 0. Now what does it mean from here when v = 0? V equal to 0 actually means that u =$\bar{u}$ and therefore, the minimum energy required, if I just say well I am just taking some energy and computing what is the minimum energy required that actually turns out to be this input only, this input u, right.

So, the minimum energy required is just given by this expression ok. So, that is a little intuition of also why we chose this particular energy here. This is not only steers from point A to point B or x or from the origin to x_1, but it also consumes the least amount of energy and this is a little proof for that.

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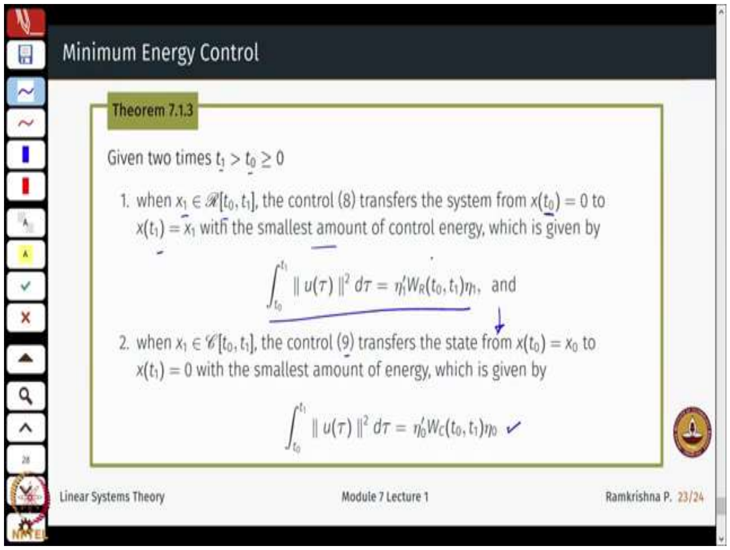

So, more formulae if I state that given two times t_1 and t_0 which are greater than 0, when x_1 is in the reachable state, the control transfers from x from the origin to x_1 with the smallest amount of control energy, right. So, what is a control 8? The equation 8 is this one, right. So, this control input transfers the system from the origin to x_1 with a least amount of control energy which is just given by this expression.

Similarly, for the controllable analogue of this when x_1 is in the controllable subspace, right so the control which is with just a negative of what is in the reachable subspace is just given by this expression. So, the idea here or the main message here is to say that the control energy which I defined over here in equations 8 and 9 is actually the best or the least energy consuming signal control signal which will transfer my state from point A to point B could either be in the reachability analysis or even in the controllability analysis, ok.

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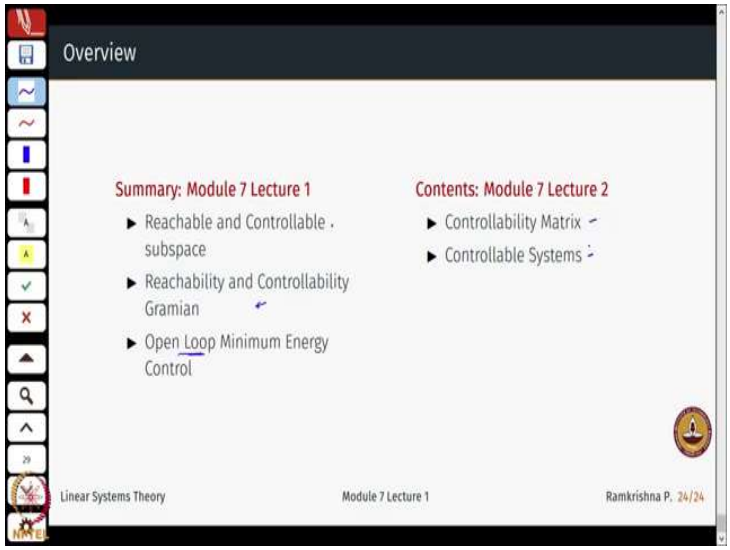

So, we conclude this lecture by just recollecting that we computed what were the reachable and controllable subspaces, we defined the notion of the gramians and then had some proofs on how the controllable and the reachable subspace are related to their respective gramians and we also quantified what is the open loop, ok. This is just a open loop, right I am not really closing any loop here. I am not doing a state feedback or something.

So, we also quantified how much is the open loop minimum energy required for the reachability and controllability analysis. And next we will go to what is the controllability matrix and what essentially how do we know define a controllable systems that comes in the next lecture.

Thanks for listening.