Linear Systems Theory

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Module - 06
Lecture - 03
Stability of Discrete Time Systems

Hello everybody. So, welcome to this last lecture on week 6 of Linear Systems Theory. We will be a rather short lecture where I will talk a bit about discrete time systems and also a little relation between linearization of non-linear systems and the stability analysis. So, does. So, the question that we will answer; again, I will not do the extensive proofs, but we already have an intuition of that from our previous lectures, what we learnt in our previous lectures, that if I take a non-linear system, if I linearize it around an equilibrium point and analyze stability in the linear sense of that equilibrium point does that have any implication on the stability or not of the original or non-linear system.

So, those two topics we will cover today, starting with stability analysis of Discrete Time Systems, ok.

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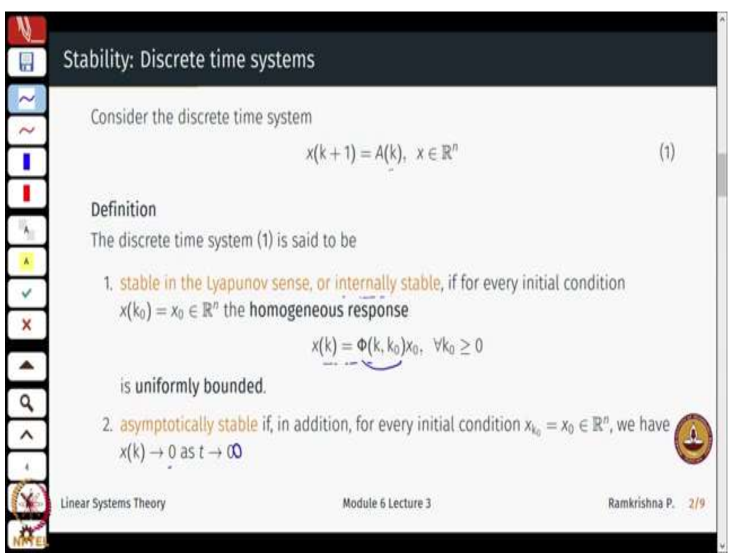

So, I start again. So, much of the definitions will look similar, so I just I just replace t by k and say x(k + 1) = Ax(k), is it is my discrete time system. So, the solution is stable in the Lyapunov sense or also called internal stability if for every initial condition the solution is uniformly bounded, again we this is our discrete time state transition matrix and so on. This is exactly similar to what was there that the solutions need to be uniformly bounded, that is what was also in the continuous time case. That does not change. What also does not change is the definition of asymptotic stability that as time progresses we would like the solution to go to origin as sorry as t goes to infinity, ok. So, this also does not change.

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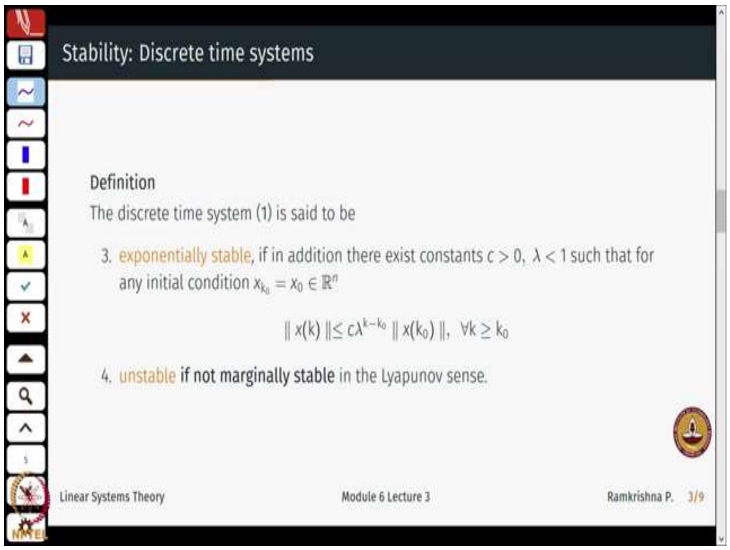

Exponentially stable, well it is again the same, just that instead of an of an exponentially decaying function we will have something like this here. So, the system is exponentially stable. If in an addition to asymptotic stability there exists constant C and λ again this $\lambda < 1$, such that for any initial condition starting over here we have a relation like this. So, this is the exponential version of exponential stability version of the vof a discrete time system. Again unstable, well if it is not stable then it is then it is unstable, ok.

So, first a little thing of; what does it mean by discrete time stability?

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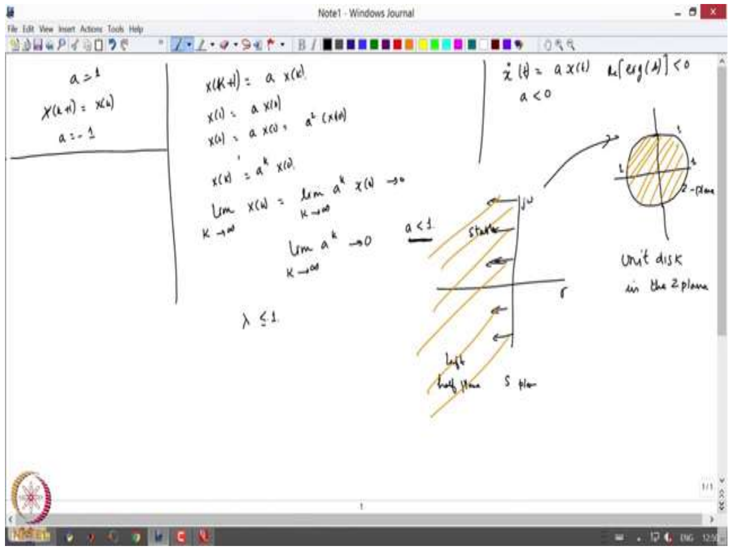

So, if I, let us me just take a scalar system say x(k + 1) is say a number ax(k). So, the solution to this, let me just write down this x(1) = ax(0), x(2) = ax(1), so this is a$x(0)^2$ and similarly x(k) is this number a^kx(0). So, what do I need for definition or for asymptotic stability that limit k goes sorry, I mean write this properly. That limit k tends to infinity, x(k) this is limit K tends to infinity, a^kx(0) should go to 0, and the initial condition is not the origin. So, I am just looking at the values of this small a for which this quantity will go to 0. And this is possible if and only if a strictly less than 1, right.

So, the discreet time stability. So, if I just say, the continuous time counterpart of this $\dot{x}$= ax(t) stability was just a should be less than 0 here, a should be strictly less than 1. What does it mean in the in the pole 0 setting? So, here I have the s plane and this is sigma and then the j omega axis. So, this was all the stable region or the left half plane. In the discrete time, I am in what is also called as the z plane, ok. So, in the continuous time I know that the entire left half including the imaginary axis is the stable region, if I am also include marginal stability.

So, this entire left half plane translates to the z plane in the form of a unit circle. So, whatever is here, let me just draw this. So, this region here corresponds to the unit disk and which is the disk of radius one, so this will be 1 and so on in the z plane, ok. So, few things about marginal stability if a = 1, then x(k + 1) = x(k) and the system would be called marginally stable, a = - 1 is a little tricky because you will have your system to be oscillating between + 1 and - 1 at each discrete time steps, ok. And what will this translate to in general then is; so, here if I was looking at the real part of eigenvalues being strictly less than 0 here I would say that the eigenvalues lambda should be strictly in the unit circle or also on the boundary. So, this is what we will talk about a little later. So, this is a little idea of discrete time stability, ok.

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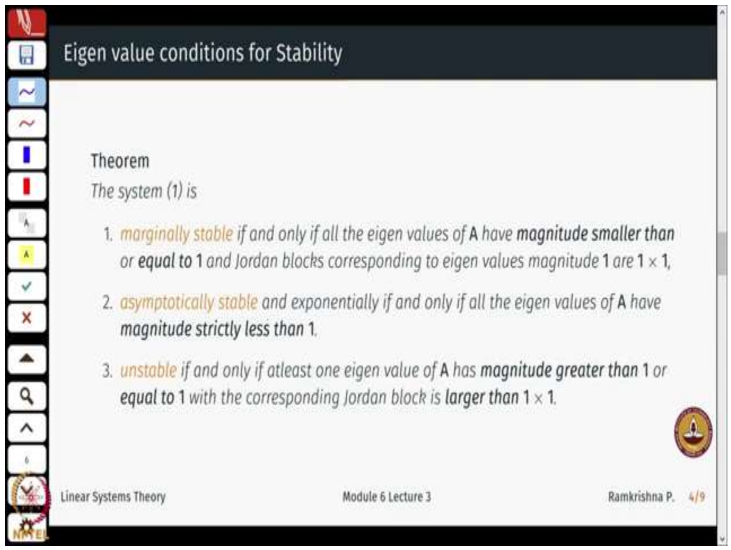

So, the eigenvalue conditions, again I will not I will not prove this these are very similar to what we had in the continuous time case. So, the eigenvalue conditions say that the system 1 is marginally stable if and only if all eigenvalues of A have magnitude smaller or equal to 1 and the same thing of the Jordan blocks, if Jordan blocks correspond to the eigenvalues of magnitude 1 are of size 1 x 1.

Asymptotically stable, if all the eigenvalues of a have magnitude strictly less than 1 and this is also equal to exponential stability. Unstable, if there is at least one eigenvalue which is on the which has magnitude larger than 1 which is the outside the unit disk, right, so I am talking here of magnitude. And what really talking of the real and imaginary part here I am just interested in the magnitude. If it rise outside the unit disk which means the magnitude is greater than 1 or when it is equal to 1 the corresponding Jordan block would be larger than 1 then 1 cross 1. So, these are the three things of stability and instability in discrete time case.

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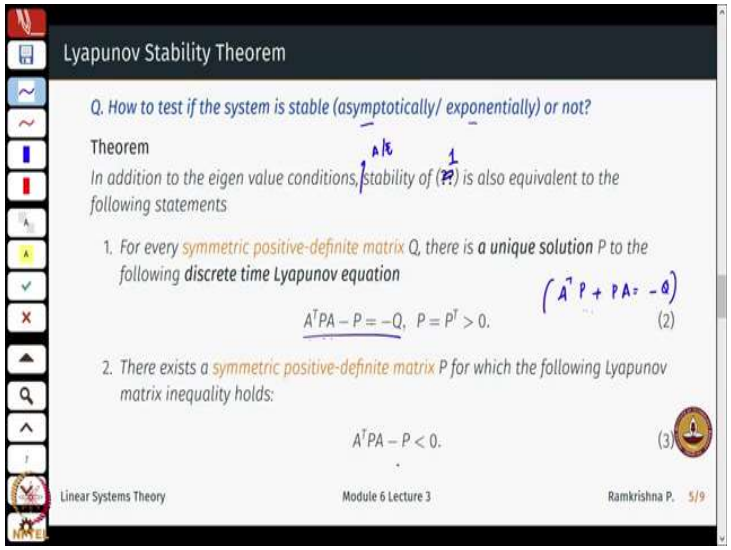

Similar in the continuous time case, ok, so, this; so, I am looking here at this is sum 1, sorry about this. So, stability in addition to the eigenvalue conditions it is also equivalent to the following statement. So, here I am looking at exponential in and asymptotic stability. So, this is not just stability, but asymptotic or exponential stability also, ok.

So, for every symmetric positive definite matrix Q, there is a unique solution P to the following discrete time Lyapunov equation. Just, the what was the contrast there? That A^TP + PA = -Q, similarly over here, right. So, I will not like do the proof of this, but give you a little idea why this expression does not exactly look the same like this, ok. So, let us do this. So, in the discrete time setting, ok.

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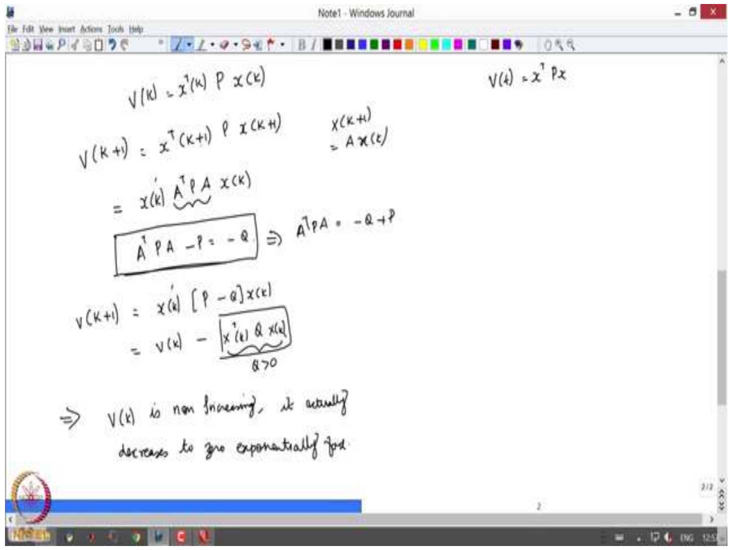

So, let me. So, we remember that we had this function, right V(t) was x^TPx, let me have again the same function V(t) is x^T(k) P x(k), sorry P of k; this, also sorry I am just write this a little better $x(k)^T$ Px(k), ok.

Now, what is V at k plus 1? V(k + 1) = $x(k+1)^T$P x(k+1), ok. Now, what is x(k + 1)? X(k + 1) = A x(k), ok. Now, if I if I substitute this back here what I will get is this will be $x(k)^T A^T$P A x(k), ok. Now, what does my Lyapunov equation tell me? In the Lyapunov equation which I would want to verify is that A^TPA - P = -Q, ok. So, V(k + 1) = $x(k)^T$(P – Q)x(k). This is equal. So, where does this come from? This is substituting for A^TPA as P minus Q from here. So, this will mean that A^TPA = -Q + P, ok. So, this will be equal to V of k minus x k P, sorry now this will be Q x(k), .

Now, what is the nature of V here? Again, this because Q > 0 this quantity here will always be greater than 0 and therefore, we can conclude that V is non-increasing, ok and not only that we can follow similar steps to show that this is not only non-increasing that it actually decrease, it actually decreases to 0 exponentially fast, ok. I will skip those steps, but just to give a little inclination to where this kind of conditions actually originate form, right.

So, ok; again back to this. So, these are the two conditions in to verify in the discrete time case. Again, I will do a supplementary lecture on what these things actually mean in the physical sense. So, can we give an interpretation to a to a physical system of what do these two conditions actually mean. So, the previous conditions of P being unique, and all those still hold here also. And the proofs will be exactly similar. So, I will just leave it to you as an exercise to verify for yourself. It will also be a good training for you to write down proofs by yourself, ok.

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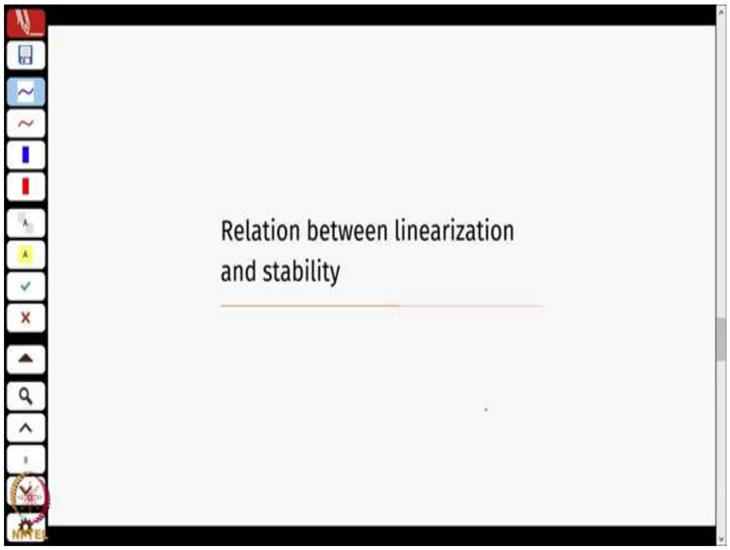

So, much of last week 5 lectures we spent on linearization, and analyzing stability of the linearization. So, linearization essentially if I just say linearization of a non-linear system, I am talking of linearization around an equilibrium point, ok.

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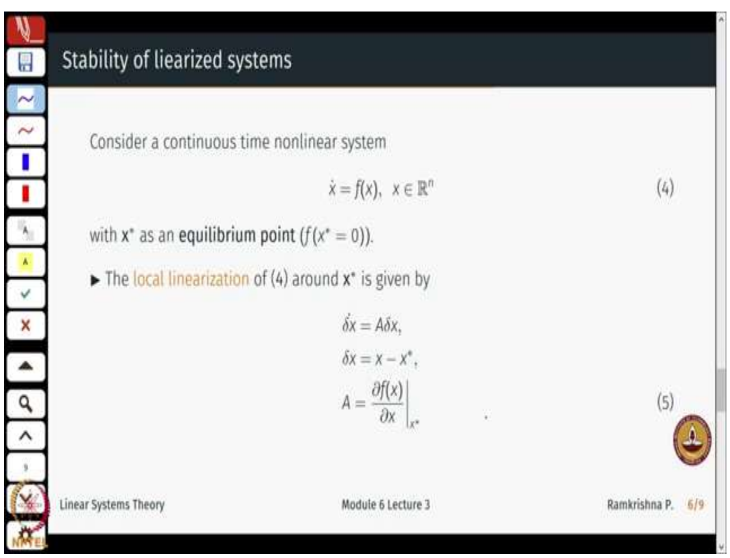

So, what was what was the relation that? So, we started off with a continuous time non-linear system, I am just talking of autonomous, time invariant, just to make the presentation a bit easy, but this will extend to the time varying case as well. So, the local linearization around x star, so we had the linear system $\delta\dot{x}$ =Aδx, where A was computed as the Jacobian of this of this function f, this vector valued function f, evaluated at the equilibrium point. What is the equilibrium point? Now, f(x^*) = 0, ok.

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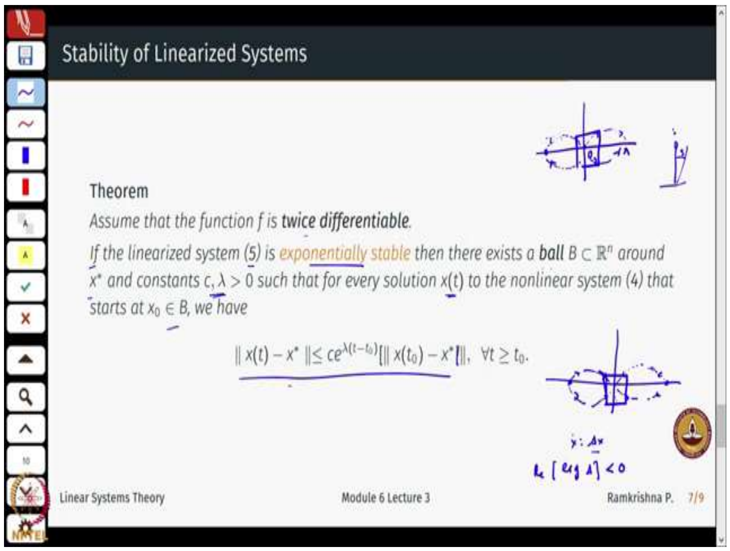

So, what does the result say? First assume that f is twice differentiable. If just differentiable once it may it may correspond to a linear system, and that is not of; you just say f is say it is a that it is 3x, right, so it is not twice differentiable. If it is x^2, it actually is its twice differentiable.

So, we are not really imposing extra conditions on f here, we just trying to infer the non-linear case in a way. So, if f is twice differentiable the linearized system, right, so the linearized version of this system $\dot{x}$ = f(x) is this is exponential. So, if the linearized system is exponentially stable then there exists a ball B around x^* and certain constants such that for every solution x(t) to the non-linear system that starts from B, we have something like this, ok, ok. Just some typos here will take care of that later, ok.

So, what does this mean? So, let us say this is my non-linear system which has some kind of a of a phase plot. Let us say if I if I talk of the inverted pendulum. So, this is unstable at Q at the upward position, so the trajectories diverge away from here, and at π I will have certain other behavior where the equilibrium will correspond to such a way trajectories around it, actually converge to that point and so on. So, if i. So, what does this mean? So, if I take, if the linearized system is exponentially stable then, ok; let us talk about the stable behavior first. So, let us me talk of the simple normal pendulum where trajectories around this come here, here, ok, they go towards origin towards origin and if I go to the upward position they diverge away, this and this, ok.

So, the linearized system here which we saw in our examples of linearization also, right that this is actually an exponentially stable behavior that, so the $\dot{x} = Ax$ in this case will have eigenvalues which have real part strictly less than 0. So, the real parts of this linearization of A will have real of eigenvalues which are strictly less than 0. And this is the linearized version is exponentially stable which means that for every solution to the non-linear system, so this phase curves tell me about the non-linear system there exists some neighborhood such that around that neighborhood for the non-linear system the solutions also converged to the to the equilibrium point, right.

So, local stability or the local linearization has a direct impact on assessing stability of the non-linear system also, right.

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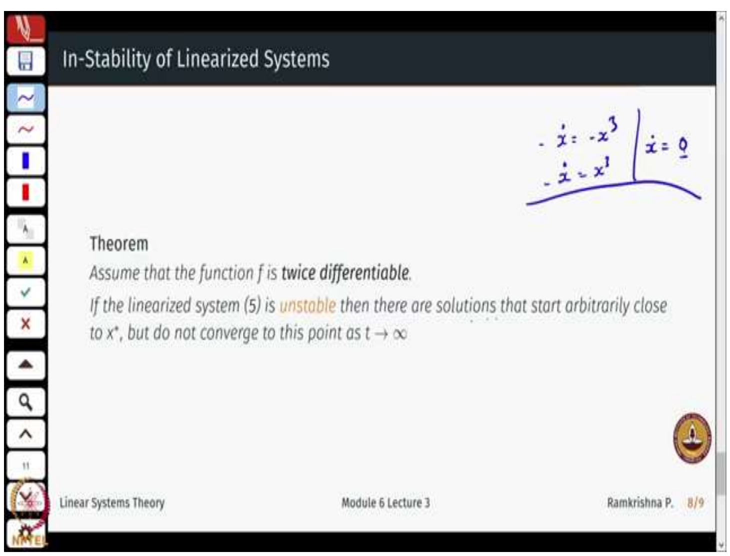

Whereas, the converse exists for unstable systems also. If the linearized system 5 is unstable, so if I if I linearize an inverted pendulum around this equilibrium point, but this is where, so, so this is how my Q is measure. So, this is Q = 0. So, if the linearize system is unstable then there are solution that start arbitrary close to x^*. So, these solutions which start arbitrarily close to x^*, they do not converge to this x^*, they just move away from x^*, ok.

So, you should also tell you that it might actually be an unstable behavior around that equilibrium and all. Solutions actually go away. They may not blow up. In some cases as we saw solutions might go away and end up in a in a limit cycle, right. So, that could; so, therefore, I am not conclusively saying that these are unstable in the way that they the solutions blow up to infinity and mathematically at least.

In this case, if I look at the stable behavior solutions that start close to the origin actually come back to the origin. So, the linearization tells me a lot of information about the local behavior of a non-linear system around an equilibrium point. And therefore, for a stability analysis in most cases I can actually do linearization, except for cases like. So, this is kind of reported very well in lot of literature, $\dot{x} = -x^3$, we will have the same linearization as $\dot{x} = x^3$ which essentially means x dot equal to 0. So, one of this is stable, one of this is unstable, therefore, the linearization method here will not work because of this of the 0 or in general if I have a 0 eigenvalue.

How to deal with this is a lot of interest in a in a non-linear literature, but we will try to avoid these cases for the moment, but as long as we avoid this case of x or being equal to 0 we can conclusively say lot of things about the local behavior of the non-linear system around that equilibrium point. I will not do the proofs of that, but intuitively we saw how these are actually true for the cases of the simple pendulum. For example where we could conclude this just by looking at the phase plots and their equivalents with the linearization of the system, right, ok.

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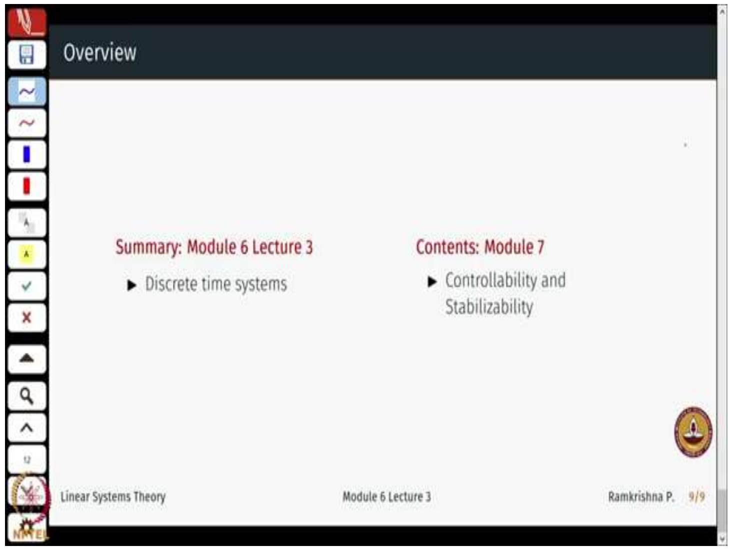

So, this concludes week 6. Of course, I will put up some supplementary materials with some with some proofs of the comparison lemma, some physical interpretation of the Lyapunov equation and so on. Module 7, we will start building up more theory towards analyzing controllability of systems, and a little weaker version, and I will tell you why this is a little weaker version of controllability called stabilizability. That is coming up next week.

Thanks for listening.